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ON FLECK QUOTIENTS

Zhi-Wei Sun[1] (Nanjing) and Daqing Wan[2] (Irvine, CA)

[1]Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
`http://math.nju.edu.cn/~zwsun`

[2]Department of Mathematics, University of California
Irvine, CA 92697-3875, USA
dwan@math.uci.edu
`http://www.math.uci.edu/~dwan`

Abstract. Let p be a prime, and let $n \geqslant 1$ and r be integers. In this paper we study Fleck's quotient

$$F_p(n,r) = (-p)^{-\lfloor (n-1)/(p-1) \rfloor} \sum_{k \equiv r \ (\mathrm{mod}\ p)} \binom{n}{k}(-1)^k \in \mathbb{Z}.$$

We determine $F_p(n,r)$ mod p completely by certain number-theoretic and combinatorial methods; consequently, if $2 \leqslant n \leqslant p$ then

$$\sum_{k=1}^{n} (-1)^{pk-1}\binom{pn-1}{pk-1} \equiv (n-1)! B_{p-n} p^n \pmod{p^{n+1}},$$

where $B_0, B_1, \ldots$ are Bernoulli numbers. We also establish the Kummer-type congruence $F_p(n+p^a(p-1),r) \equiv F_p(n,r) \pmod{p^a}$ for $a = 1,2,3,\ldots$, and reveal some connections between Fleck's quotients and class numbers of the quadratic fields $\mathbb{Q}(\sqrt{\pm p})$ and the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$. In addition, generalized Fleck quotients are also studied in this paper.

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1. Introduction and main results

Let $m \in \mathbb{Z}^+ = \{1, 2, \dots\}$, $n \in \mathbb{N} = \{0, 1, \dots\}$ and $r \in \mathbb{Z}$, and define

$$C_m(n,r) = \sum_{k \equiv r \pmod{m}} \binom{n}{k}(-1)^k. \tag{1.0}$$

This sum has been studied by various authors and many applications have been found (cf. [S02] and its references). The following well-known observation is fundamental:

$$mC_m(n,r) = \sum_{k=0}^{n} \binom{n}{k}(-1)^k \sum_{\gamma^m=1} \gamma^{k-r} = \sum_{\gamma^m=1} \gamma^{-r}(1-\gamma)^n.$$

Note that

$$C_m(n+1,r) = C_m(n,r) - C_m(n,r-1)$$

since $x^{-r}(1-x)^{n+1} = x^{-r}(1-x)^n - x^{-r+1}(1-x)^n$.

Let p be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. In 1913 A. Fleck (cf. [D, p. 274]) showed that

$$\mathrm{ord}_p(C_p(n,r)) \geqslant \left\lfloor \frac{n-1}{p-1} \right\rfloor,$$

where $\mathrm{ord}_p(\alpha)$ denotes the p-adic order of a p-adic number α, and $\lfloor \cdot \rfloor$ is the well-known floor function. Fleck's result is fundamental in the recent investigation of the ψ-operator related to Fontaine's theory, Iwasawa's theory, and p-adic Langlands correspondence (cf. [Co], [SW] and [W]); it also plays an indispensable role in Davis and Sun's study of homotopy exponents of special unitary groups (cf. [DS] and [SD]). In this paper we are interested in the *Fleck quotient*

$$F_p(n,r) := (-p)^{-\lfloor (n-1)/(p-1) \rfloor} C_p(n,r) + [\![n=0]\!]. \tag{1.1}$$

(Throughout this paper, for an assertion A we let $[\![A]\!]$ take 1 or 0 according as A holds or not.)

For $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, we use $\{a\}_m$ to denote the least nonnegative residue of a mod m (thus $\{a\}_m/m$ is the fractional part $\{a/m\}$ of a/m). For a prime p and an integer a, we define $q_p(a) = (a^{p-1}-1)/p$ which is an integer if $a \not\equiv 0 \pmod{p}$.

By a number-theoretic approach related to Gauss sums, we establish the following explicit result.

Theorem 1.1. *Let p be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Set $n_0 = \{n\}_p$ and $n_1 = \{n_0 - n\}_{p-1} = \{-\lfloor n/p \rfloor\}_{p-1}$. If $n_0 \leqslant n_1$, then*

$$F_p(n,r) \equiv \frac{(-1)^{n_1}}{n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k}(-1)^k (k-r)^{n_1} \pmod{p}. \tag{1.2}$$

If $n_0 > n_1 = 0$, *then*

$$F_p(n,r) \equiv (-1)^{\{r\}_p}\binom{n_0}{\{r\}_p} \pmod p. \tag{1.3}$$

If $n_0 > n_1 > 0$, *then*

$$F_p(n,r) \equiv \frac{(-1)^{n_1-1}}{(n_1-1)!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{n_1}q_p(k-r) \pmod p. \tag{1.4}$$

Corollary 1.1. *Let* p *be a prime and let* $n \in \mathbb{N}$ *and* $r \in \mathbb{Z}$. *Then*

$$F_p(pn,r) \equiv \frac{r^{n^*}}{n^*!} \pmod p \tag{1.5}$$

where $n^* = \{-n\}_{p-1}$. *Consequently,*

$$F_p\left(p\frac{p-1}{2},r\right) \equiv \begin{cases} (-1)^{(h(-p)+1)/2}(\frac{r}{p}) \pmod p & \text{if } p \neq 3 \ \&\ 4 \mid p+1, \\ (-1)^{(h(p)-1)/2}(\frac{r}{p})\frac{v}{2} \pmod p & \text{if } 4 \mid p-1, \end{cases} \tag{1.6}$$

where $(\frac{\cdot}{p})$ *is the Legendre symbol, and* $h(-p)$ *and* $h(p)$ *are the class numbers of the quadratic fields* $\mathbb{Q}(\sqrt{-p})$ *and* $\mathbb{Q}(\sqrt{p})$ *respectively, and for* $p \equiv 1 \pmod 4$ *we write the fundamental unit of* $\mathbb{Q}(\sqrt{p})$ *in the form* $(v+u\sqrt{p})/2$ *with* $u,v \in \mathbb{Z}$ *and* $u \equiv v \pmod 2$.

Proof. Note that $\{pn\}_p = 0$. By Theorem 1.1,

$$F_p(pn,r) \equiv \frac{(-1)^{n^*}}{n^*!}\sum_{k=0}^{0}\binom{0}{k}(-1)^k(k-r)^{n^*} = \frac{r^{n^*}}{n^*!} \pmod p.$$

When $p \neq 2$ and $n = (p-1)/2$, we have $n^* = (p-1)/2$ and hence

$$\begin{aligned} F_p\left(p\frac{p-1}{2},r\right) \equiv& r^{(p-1)/2}(-1)^{(p-1)/2}\frac{((p-1)/2)!}{\prod_{k=1}^{(p-1)/2}k(p-k)} \\ \equiv& \left(\frac{r}{p}\right)(-1)^{(p-1)/2}\frac{((p-1)/2)!}{(p-1)!} \text{ (by Euler's criterion)} \\ \equiv& (-1)^{(p+1)/2}\left(\frac{r}{p}\right)\frac{p-1}{2}! \pmod p \text{ (by Wilson's theorem)}. \end{aligned}$$

If $p > 3$ and $p \equiv 3 \pmod 4$, then

$$\frac{p-1}{2}! \equiv (-1)^{(h(-p)+1)/2} \pmod p$$

by a result of L. J. Mordell [M]. When $p \equiv 1 \pmod 4$ and $\varepsilon_p = (v + u\sqrt{p})/2 > 1$ is the fundamental unit of $\mathbb{Q}(\sqrt{p})$ with $u, v \in \mathbb{Z}$ and $u \equiv v \pmod 2$, by S. Chowla [C] we have

$$\frac{p-1}{2}! \equiv (-1)^{(h(p)+1)/2}\,\frac{v}{2} \pmod p.$$

Combining the above we immediately obtain (1.6). □

Remark. Let n be a positive integer and $p > 2n+1$ be a prime. By the first part of Corollary 1.1 in the case $r = 0$, we have

$$\binom{2pn}{pn}(-1)^n + 2\sum_{k=0}^{n-1}\binom{2pn}{pk}(-1)^k = \sum_{k=0}^{2n}\binom{2pn}{pk}(-1)^{pk} \equiv 0 \pmod{p^{2n+1}}$$

and hence

$$\binom{2pn-1}{pn-1} = \frac{1}{2}\binom{2pn}{pn} \equiv \sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{2pn}{pk} \pmod{p^{2n+1}}. \tag{1.7}$$

When $n = 1$ and $p > 3$, this gives the Wolstenholme congruence

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

When $n = 2$ and $p > 5$, (1.7) yields the following new congruence

$$\binom{4p-1}{2p-1} = \frac{1}{2}\binom{4p}{2p} \equiv \binom{4p}{p} - 1 \pmod{p^5}.$$

Our second approach to Fleck quotients is of combinatorial nature. It involves Stirling numbers of the second kind as well as higher-order Bernoulli polynomials.

