# PROGRAMMING OF WEAK GALERKIN METHOD

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## 1. POISSON TYPE EQUATIONS

1.1. **2-D** $(P_0, P_0) - RT_0$**.** We chose piecewise constant bases for boundary edges and interior of triangles. The four bases are denoted by $\phi_0, \phi_{b_1}, \phi_{b_2}, \phi_{b_3}$ as shown in Fig 1. The weak gradient is $\nabla_w \phi = Q_T(\nabla \phi)$. Here $\nabla \phi$ is understood in the distribution sense and $Q_T$ is the $L^2$ projection to $RT_0$ space. Chose a bases $\{\chi_1, \chi_2, \chi_3\}$ of $RT_0$, the computation of $\nabla_w \phi_i = Q_T(\nabla \phi_i)$ will involve the assembling of the corresponding mass matrix and the evaluation of the action $\langle \nabla \phi_i, \chi_j \rangle$.

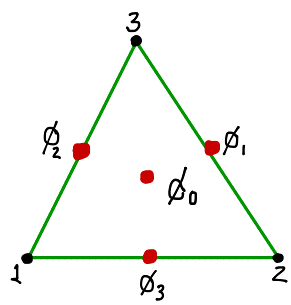


FIGURE 1. Bases of WG element

Since the inverse of the mass matrix is needed, we chose a $L^2$-orthogonal bases of $RT_0(T)$ as the following

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \chi_3 = \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}. \tag{1}$$

where $(\bar{x}, \bar{y})$ is the barycenter of triangle $T$. The mass matrix is

$$M = \operatorname{diag}(|T|, |T|, C_T^{-1}|T|),$$

where $|T|$ is the area of triangle $T$ and

$$C_T = \left[ \frac{1}{T} \int_T (x - \bar{x})^2 + (y - \bar{y})^2 \, \mathrm{d}x \, \mathrm{d}y \right]^{-1}.$$

The quantity $C_T$ can be computed using numerical quadrature, e.g., three middle points rule.

For a weak function $\phi = (\phi_0, \phi_b)$, we now compute

$$\boldsymbol{q} = (q_j) = \langle \nabla \phi, \chi_j \rangle := -(\phi_0, \nabla \cdot \chi_j)_T + (\phi_b, \chi_j \cdot n)_{\partial T}.$$

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For the basis $\phi_0$, the boundary part is vanished. Since $\nabla\cdot\chi_1 = \nabla\cdot\chi_2 = 0$, the only nonzero is $q_3^0 = -\int_T \nabla\cdot\chi_3 = -2|T|$. Therefore we obtain

$$\boldsymbol{q}^0 = \begin{pmatrix} 0 \\ 0 \\ -2|T| \end{pmatrix}, \quad \nabla\phi^0 = M^{-1}\boldsymbol{q}^0 = \begin{pmatrix} 0 \\ 0 \\ -2C_T \end{pmatrix}.$$

For the basis $\phi_{b_i}, i = 1,2,3$, only need to compute the boundary part. We compute the first two components as follows

$$q_1^i = (\phi_{b_i}, \chi_1\cdot n)_{\partial T} = \int_{e_i} \chi_1\cdot n_i dS = |e_i| n_i\cdot(1,0),$$
$$q_2^i = (\phi_{b_i}, \chi_2\cdot n)_{\partial T} = \int_{e_i} \chi_2\cdot n_i dS = |e_i| n_i\cdot(0,1).$$

Now we use the formula of gradient of barycentric coordinate $\nabla\lambda_i$

$$\nabla\lambda_i = -\frac{n_i}{d_i} = -\frac{n_i|e_i|}{2|T|}$$

to express $(q_1^i, q_2^i) = -2\nabla\lambda_i|T|$. The computation of the third component is a little bit subtle.

$$q_3^i = (\phi_{b_i}, \chi_3\cdot n)_{\partial T} = \int_{e_i} \chi_3\cdot n_i dS = (x_{im}-\bar{x}, y_{im}-\bar{y})\cdot n_i|e_i|$$
$$= \frac{1}{3}(x_{im}-x_i, y_{im}-y_i)\cdot n_i|e_i| = \frac{1}{3}d_i|e_i| = -\frac{2}{3}|T|.$$

We summarize as for $i = 1,2,3$

$$\boldsymbol{q}^i = \begin{pmatrix} -2\nabla\lambda_i|T| \\ \\ \frac{2}{3}|T| \end{pmatrix}, \quad \nabla_w\phi_{b_i} = M^{-1}\boldsymbol{q}^i = \begin{pmatrix} -2\nabla\lambda_i \\ \\ \frac{2}{3}C_T \end{pmatrix}.$$

**Remark 1.1.** Due to the nonlinear term $C_T$, the weak gradient is not affine invariant. The traditional way of computing gradient and local stiffness matrix using affine map is no longer valid.

**Remark 1.2.** It is interesting to note that the first two components of $\nabla_w\phi_{b_i}$ corresponds to the gradient of nonconforming CR element. For CR element, the three bases are $\{1-2\lambda_i\}$ and the element-wise gradient is $\{-2\nabla\lambda_i\}$.