Let $n \in \mathbb{N}$. The Stirling numbers $S(n,k)$ $(k \in \mathbb{N})$ of the second kind are given by

$$x^n = \sum_{k \in \mathbb{N}} S(n,k)(x)_k,$$

where

$$(x)_0 = 1 \ \text{ and } \ (x)_k = x(x-1)\cdots(x-k+1) \text{ for } k = 1, 2, \ldots.$$

Clearly, $S(n,n) = 1$, and $S(n,k) = 0$ if $k > n$. When $n + k > 0$, $S(n,k)$ is actually the number of ways to partition a set of cardinality n into k

nonempty subsets. Here is an explicit formula (cf. [LW, p. 126]) for Stirling numbers of the second kind:

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}j^n.$$

As $S(i,k)=0$ for all those $i\in\mathbb{N}$ with $i<k$, we have *Euler's identity*

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^jP(j)=0,$$

where $P(x)$ is any polynomial with $\deg P<k$ having complex number coefficients. It is known (cf. [LW, p. 126]) that

$$\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}=\frac{(e^x-1)^k}{k!};$$

in other words,

$$(e^x-1)^k=\sum_{n=k}^{\infty}\bar{S}(n,k)x^n\quad\text{with}\quad\bar{S}(n,k)=\frac{k!}{n!}S(n,k).$$

For $m=0,1,\dots$, the m-th order Bernoulli polynomials $B_n^{(m)}(t)$ $(n\in\mathbb{N})$ are defined by

$$\frac{x^me^{tx}}{(e^x-1)^m}=\sum_{n=0}^{\infty}B_n^{(m)}(t)\frac{x^n}{n!},\tag{1.8}$$

and those $B_n^{(m)}=B_n^{(m)}(0)$ are called the m-th order Bernoulli numbers. The usual Bernoulli polynomials and numbers are $B_n(t)=B_n^{(1)}(t)$ and $B_n=B_n(0)=B_n^{(1)}$ respectively. (It is well known that $B_0=1$, $B_1=-1/2$ and $B_{2k+1}=0$ for $k=1,2,\dots$; the reader may consult [IR, pp. 228–248] for the basic properties of Bernoulli numbers.) For a formal power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$, we use $[x^n]f(x)$ to denote the coefficient a_n of the monomial x^n in $f(x)$. Thus

$$\begin{aligned}B_n^{(m)}(t)=&[x^n]n!\left(\frac{x}{e^x-1}\right)^m e^{tx}\\=&[x^n]n!\sum_{k=0}^{\infty}B_k^{(m)}\frac{x^k}{k!}\sum_{j=0}^{\infty}\frac{(tx)^j}{j!}=\sum_{k=0}^{n}\binom{n}{k}B_k^{(m)}t^{n-k}.\end{aligned}$$

It is also easy to verify that $B_n^{(m)}(m-t)=(-1)^nB_n^{(m)}(t)$, and

$$\frac{B_n^{(m)}(t)}{n!}=\sum_{k_0+\cdots+k_{m-1}=n}\frac{B_{k_0}(t)}{k_0!}\prod_{0<i<m}\frac{B_{k_i}}{k_i!}\quad\text{provided } m>0.$$

If $0\leqslant n<p-1$, then $B_0,\ldots,B_n$ are p-adic integers by the von Staudt-Clausen theorem (cf. [IR, p. 233]) or the recurrence $\sum_{k=0}^{l}\binom{l+1}{k}B_k=0$ ($l=1,2,\ldots$), therefore $B_n^{(m)}(t)\in\mathbb{Z}_p[t]$ where $\mathbb{Z}_p$ is the ring of p-adic integers.

Our discovery of the next theorem was actually motivated by Theorem 1.1.

Theorem 1.2. *Let p be a prime, and let $n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Set $n^*=\{-n\}_{p-1}$. For any integer $m\equiv n\pmod p$, if $m\geqslant 0$ then $(-1)^nF_p(n,r)$ is congruent to*

$$\begin{aligned}\sum_{k=0}^{n^*}\bar S(n^*-k+m,m)\frac{(-r)^k}{k!}&=\sum_{k=0}^{n^*}\bar S(m+n^*,m+k)\binom{-r}{k}\\&=\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{(k-r)^{m+n^*}}{(m+n^*)!}\end{aligned}\tag{1.9}$$

modulo p; if $m\leqslant 0$ then we have

$$F_p(n,r)\equiv\frac{(-1)^{n^*}}{n^*!}B_{n^*}^{(-m)}(-r)\equiv-(p-1-n^*)!B_{n^*}^{(-m)}(-r)\pmod p.\tag{1.10}$$

The following consequence determines $B_n^{(m)}(a)$ modulo a prime p for $m\in\{1,\ldots,p\}$, $n\in\{0,\ldots,p-2\}$ and $a\in\mathbb{Z}$.

Corollary 1.2. *Let p be a prime and $r\in\mathbb{Z}$. Let $n_0\in\{0,\ldots,p-1\}$ and $n_1\in\{0,\ldots,p-2\}$. If $n_0\leqslant n_1$, then*

$$B_{n_1-n_0}^{(p-n_0)}(-r)\equiv\frac{1}{(n_1)_{n_0}}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^{n_0-k}(k-r)^{n_1}\pmod p.\tag{1.11}$$

If $n_0>n_1=0$, then

$$B_{p-n_0+n_1-1}^{(p-n_0)}(-r)\equiv\frac{(-1)^{\{r\}_p-1}}{n_0!}\binom{n_0}{\{r\}_p}\pmod p.\tag{1.12}$$

If $n_0>n_1>0$, then

$$\begin{aligned}B_{p-n_0+n_1-1}^{(p-n_0)}(-r)\equiv&\frac{(-1)^{n_1}}{(n_0-n_1)!(n_1-1)!}\\&\times\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{n_1}q_p(k-r)\pmod p.\end{aligned}\tag{1.13}$$

Proof. Let n be a nonnegative integer with $n \equiv n_0 - pn_1 \pmod{p(p-1)}$. Applying (1.10) with $m = n_0 - p$ we obtain

$$F_p(n,r) \equiv \frac{(-1)^{n^*}}{n^*!} B_{n^*}^{(p-n_0)}(-r) \equiv -(p-1-n^*)! B_{n^*}^{(p-n_0)}(-r) \pmod{p},$$

where $n^* = \{-n\}_{p-1}$.

If $n_0 \leqslant n_1$, then $n^* = n_1 - n_0$ and hence

$$B_{n_1-n_0}^{(p-n_0)}(-r) \equiv (-1)^{n_1-n_0}(n_1-n_0)! F_p(n,r) \pmod{p},$$

which implies (1.11) with the help of (1.2).

Now we consider the case $n_0 > n_1$. Clearly $n^* = n_1 - n_0 + p - 1$ and $p - 1 - n^* = n_0 - n_1$. Therefore

$$F_p(n,r) \equiv -(n_0-n_1)! B_{n_1-n_0+p-1}^{(p-n_0)}(-r) \pmod{p}.$$

The case $n_1 = 0$ of this, together with (1.3), yields (1.12). When $n_1 > 0$, combining the last congruence with (1.4) we obtain (1.13). $\square$

Corollary 1.3. *Let p be a prime and let $n \in \mathbb{Z}^+$. Then* $\operatorname{ord}_p(C_p(n,r)) = \lfloor (n-1)/(p-1) \rfloor$ *for at least $p - n^* \geqslant 2$ values of $r \in \{0, \ldots, p-1\}$, where $n^* = \{-n\}_{p-1}$.*

Proof. For any $r \in \mathbb{Z}$, $\operatorname{ord}_p(C_p(n,r)) = \lfloor (n-1)/(p-1) \rfloor$ if and only if $F_p(n,r) \not\equiv 0 \pmod{p}$. By Theorem 1.2,

$$F_p(n,r) \equiv \frac{(-1)^{n^*}}{n^*!} B_{n^*}^{(p-\{n\}_p)}(-r) \pmod{p} \quad \text{for all } r = 0, \ldots, p-1.$$

Recall that $B_{n^*}^{(p-\{n\}_p)}(x) \in \mathbb{Z}_p[x]$ is monic and of degree n^*. Also, a polynomial of degree n^* over the field $\mathbb{Z}/p\mathbb{Z}$ cannot have more than n^* distinct zeroes in the field (cf. [IR, p. 39]). So the congruence equation $F_p(n,r) \equiv 0 \pmod{p}$ has at most n^* solutions with $r \in \{0, \ldots, p-1\}$. This yields the desired result. $\square$

Corollary 1.4. *Let p be a prime, and let $n \in \mathbb{N}$ and $n^* = \{-n\}_{p-1}$. Then*

$$(-1)^n F_p(n,0) \equiv \bar{S}(n^* + \{n\}_p, \{n\}_p) \equiv \frac{B_{n^*}^{(m)}}{n^*!} \pmod{p}, \tag{1.14}$$

where m is any nonnegative integer with $m + n \equiv 0 \pmod{p}$. Also,

$$(-1)^n F_p(pn+p-1,r) \equiv \frac{B_{n^*}(-r)}{n^*!} \equiv -(p-1-n^*)! B_{n^*}(r+1) \pmod{p} \tag{1.15}$$

for all $r \in \mathbb{Z}$, and in particular

$$\binom{2p-1}{p+r} + (-1)^p \binom{2p-1}{r} \equiv (-1)^r p^2 B_{p-2}(-r) \pmod{p^3} \tag{1.16}$$

for every $r = 0, \ldots, p-1$.

Proof. Applying Theorem 1.2 with $r = 0$ we immediately get (1.14).

As $pn + p - 1 \equiv -1 \pmod p$ and $n^* = \{-(pn+p-1)\}_{p-1}$, by the second part of Theorem 1.2 and the identity $(-1)^{n^*} B_{n^*}(x) = B_{n^*}(1-x)$, whenever $r \in \mathbb{Z}$ we have

$$\begin{aligned}(-1)^{n^*} F_p(pn+p-1, r) \equiv & \frac{B_{n^*}(-r)}{n^*!} \equiv (-1)^{n^*+1}(p-1-n^*)! B_{n^*}(-r) \\ \equiv & -(p-1-n^*)! B_{n^*}(r+1) \pmod p\end{aligned}$$

and hence (1.15) holds.

Now let $r \in \{0, \ldots, p-1\}$. By (1.15) in the case $n = 1$,

$$-F_p(2p-1, r) \equiv -(p-1-(p-2))! B_{p-2}(r+1) \pmod p$$

and hence

$$F_p(2p-1, r) \equiv B_{p-2}(1-(-r)) = (-1)^{p-2} B_{p-2}(-r) \pmod p$$

which is equivalent to (1.16). We are done. □

Let p be an odd prime, and let h_p and h_p^+ denote the class numbers of the cyclotomic field $\mathbb{Q}(\zeta_p)$ and its maximal real subfield $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ respectively, where ζ_p is a primitive p-th root of unity in the complex field $\mathbb{C}$. It is well known that $h_p^- = h_p / h_p^+$ is an integer. If p divides none of the numerators of the Bernoulli numbers $B_0, B_2, \ldots, B_{p-3} \in \mathbb{Z}_p$, then p is said to be a *regular* prime. In 1850 E. Kummer proved that

$$\begin{aligned}p \nmid h_p \iff & p \nmid h_p^- \iff p \text{ is regular} \\ \implies & x^p + y^p = z^p \text{ has no integer solution with } xyz \neq 0.\end{aligned}$$

Furthermore,

$$h_p^- \equiv \prod_{0 < n \leqslant (p-3)/2} \left(-\frac{B_{2n}}{4n}\right) \pmod p$$

by the proof of Theorem 5.16 in [Wa, p. 62].

Corollary 1.5. *Let p be a prime.*

(i) *For every $n = 2, \dots , p$ we have*

$$\sum_{k=1}^{n}(-1)^{pk-1}\binom{pn-1}{pk-1} \equiv (n-1)!B_{p-n}p^n \pmod{p^{n+1}}. \tag{1.17}$$

(ii) *Suppose that $p > 3$. Then p does not divide the class number h_p of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$, if and only if*

$$\mathrm{ord}_p\left(\sum_{k=1}^{n}(-1)^k\binom{pn-1}{pk-1}\right) = n \quad \textit{for all } n = 3, 5, \dots , p-2.$$

Also,

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}(-1)^{k-1}\binom{p(p-1)/2-1}{pk-1}\\ \equiv&[\![4 \mid p+1]\!](-1)^{(h(-p)+1)/2}h(-p)p^{(p-1)/2} \left(\mathrm{mod}\ p^{(p+1)/2}\right),\end{aligned} \tag{1.18}$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

Proof. (i) Let $n \in \{2, \dots , p\}$. Then $\lfloor (pn-1-1)/(p-1)\rfloor = n$ and hence

$$F_p(pn-1,-1) = (-p)^{-n}C_p(pn-1,-1) = (-p)^{-n}\sum_{k=1}^{n}\binom{pn-1}{pk-1}(-1)^{pk-1}.$$

By Corollary 1.4, $(-1)^nF_p(pn-1,-1)$ is congruent to

$$(p-1-\{-(n-1)\}_{p-1})!B_{\{-(n-1)\}_{p-1}}(-1+1) = (n-1)!B_{p-n}$$

modulo p. Therefore (1.17) holds.

(ii) In view of part (i),

$$\begin{aligned}&\mathrm{ord}_p\left(\sum_{k=1}^{n}(-1)^k\binom{pn-1}{pk-1}\right) = n \quad \text{for } n = 3, 5, \dots , p-2\\ \iff& B_{p-n} \not\equiv 0 \pmod p \quad \text{for } n = 3, 5, \dots , p-2\\ \iff& p \text{ is regular}\\ \iff& h_p \not\equiv 0 \pmod p.\end{aligned}$$

Taking $n = (p-1)/2$ in (1.17) we get

$$\begin{aligned}&\sum_{k=1}^{(p-1)/2}(-1)^{k-1}\binom{p(p-1)/2-1}{pk-1}\\ \equiv&\frac{((p-1)/2)!}{(p-1)/2}p^{(p-1)/2}B_{(p+1)/2} \left(\mathrm{mod}\ p^{(p+1)/2}\right).\end{aligned}$$

If $p \equiv 1 \pmod 4$, then $B_{(p+1)/2} = 0$ since $(p+1)/2 \in \{3, 5, \dots\}$. If $p \equiv 3 \pmod 4$, then $h(-p) \equiv -2B_{(p+1)/2} \pmod p$ (cf. [IR, p. 238]), and $((p-1)/2)! \equiv (-1)^{(h(-p)+1)/2} \pmod p$ by Mordell [M]. So (1.18) follows from the above. This concludes the proof. $\square$

Remark. Let p be an odd prime. If $p \geqslant 5$, then (1.17) in the case $n = 2$ reduces to Wolstenholme's congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ since $B_{p-2} = 0$. Taking $n = 3$ in (1.17) we get

$$\binom{3p-1}{p-1} - \binom{3p-1}{2p-1} + \binom{3p-1}{3p-1} \equiv 2B_{p-3}p^3 \pmod{p^4};$$

as $\binom{3p-1}{2p-1} = 2\binom{3p-1}{p-1}$ this yields the congruence

$$\binom{3p-1}{p-1} \equiv 1 - 2p^3 B_{p-3} \pmod{p^4}.$$

This was first obtained by J.W.L. Glaisher (cf. [G1, p. 21] and [G2, p. 323]) who showed that

$$\binom{pn-1}{p-1} \equiv 1 - \frac{n(n-1)}{3} p^3 B_{p-3} \pmod{p^4} \quad \text{for } n = 1, 2, 3, \dots .$$

Corollary 1.6. *Let p be an odd prime, and let $n \in \{3, \dots, p\}$ and $r \in \mathbb{Z}$. Then*

$$F_p(pn-2, r) \equiv -n! \left(\frac{B_{p-n+1}(-r)}{n-1} + (r+1)\frac{B_{p-n}(-r)}{n} \right) \pmod p. \quad (1.19)$$

Proof. Clearly $\{-(pn-2)\}_{p-1} = p - n + 1$. By Theorem 1.2, $F_p(pn-2, r)$ is congruent to

$$-(p-1-(p-n+1))! B^{(2)}_{p-n+1}(-r) = -(n-2)! B^{(2)}_{p-n+1}(-r)$$

modulo p.