With the formulae of weak gradient, we can compute the local stiffness by the standard formulae

$$A_{ij} = (\nabla_w\phi_i, \nabla_w\phi_j) = (\nabla_w\phi_j)^T M \nabla_w\phi_i = \boldsymbol{q}^j\cdot\nabla_w\phi_i.$$

We write the formulae for different block of the local stiffness matrix:

$$A_{b_ib_j} = 4\nabla\lambda_i\cdot\nabla\lambda_j|T| + \frac{4}{9}C_T|T|,$$
$$A_{0,b_i} = -\frac{4}{3}C_T|T|$$
$$A_{00} = 4C_T|T|.$$

If we eliminate the interior basis $\phi_0$ and form the Schur complement

$$S = A_{bb} - A_{b0}A_{00}^{-1}A_{0b} = A_{CR}$$

which is exactly the stiffness matrix for the CR nonconforming element. The difference will be the right hand side $\frac{1}{3}\int_T f$ comparing with $\int_T f(1-2\lambda_i)$.

Locally the weak function space $(P_0, P_0)$ is of dimension 4 and its gradient space $RT_0$ is dimension 3. The weak gradient $\nabla_w : (P_0, P_0) \to RT_0$ maps a $4\times 1$ vector to a $3\times 1$ vector. The matrix representation $G$ is formed by using $\nabla_w \phi_i$ as column vectors, i.e,

$$G = \left(\nabla_w\phi_0, \nabla_w\phi_1, \nabla_w\phi_2, \nabla_w\phi_3\right) = \begin{pmatrix} 0 & -2\nabla\lambda_1 & -2\nabla\lambda_2 & -2\nabla\lambda_3 \\ -2C_T & \frac{2}{3}C_T & \frac{2}{3}C_T & \frac{2}{3}C_T \end{pmatrix}.$$

It is easy to see the rank of $G$ is 3 and the null space of $G$ is the constant vector which reflects to the important property of the weak gradient

$$\nabla_w\phi = 0 \iff \phi = \text{constant}.$$

Evaluation of the weak gradient. Suppose four coefficients $\boldsymbol{u} = (u_0, u_1, u_2, u_3)^T$ are given, the product $G\boldsymbol{u}$ will give the coefficients in the bases $\boldsymbol{\chi} = (\chi_1, \chi_2, \chi_3)^T$. Then the function $\nabla_w u = \boldsymbol{\chi}^T G\boldsymbol{u}$. Using the formulae of $\chi$, we can write the weak gradient in two parts:

$$\nabla_w u = \nabla_{CR} u + h.o.t.$$

The constant vector $\nabla_{CR} u = -2\sum u_i \nabla\lambda_i$ is exactly the gradient of CR element. The h.o.t term corresponds to the contribution of $\chi_3$ whose coefficient is

$$\left[\frac{1}{3}(u_1+u_2+u_3) - u_0\right] 2C_T\chi_3.$$

The term $(u_1+u_2+u_3)/3 - u_0$ is zero for linear polynomial and interpolant and thus in general it is of order $\mathcal{O}(h^2)$. This term can be safely skipped.

We check the scaling as follows: $\nabla\lambda_i = \mathcal{O}(1/h)$ and $C_T\chi_3 = \mathcal{O}(1/h)$, i.e, as a gradient of bases, they are in $(1/h)$ scaling. The coefficient $u_i$ are $\mathcal{O}(1)$ for $\nabla_{CR} u$ and the coefficient for linear part is $\mathcal{O}(h^2)$. Two orders higher.

**Remark 1.3.** For general Poisson equation with scalar coefficient $\int_T K|\nabla u|^2$, since the lowest order scheme is used, we compute the average of $K$ over $T$ and multiply to the local stiffness matrix. For highly oscillatory or tensor coefficient, we need to compute the mass matrix

$$M_K = \left(\int_T K\chi_i\chi_j \,\mathrm{d}V\right).$$

The local stiffness matrix will be given by

$$A_{4\times 4} = G^t_{4\times 3} M_{K,3\times 3} G_{3\times 4}.$$

Note that $M_K$ may not be diagonal and the formulae of $A_{ij}$ is not concise and not necessary.

1.2. **3D $(P_0, P_0) - RT_0$.** The computation is similar. We collect the computation result and skip details.

- Bases of weak function: $\phi_0, \phi_{b_1}, \phi_{b_2}, \phi_{b_3}, \phi_{b_4}$.
- Bases of $RT_0$:

$$\chi_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \chi_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \chi_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \chi_4 = \begin{pmatrix}x-\bar{x}\\y-\bar{y}\\z-\bar{z}\end{pmatrix}.$$

- The mass matrix of $RT_0$ is
$$M = \mathrm{diag}(|T|, |T|, |T|, C_T^{-1}|T|),$$
where $|T|$ is the area of triangle $T$ and
$$C_T = \left[\frac{1}{T}\int_T (x-\bar{x})^2 + (y-\bar{y})^2 + (z-\bar{z})^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right]^{-1}.$$
- The weak gradient is
$$\begin{aligned}&\big(\nabla_w\phi_0, \nabla_w\phi_1, \nabla_w\phi_2, \nabla_w\phi_3, \nabla_w\phi_4\big)\\ =&\begin{pmatrix} 0 & -3\nabla\lambda_1 & -3\nabla\lambda_2 & -3\nabla\lambda_3 & -3\nabla\lambda_4\\ -3C_T & \frac{3}{4}C_T & \frac{3}{4}C_T & \frac{3}{4}C_T & \frac{3}{4}C_T\end{pmatrix}.\end{aligned}$$
- Local stiffness matrix
$$\begin{aligned}A_{b_ib_j} &= 9\nabla\lambda_i\cdot\nabla\lambda_j|T| + \frac{9}{16}C_T|T|,\\ A_{0,b_i} &= -\frac{9}{4}C_T|T|,\\ A_{00} &= 9C_T|T|.\end{aligned}$$