Let $m = p - n + 1$. By [PS, (2.14)] or [SP, (1.12)],

$$\begin{aligned}
&\frac{(-1)^m}{m} \sum_{k=0}^{m} \binom{m}{k} B_k B_{m-k}(x) - \frac{B_m(1-x)}{m} B_0 \\
=& - \sum_{k=0}^{1} \binom{1}{k} B_{1-k}(x) B_{m-1+k}(1-x) - B_1 B_{m-1}(1-x) \\
=& - B_1(x) B_{m-1}(1-x) - B_0(x) B_m(1-x) - B_1 B_{m-1}(1-x) \\
=& (-1)^m \left((B_1(x) + B_1) B_{m-1}(x) - B_m(x) \right) \\
=& (-1)^m \left((x-1) B_{m-1}(x) - B_m(x) \right).
\end{aligned}$$

It follows that

$$\begin{aligned}B_m^{(2)}(-r)=&\sum_{k=0}^m\binom{m}{k}B_kB_{m-k}(-r)\\=&(1-m)B_m(-r)+m(-r-1)B_{m-1}(-r)\\\equiv&(1+n-1)B_{p-n+1}(-r)-(r+1)(-n+1)B_{p-n}(-r)\\\equiv&n(n-1)\left(\frac{B_{p-n+1}(-r)}{n-1}+(r+1)\frac{B_{p-n}(-r)}{n}\right)\pmod p.\end{aligned}$$

Combining the above we immediately obtain (1.19). □

By Theorem 1.1 or 1.2, for any prime p the Fleck quotient $F_p(n,r)$ (with $n\in\mathbb{N}$ and $r\in\mathbb{Z}$) modulo p only depends on p and r and the remainder of n modulo $p(p-1)$. This observation can be further extended as follows.

Theorem 1.3. *Let p be a prime, and let $a,l,n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Then*

$$\begin{aligned}&\sum_{k=0}^n\binom{n}{k}(-1)^kF_p\left(kp^a(p-1)+l,r\right)\\&\equiv 0\ \left(\mathrm{mod}\ p^{an+\lceil (n-l^*)/(p-1)\rceil}\right),\end{aligned}\tag{1.20}$$

where $l^*=\{-l\}_{p-1}$ and $\lceil\cdot\rceil$ is the ceiling function.

The following consequence is somewhat similar to Kummer's congruence for Bernoulli numbers (cf. [IR, pp. 238–241]).

Corollary 1.7. *Let p be a prime, and let $a,l\in\mathbb{N}$ and $r\in\mathbb{Z}$. Then*

$$\begin{aligned}F_p(p^a(p-1)+l,r)\equiv&F_p(l,r)\pmod{p^a},\\F_p(2p^a(p-1)+l,r)\equiv&2F_p(p^a(p-1)+l,r)-F_p(l,r)\pmod{p^{2a}},\\F_p(3p^a(p-1)+l,r)\equiv&3F_p(2p^a(p-1)+l,r)-3F_p(p^a(p-1)+l,r)\\&+F_p(l,r)\pmod{p^{3a}}.\end{aligned}$$

Proof. Simply apply (1.20) with $n=1,2,3$. □

Let p be a prime, and let $a\in\mathbb{Z}^+$ and $r\in\mathbb{Z}$. In 1977 C. S. Weisman [We] extended Fleck's result by showing that if $n\geqslant p^{a-1}$ then

$$C_{p^a}(n,r)\equiv 0\ \left(\mathrm{mod}\ p^{\lfloor (n-p^{a-1})/\varphi(p^a)\rfloor}\right),$$

where φ is Euler's totient function. In view of this, we define the *generalized Fleck quotient*

$$F_{p^a}(n,r)=(-p)^{-\lfloor (n-p^{a-1})/\varphi(p^a)\rfloor}C_{p^a}(n,r)+[\![n<p^{a-1}]\!]\in\mathbb{Z}.$$

Note that $F_{p^a}(n,r)\equiv 1\pmod p$ for $n=0,\dots,p^{a-1}-1$.

Theorem 1.4. *Let p be a prime, and let $a, n \in \mathbb{Z}^+$ with $n \geqslant p^{a-1}$.*

(i) *For any $r \in \mathbb{Z}$ we have*

$$F_{p^a}(n,r) \equiv \sum_{k=0}^{d} \binom{r+k-1}{k} F_{p^a}(n+k,0) \pmod{p}, \tag{1.21}$$

where $d = \{p^{a-1}-1-n\}_{\varphi(p^a)}$ is the least nonnegative integer with $n+d \equiv p^{a-1}-1 \pmod{\varphi(p^a)}$.

(ii) *We have*

$$\mathrm{ord}_p\left(C_{p^a}(n,r)\right) = \left\lfloor \frac{n-p^{a-1}}{\varphi(p^a)} \right\rfloor \ \ (i.e.,\ p \nmid F_{p^a}(n,r)) \quad \text{for some } r \in \mathbb{Z}. \tag{1.22}$$

If $n \geqslant 2p^{a-1}$, then

$$F_{p^a}\left(n + p^a(p-1), r\right) \equiv F_{p^a}(n,r) \pmod{p} \quad \text{for all } r \in \mathbb{Z}. \tag{1.23}$$

In view of the first congruence in Corollary 1.7 and the last congruence in Theorem 1.4, we propose the following conjecture.

Conjecture 1.1. *Let p be a prime, and let $a, b, n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. If $n \geqslant 2p^{a+b-2}$, then*

$$F_{p^a}\left(n + \varphi(p^{a+b}), r\right) \equiv F_{p^a}(n,r) \pmod{p^b}.$$

Theorems 1.1, 1.2 and 1.3 will be proved in Sections 2, 3 and 4 respectively. In Section 5 we will first give a new proof of Weisman's congruence via roots of unity, and then establish Theorem 1.4.

2. Proof of Theorem 1.1

Lemma 2.1. *Let p be a prime, and let $n \in \mathbb{N}$ and $n^* = \{-n\}_{p-1}$. Define $G(n) = \sum_{a=1}^{p-1} a^n \zeta_p^a$ and $\pi = 1 - \zeta_p$, where ζ_p is a primitive p-th root of unity in the complex field $\mathbb{C}$. Then*

$$G(n) \equiv (-1)^{n^*-1} \sum_{m=n^*}^{p-2} s(m,n^*) \frac{\pi^m}{m!} \pmod{p}, \tag{2.1}$$

where $s(m,0), \ldots, s(m,m)$ are Stirling numbers of the first kind defined by $(x)_m = \sum_{k=0}^{m} (-1)^{m-k} s(m,k) x^k$.

Proof. Clearly,

$$\begin{aligned}
G(n) &= \sum_{a=1}^{p-1} a^n (1-\pi)^a = \sum_{a=1}^{p-1} a^n \sum_{m=0}^{a} \binom{a}{m} (-\pi)^m \\
&= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n (a)_m \\
&= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n \sum_{k=0}^{m} (-1)^{m-k} s(m,k) a^k \\
&= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{k=0}^{m} (-1)^{m-k} s(m,k) \sum_{a=1}^{p-1} a^{n+k}.
\end{aligned}$$

Since

$$1 + x + \cdots + x^{p-1} = \frac{x^p - 1}{x - 1} = \prod_{a=1}^{p-1} (x - \zeta_p^a),$$

we have

$$\frac{p}{\pi^{p-1}} = \prod_{a=1}^{p-1} \frac{1-\zeta_p^a}{\pi} = \prod_{a=1}^{p-1} \frac{1-(1-\pi)^a}{\pi} \equiv \prod_{a=1}^{p-1} a \equiv -1 \pmod{\pi}$$

with the help of Wilson's theorem. Note also that

$$\sum_{a=1}^{p-1} a^{n+k} \equiv -[\![p-1 \mid n+k]\!] \pmod{p}$$

by elementary number theory (see, e.g., [IR, pp. 235–236]). Therefore

$$\begin{aligned}
G(n) &\equiv \sum_{m=0}^{p-2} \frac{\pi^m}{m!} \sum_{k=0}^{m} (-1)^k s(m,k) (-[\![k = n^*]\!]) \\
&\equiv (-1)^{n^*-1} \sum_{m=n^*}^{p-2} s(m, n^*) \frac{\pi^m}{m!} \pmod{p}.
\end{aligned}$$

This concludes the proof. □

Remark. Let p be an odd prime. For each $a \in \mathbb{Z}$ let $\bar{a} = a + p\mathbb{Z} \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let ω be the Teichmüller character of the multiplicative group $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{\bar{0}\}$. For $\bar{a} \in \mathbb{F}_p^*$, $\omega(\bar{a})$ is just the $(p-1)$-th root of unity in the unique unramified extension of the p-adic field $\mathbb{Q}_p$ with $\omega(\bar{a}) \equiv a \pmod{p}$. (See, e.g., [Wa, p. 51].) If ζ_p is a primitive p-th root of unity in the algebraic closure of $\mathbb{Q}_p$, then for $n \in \mathbb{N}$ and $\pi = 1 - \zeta_p$ we have

$$\sum_{a=1}^{p-1} a^n \zeta_p^a \equiv \sum_{a=1}^{p-1} \omega^n(\bar{a}) \zeta_p^a \equiv -\frac{(-\pi)^{n^*}}{n^*!} \pmod{\pi^{n^*+1}}$$

with $n^* = \{-n\}_{p-1}$, by Stickelberger's congruence for Gauss' sums (cf. [BEW, pp. 344–345]).

Lemma 2.2. *Let p be a prime, and let ζ_p be a primitive p-th root of unity in $\mathbb{C}$. Let $n = p^a m + n_0 > 0$ with $a \in \mathbb{Z}^+$ and $m, n_0 \in \mathbb{N}$. Then, for any $r \in \mathbb{Z}$ we have*

$$\begin{aligned}&\pi^{-p^a m} C_p(n,r) - [\![p-1 \mid m]\!] C_p(n_0, r)\\ \equiv&\frac{G(p^a m)}{p} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{p^a m^*} \pmod{p^{a-1}\pi^{\min\{n_0+1,\, p-1\}}},\end{aligned}$$

where $\pi = 1 - \zeta_p$ and $m^* = \{-m\}_{p-1}$.

Proof. Let $j \in \{1, \ldots, p-1\}$. Then

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^m = \left(\frac{1-(1-\pi)^j}{\pi}\right)^m = \left(\sum_{i=1}^{j} \binom{j}{i} (-\pi)^{i-1}\right)^m = j^m + \beta_j \pi,$$

where β_j is a suitable element in the ring $\overline{\mathbb{Z}}$ of algebraic integers. For $i = 0, 1, \ldots,$ if

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p^i m} = j^{p^i m} + p^i \pi \beta_j^{(i)}$$

for some $\beta_j^{(i)} \in \overline{\mathbb{Z}}$, then

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p^{i+1} m} = \left(j^{p^i m} + p^i \pi \beta_j^{(i)}\right)^p = j^{p^{i+1} m} + p^{i+1} \pi \beta_j^{(i+1)}$$

for some $\beta_j^{(i+1)} \in \overline{\mathbb{Z}}$. So

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p^a m} \equiv j^{p^a m} \pmod{p^a \pi}.$$

Observe that

$$pC_p(n,r) = \sum_{j=0}^{p-1} \zeta_p^{-jr} (1-\zeta_p^j)^n = \pi^{p^a m} \sum_{j=1}^{p-1} \zeta_p^{-jr} \left(\frac{1-\zeta_p^j}{\pi}\right)^{p^a m} (1-\zeta_p^j)^{n_0}.$$

As π^{n_0} divides $(1-\zeta_p^j)^{n_0}$ in the ring $\overline{\mathbb{Z}}$, by the above $\pi^{-p^a m} pC_p(n,r)$ is congruent to

$$\sum_{j=1}^{p-1} \zeta_p^{-jr} j^{p^a m} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \zeta_p^{jk} = \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k S_{k-r}$$

modulo $p^a\pi^{n_0+1}$, where

$$S_{k-r}=\sum_{j=1}^{p-1} j^{p^a m}\zeta_p^{j(k-r)}.$$

If $k\not\equiv r \pmod p$, then

$$\begin{aligned}S_{k-r}=&(k-r)^{-p^a m}\sum_{j=1}^{p-1}(j(k-r))^{p^a m}\zeta_p^{j(k-r)}\\ \equiv&(k-r)^{p^a m^*}\sum_{t=1}^{p-1} t^{p^a m}\zeta_p^t=(k-r)^{p^a m^*}G(p^a m) \pmod{p^{a+1}}.\end{aligned}$$

(Note that if $j(k-r)\equiv t \pmod p$ then $(j(k-r))^{p^a}\equiv t^{p^a} \pmod{p^{a+1}}$.)
Choose a primitive root g modulo p. Since

$$(g^{p^a m}-1)\sum_{j=1}^{p-1} j^{p^a m}=\sum_{j=1}^{p-1}(gj)^{p^a m}-\sum_{t=1}^{p-1} t^{p^a m}\equiv 0 \pmod{p^{a+1}},$$

if $p-1\nmid m$ then $g^{p^a m}-1\not\equiv 0 \pmod p$ and so $\sum_{j=1}^{p-1} j^{p^a m}\equiv 0 \pmod{p^{a+1}}$. Thus, when $k\equiv r \pmod p$ we have

$$S_{k-r}=\sum_{j=1}^{p-1} j^{p^a m}\equiv (p-1)[\![p-1\mid m]\!] \pmod{p^{a+1}}.$$

Recall that $p/\pi^{p-1}\equiv -1 \pmod \pi$. In view of the above,

$$\begin{aligned}&\pi^{-p^a m}pC_p(n,r)-\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{p^a m^*}G(p^a m)\\ \equiv&\sum_{\substack{k=0\\ p\mid k-r}}^{n_0}\binom{n_0}{k}(-1)^k\left([\![p-1\mid m]\!](p-1)-(k-r)^{p^a m^*}G(p^a m)\right)\\ \equiv&C_p(n_0,r)[\![p-1\mid m]\!]p \pmod{p^a\pi^{\min\{n_0+1,\,p-1\}}},\end{aligned}$$

where we have noted that if $p-1\mid m$ (i.e., $m^*=0$) then

$$p-1-G(p^a m)\equiv p-\sum_{t=0}^{p-1}\zeta_p^t=p-\frac{1-\zeta_p^p}{1-\zeta_p}=p \pmod{p^{a+1}}.$$

Therefore the desired congruence follows. □

Proof of Theorem 1.1. In the case $n=0$, (1.2) holds since $n_1=n_0=0$ and $F_p(n,r)=-pC_p(0,r)+1$. Below we assume $n>0$.

Let ζ_p be a primitive p-th root of unity in $\mathbb{C}$, and set $\pi=1-\zeta_p$. By Lemma 2.2 in the case $a=1$,

$$\begin{aligned}&\pi^{-p\lfloor n/p\rfloor}C_p(n,r)-[\![n_1=0]\!]C_p(n_0,r)\\ \equiv&\frac{G(p\lfloor n/p\rfloor)}{p}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\ \left(\mathrm{mod}\ \pi^{\min\{n_0+1,\,p-1\}}\right).\end{aligned}$$

In view of Lemma 2.1,

$$G\left(p\left\lfloor\frac{n}{p}\right\rfloor\right)\equiv G\left(\left\lfloor\frac{n}{p}\right\rfloor\right)\equiv(-1)^{n_1-1}\sum_{m=n_1}^{p-2}s(m,n_1)\frac{\pi^m}{m!}\ (\mathrm{mod}\ p).$$

If $n_0>n_1$, then

$$\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\equiv\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{n_1}=0\ (\mathrm{mod}\ p),$$

where we have applied Fermat's little theorem and Euler's identity (mentioned in Section 1). Therefore

$$\begin{aligned}&\pi^{-p\lfloor n/p\rfloor}C_p(n,r)-[\![n_1=0]\!]C_p(n_0,r)\\ \equiv&\frac{(-1)^{n_1-1}}{p}\sum_{m=n_1}^{p-2}s(m,n_1)\frac{\pi^m}{m!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\\ &\qquad\left(\mathrm{mod}\ \pi^{[\![n_0>n_1]\!]\min\{n_0+1,\,p-1\}}\right).\end{aligned}$$

Recall that $-p/\pi^{p-1}\equiv1\ (\mathrm{mod}\ \pi)$. Since $s(n_1,n_1)=1$ and

$$\frac{p^{[\![n_0\leqslant n_1]\!]}}{\pi^{n_1}}\pi^{[\![n_0>n_1]\!]\min\{n_0+1,\,p-1\}}\equiv0\ (\mathrm{mod}\ \pi),$$

by the above we have

$$\begin{aligned}&\frac{p^{[\![n_0\leqslant n_1]\!]}C_p(n,r)}{\pi^{p\lfloor n/p\rfloor+n_1}}-p^{[\![n_0=0]\!]}[\![n_1=0]\!]C_p(n_0,r)\\ \equiv&\frac{(-1)^{n_1-1}/n_1!}{p^{[\![n_0>n_1]\!]}}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\ (\mathrm{mod}\ \pi).\end{aligned}$$

Note that

$$\left\lfloor\frac{n-1}{p-1}\right\rfloor=\left\lfloor\frac{p\lfloor n/p\rfloor+n_0-1}{p-1}\right\rfloor=\frac{p\lfloor n/p\rfloor+n_1}{p-1}-[\![n_0\leqslant n_1]\!]$$

and hence

$$\begin{aligned}\frac{(-p)^{[\![n_0\leqslant n_1]\!]}C_p(n,r)}{\pi^{p\lfloor n/p\rfloor+n_1}} =&\frac{C_p(n,r)}{(-p)^{\lfloor (n-1)/(p-1)\rfloor}}\left(\frac{-p}{\pi^{p-1}}\right)^{(p\lfloor n/p\rfloor+n_1)/(p-1)}\\ \equiv&F_p(n,r) \pmod{\pi}.\end{aligned}$$

In view of the above,

$$\begin{aligned}&(-1)^{[n_0\leqslant n_1]}F_p(n,r)-[\![n_0>n_1=0]\!]C_p(n_0,r)\\ \equiv&\frac{(-1)^{n_1-1}/n_1!}{p^{[\![n_0>n_1]\!]}}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1} \pmod{\pi}.\end{aligned}$$

As the rational p-adic integer

$$\begin{aligned}D=&F_p(n,r)-[\![n_0>n_1=0]\!]C_p(n_0,r)\\ &-\frac{(-1)^{n_1}}{(-p)^{[\![n_0>n_1]\!]}\cdot n_1!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1}\end{aligned}$$

is divisible by π, we have $D^{p-1}\equiv 0 \pmod{p}$ and hence $D\equiv 0 \pmod{p}$. Thus

$$\begin{aligned}&F_p(n,r)-[\![n_0>n_1=0]\!]C_p(n_0,r)\\ \equiv&\frac{(-1)^{n_1}}{(-p)^{[\![n_0>n_1]\!]}\cdot n_1!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k(k-r)^{pn_1} \pmod{p}.\end{aligned}\tag{2.2}$$

In the case $n_0\leqslant n_1$, (2.2) reduces to (1.2). When $n_0>n_1=0$, (2.2) yields (1.3) since $C_p(n_0,r)=(-1)^{\{r\}_p}\binom{n_0}{\{r\}_p}$ and $\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k=(1-1)^{n_0}=0$.

Now assume that $n_0>n_1>0$. As $\sum_{k=0}^{n_0}\binom{n_0}{k}(k-r)^{n_1}=0$ by Euler's identity, (2.2) implies that

$$F_p(n,r)\equiv\frac{(-1)^{n_1-1}}{n_1!}\sum_{k=0}^{n_0}\binom{n_0}{k}(-1)^k\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p} \pmod{p}.$$

If $n_1=1$, then

$$\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p}=(k-r)^{n_1}n_1q_p(k-r);$$

if $n_1\geqslant 2$ and $k\equiv r \pmod{p}$, then

$$\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p}\equiv 0\equiv (k-r)^{n_1}n_1q_p(k-r) \pmod{p};$$

if $a=k-r\not\equiv 0 \pmod{p}$, then

$$\frac{(k-r)^{pn_1}-(k-r)^{n_1}}{p}=a^{n_1}\frac{(1+p\cdot q_p(a))^{n_1}-1}{p}\equiv a^{n_1}n_1q_p(a) \pmod{p}.$$

Therefore (1.4) follows.

The proof is now complete. □

3. Proof of Theorem 1.2

The following lemma is a refinement of an induction technique used by Sun [S06].

Lemma 3.1. *Let p be a prime, and let $n \in \mathbb{N}$ with $n \geqslant p$. Then*

$$F_p(n,r) \equiv -\sum_{j=1}^{p-1}\frac{1}{j}\sum_{i=0}^{j-1}F_p(n-p+1,r-i) \pmod p. \tag{3.1}$$

Proof. Set $n' = n-(p-1) > 0$. By the Chu-Vandermonde convolution identity (cf. [GKP, (5.27)]),

$$\begin{aligned}
F_p(n,r) =&(-p)^{-\lfloor (n-1)/(p-1)\rfloor}\sum_{\substack{0\leqslant k\leqslant n\\ k\equiv r \ (\mathrm{mod}\ p)}}\sum_{j=0}^{k}\binom{p-1}{j}\binom{n'}{k-j}(-1)^k\\
=&-\frac{1}{p}\sum_{j=0}^{p-1}\binom{p-1}{j}(-p)^{-\lfloor (n'-1)/(p-1)\rfloor}\sum_{\substack{j\leqslant k\leqslant n\\ p\mid k-r}}\binom{n'}{k-j}(-1)^k\\
=&-\frac{1}{p}\sum_{j=0}^{p-1}\binom{p-1}{j}(-1)^jF_p(n',r-j).
\end{aligned}$$

For any $j=0,\ldots,p-1$, clearly

$$\begin{aligned}
&\binom{p-1}{j}(-1)^j=\prod_{0<i\leqslant j}\left(1-\frac{p}{i}\right)\\
\equiv&1-\sum_{0<i\leqslant j}\frac{p}{i}\equiv(-1)^{p-1}+p\sum_{j<k<p}\frac{1}{k} \pmod{p^2}.
\end{aligned}$$

(Note that $2\sum_{k=1}^{p-1}1/k=\sum_{k=1}^{p-1}(1/k+1/(p-k))\equiv 0 \pmod p$.) Also,

$$\sum_{j=0}^{p-1}F_p(n',r-j)=(-p)^{-\lfloor (n'-1)/(p-1)\rfloor}\sum_{k=0}^{n'}\binom{n'}{k}(-1)^k=0.$$

Therefore

$$F_p(n,r)\equiv-\sum_{j=0}^{p-1}\sum_{j<k<p}\frac{F_p(n',r-j)}{k}=-\sum_{k=1}^{p-1}\frac{1}{k}\sum_{j=0}^{k-1}F_p(n',r-j) \pmod p.$$

This proves (3.1). $\square$

Proof of Theorem 1.2. (i) Suppose $m \geqslant 0$. Then

$$\begin{aligned}
&\sum_{k=0}^{n^*} \bar{S}(m+n^*-k,m)\frac{(-r)^k}{k!}\\
=&[x^{m+n^*}]\sum_{l=m}^{\infty}\bar{S}(l,m)x^l\sum_{k=0}^{\infty}\frac{(-rx)^k}{k!}\\
=&[x^{m+n^*}](e^x-1)^m e^{-rx} = [x^{n^*}]\left(\frac{e^x-1}{x}\right)^m e^{-rx}\\
=&[x^{m+n^*}]\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}e^{(k-r)x} = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{(k-r)^{m+n^*}}{(m+n^*)!}.
\end{aligned}$$

By the identity (2.4) of Sun [S03], for any $l = 0, 1, \ldots$ we have

$$\begin{aligned}
\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(k+l)^{m+n^*} =&\sum_{j=0}^{l}\binom{l}{j}(m+j)!S(m+n^*,m+j)\\
=&\sum_{j=0}^{n^*}\binom{l}{j}(m+j)!S(m+n^*,m+j).
\end{aligned}$$

Thus

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(k+x)^{m+n^*} = \sum_{j=0}^{n^*}\binom{x}{j}(m+j)!S(m+n^*,m+j)$$

and hence

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{(k-r)^{m+n^*}}{(m+n^*)!} = \sum_{j=0}^{n^*}\binom{-r}{j}\bar{S}(m+n^*,m+j).$$

If $m \leqslant 0$, then

$$\frac{B_{n^*}^{(-m)}(-r)}{n^*!} = [x^{n^*}]\left(\frac{x}{e^x-1}\right)^{-m} e^{-rx} = [x^{n^*}]\left(\frac{e^x-1}{x}\right)^m e^{-rx}.$$

Note also that

$$\frac{1}{n^*!} = \frac{\prod_{j=1}^{p-1-n^*}(p-j)}{(p-1)!} \equiv (-1)^{n^*+1}(p-1-n^*)! \pmod p$$

by Wilson's theorem.

In view of the above, whether $m \geqslant 0$ or $m \leqslant 0$, we only need to show that
$$(-1)^n F_p(n,r) \equiv [x^{n^*}]\left(\frac{e^x-1}{x}\right)^m e^{-rx} \pmod p.$$

(ii) All those formal power series $f(x)=\sum_{k=0}^\infty a_k x^k$ with $a_k \in \mathbb{Q}$ and $a_0,\ldots,a_{n^*} \in \mathbb{Z}_p$ form a ring R_{n^*} under the usual addition and multiplication. In particular, this ring contains
$$e^{-rx}=\sum_{k=0}^\infty (-r)^k \frac{x^k}{k!},\quad \frac{e^x-1}{x}=\sum_{k=0}^\infty \frac{x^k}{(k+1)!} \text{ and } \frac{x}{e^x-1}=\sum_{k=0}^\infty B_k\frac{x^k}{k!}.$$
(Recall that $n^*<p-1$ and $B_0,\ldots,B_{n^*}\in\mathbb{Z}_p$.) If $f(x)=\sum_{k=0}^\infty a_k x^k$ and $g(x)=\sum_{k=0}^\infty b_k x^k$ belong to R_{n^*}, then
$$\begin{aligned}&[x^{n^*}]f(x)g(x)^p=[x^{n^*}]\sum_{j=0}^{n^*}a_jx^j\left(\sum_{k=0}^{n^*}b_kx^k\right)^p\\ \equiv&[x^{n^*}]\sum_{j=0}^{n^*}a_jx^j\sum_{k=0}^{n^*}b_k^px^{pk}=a_{n^*}b_0^p\equiv[x^{n^*}]f(x)[x^0]g(x) \pmod p.\end{aligned}$$
Consequently, for any $a\in\mathbb{Z}$ we have
$$[x^{n^*}]\left(\frac{e^x-1}{x}\right)^m e^{ax}\equiv[x^{n^*}]\left(\frac{e^x-1}{x}\right)^n e^{ax} \pmod p$$
since $m\equiv n \pmod p$. By this and part (i), it suffices to use induction on n to show that
$$(-1)^n F_p(n,r) \equiv [x^{n^*}]\left(\frac{e^x-1}{x}\right)^n e^{-rx} \pmod p. \tag{3.2}$$

(iii) Obviously
$$(-1)^0F_p(0,r)=-pC_p(0,r)+1\equiv 1=[x^0]\left(\frac{e^x-1}{x}\right)^0 e^{-rx} \pmod p.$$
So (3.2) holds for $n=0$.

Suppose that $0<n\leqslant p-1$. Then $n^*=p-1-n$ and
$$\begin{aligned}&[x^{n^*}]\left(\frac{e^x-1}{x}\right)^n e^{-rx}=[x^{p-1}](e^x-1)^ne^{-rx}\\ =&\sum_{k=0}^n\binom{n}{k}(-1)^{n-k}[x^{p-1}]e^{(k-r)x}=\sum_{k=0}^n\binom{n}{k}(-1)^{n-k}\frac{(k-r)^{p-1}}{(p-1)!}\\ \equiv&(-1)^{n-1}\sum_{k\not\equiv r \,(\mathrm{mod}\ p)}\binom{n}{k}(-1)^k \pmod p.\end{aligned}$$

(To get the last congruence we have applied Wilson's theorem and Fermat's little theorem.) Since

$$-\sum_{k\not\equiv r \,(\mathrm{mod}\ p)}\binom{n}{k}(-1)^k=\sum_{k\equiv r\,(\mathrm{mod}\ p)}\binom{n}{k}(-1)^k=F_p(n,r),$$

the desired (3.2) follows.

Now fix $n\geqslant p$ and assume that (3.2) holds for smaller values of n. Clearly $n'=n-(p-1)>0$ and $\{-n'\}_{p-1}=n^*$. In light of Lemma 3.1,

$$F_p(n,r)\equiv-\sum_{j=1}^{p-1}\frac{1}{j}\sum_{k=0}^{j-1}F_p(n',r-k)\ (\mathrm{mod}\ p).$$

By the induction hypothesis and part (ii),

$$\begin{aligned}(-1)^{n'}F_p(n',r-k)\equiv&[x^{n^*}]\left(\frac{e^x-1}{x}\right)^{n'}e^{-(r-k)x}\\ \equiv&[x^{n^*}]\left(\frac{e^x-1}{x}\right)^{n+1}e^{(k-r)x}\ (\mathrm{mod}\ p).\end{aligned}$$

Thus $(-1)^{n-1}F_p(n,r)$ is congruent to

$$\begin{aligned}&\sum_{j=1}^{p-1}\frac{1}{j}\sum_{k=0}^{j-1}\left([x^{n^*}]\left(\frac{e^x-1}{x}\right)^{n+1}e^{(k-r)x}\right)\\ =&[x^{n^*}]\left(\frac{e^x-1}{x}\right)^{n+1}e^{-rx}\sum_{j=1}^{p-1}\left(\frac{1}{j}\cdot\frac{e^{jx}-1}{e^x-1}\right)\\ =&[x^{n^*}]\left(\frac{e^x-1}{x}\right)^{n}e^{-rx}\sum_{j=1}^{p-1}\frac{e^{jx}-1}{jx}\end{aligned}$$

modulo p. This yields

$$\begin{aligned}(-1)^nF_p(n,r)\equiv&-[x^{n^*}]\left(\frac{e^x-1}{x}\right)^{n}e^{-rx}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}\frac{(jx)^{k-1}}{k!}\\ \equiv&[x^{n^*}]\left(\frac{e^x-1}{x}\right)^{n}e^{-rx}\ (\mathrm{mod}\ p),\end{aligned}$$

since $n^*<p-1$ and $\sum_{j=1}^{p-1}j^{k-1}\equiv-[\![p-1\mid k-1]\!]\ (\mathrm{mod}\ p)$.

In view of the above, we have completed the proof. $\square$

4. Proof of Theorem 1.3

Proof of Theorem 1.3. Let ζ_p be a primitive p-th root of unity in $\mathbb{C}$, and set $\pi = 1-\zeta_p$. For any $k=0,\ldots,n$, we have

$$\begin{aligned} pC_p(kp^a(p-1)+l,r) =& \sum_{j=0}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^{kp^a(p-1)+l} \\ =& \sum_{j=1}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^{kp^a(p-1)+l} + [\![k=l=0]\!] \end{aligned}$$

and thus

$$\begin{aligned} & F_p(kp^a(p-1)+l,r) \\ =&(-p)^{-\lfloor (kp^a(p-1)+l-1)/(p-1)\rfloor}C_p(kp^a(p-1)+l,r)+[\![k=l=0]\!] \\ =&-(-p)^{-kp^a-\lfloor (l-1)/(p-1)\rfloor-1}\sum_{j=1}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^{kp^a(p-1)+l}. \end{aligned}$$

Therefore, for $S_n=\sum_{k=0}^n\binom{n}{k}(-1)^kF_p(kp^a(p-1)+l,r)$ we have

$$S_n = -\sum_{j=1}^{p-1}\zeta_p^{-jr}(1-\zeta_p^j)^l(-p)^{-\lfloor (l-1)/(p-1)\rfloor-1}c_{n,j}, \tag{4.1}$$

where

$$\begin{aligned} c_{n,j} =& \sum_{k=0}^n\binom{n}{k}(-1)^k(-p)^{-kp^a}(1-\zeta_p^j)^{kp^a(p-1)} \\ =& \left(1-(-p)^{-p^a}(1-\zeta_p^j)^{p^a(p-1)}\right)^n. \end{aligned}$$

Let $j\in\{1,\ldots,p-1\}$. Clearly

$$\left(\frac{1-\zeta_p^j}{\pi}\right)^{p-1} = \left(\frac{1-(1-\pi)^j}{\pi}\right)^{p-1} \equiv j^{p-1}\equiv 1 \pmod{\pi}$$

and hence

$$b_j := \frac{(1-\zeta_p^j)^{p-1}}{-p} = \left(\frac{1-\zeta_p^j}{\pi}\right)^{p-1}\frac{\pi^{p-1}}{-p}\equiv 1 \pmod{\pi}.$$

(Recall the congruence $p/\pi^{p-1}\equiv -1 \pmod{\pi}$.) It follows that $b_j^{p^a}\equiv 1 \pmod{p^a\pi}$ and

$$c_{n,j} = \left(1-b_j^{p^a}\right)^n \equiv 0 \pmod{p^{an}\pi^n}. \tag{4.2}$$

Since $(1-\zeta_p^j)^l \equiv 0 \pmod{\pi^l}$ and $\mathrm{ord}_p(\pi) = 1/(p-1)$, in view of (4.1) and (4.2) we have

$$\mathrm{ord}_p(S_n) \geqslant \frac{l+n}{p-1} + an - \left\lfloor \frac{l-1}{p-1} \right\rfloor - 1 = an + \frac{l+n}{p-1} - \frac{l+l^*}{p-1} = an + \frac{n-l^*}{p-1}$$

and hence $\mathrm{ord}_p(S_n) \geqslant an + \lceil (n-l^*)/(p-1) \rceil$. This proves (1.20). $\square$

5. On generalized Fleck quotients

Lemma 5.1. *Let $d, q \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Let ζ_{dq} be a primitive dq-th root of unity in $\mathbb{C}$. Then*

$$C_{dq}(n,r) = \frac{1}{d}\sum_{k=0}^{n}\binom{n}{k}C_q(k,r)\sum_{j=0}^{d-1}\zeta_{dq}^{j(k-r)}\left(1-\zeta_{dq}^{j}\right)^{n-k}. \tag{5.1}$$

Proof. Note that $\zeta = \zeta_{dq}^d$ is a primitive q-th root of unity. Thus

$$\begin{aligned}
&q\sum_{k=0}^{n}\binom{n}{k}C_q(k,r)\sum_{j=0}^{d-1}\zeta_{dq}^{j(k-r)}\left(1-\zeta_{dq}^{j}\right)^{n-k}\\
=&\sum_{k=0}^{n}\binom{n}{k}\sum_{s=0}^{q-1}\zeta^{-sr}(1-\zeta^s)^k\sum_{j=0}^{d-1}\zeta_{dq}^{j(k-r)}\left(1-\zeta_{dq}^{j}\right)^{n-k}\\
=&\sum_{s=0}^{q-1}\sum_{j=0}^{d-1}\zeta_{dq}^{-(ds+j)r}\sum_{k=0}^{n}\binom{n}{k}\left(\zeta_{dq}^{j}(1-\zeta_{dq}^{ds})\right)^k\left(1-\zeta_{dq}^{j}\right)^{n-k}\\
=&\sum_{s=0}^{q-1}\sum_{j=0}^{d-1}\zeta_{dq}^{-(ds+j)r}\left(1-\zeta_{dq}^{ds+j}\right)^{n}\\
=&\sum_{t=0}^{dq-1}\zeta_{dq}^{-tr}\left(1-\zeta_{dq}^{t}\right)^{n} = dqC_{dq}(n,r).
\end{aligned}$$

So we have (5.1). $\square$

With the help of Lemma 5.1 we can prove the following result via roots of unity.

Theorem 5.1 (Weisman, 1977). *Let p be a prime, and let $a \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then $F_{p^a}(n,r) \in \mathbb{Z}$.*

Proof. We use induction on a.

The case $a = 1$ reduces to Fleck's result. A proof of Fleck's result via roots of unity was given by A. Granville [Gr].

Now let $a \geqslant 2$ and assume that $F_{p^{a-1}}(n', r') \in \mathbb{Z}$ for all $n' \in \mathbb{N}$ and $r' \in \mathbb{Z}$. If $n < p^a$, then $\lfloor (n - p^{a-1})/\varphi(p^a) \rfloor \leqslant 0$ and hence $F_{p^a}(n, r) \in \mathbb{Z}$. Below we suppose $n \geqslant p^a$ and let ζ_{p^a} be a primitive p^a-th root of unity in $\mathbb{C}$.

By Lemma 5.1,

$$C_{p^a}(n, r) = \frac{1}{p} \sum_{k=0}^{n} \binom{n}{k} C_{p^{a-1}}(k, r) \sum_{j=0}^{p-1} \zeta_{p^a}^{j(k-r)} \left(1 - \zeta_{p^a}^{j}\right)^{n-k}. \tag{5.2}$$

Observe that

$$\prod_{\substack{j=1 \\ p \nmid j}}^{p^a - 1} \left(1 - \zeta_{p^a}^{j}\right) = \prod_{\substack{\gamma^{p^a} = 1 \\ \gamma^{p^{a-1}} \neq 1}} (1 - \gamma) = \lim_{x \to 1} \frac{x^{p^a} - 1}{x^{p^{a-1}} - 1} = \frac{p^a}{p^{a-1}} = p.$$

If $p \nmid j$, then $(1 - \zeta_{p^a}^{j})/(1 - \zeta_{p^a})$ is a unit in the ring $\mathbb{Z}[\zeta_{p^a}]$ and thus

$$\mathrm{ord}_p(1 - \zeta_{p^a}^{j}) = \mathrm{ord}_p(1 - \zeta_{p^a}) = \frac{1}{\varphi(p^a)}.$$

By this and the induction hypothesis, for any $k = 0, \ldots, n$ we have

$$\begin{aligned}
&\mathrm{ord}_p\left(C_{p^{a-1}}(k, r) \sum_{j=0}^{p-1} \zeta_{p^a}^{j(k-r)} \left(1 - \zeta_{p^a}^{j}\right)^{n-k}\right) \\
&\geqslant \max\left\{0, \left\lfloor \frac{k - p^{a-2}}{\varphi(p^{a-1})} \right\rfloor\right\} + \frac{n-k}{\varphi(p^a)} \\
&= \max\left\{0, \frac{pk - p^{a-1}}{\varphi(p^a)} - \left\{\frac{k - p^{a-2}}{\varphi(p^{a-1})}\right\}\right\} + \frac{n-k}{\varphi(p^a)} \\
&= \max\left\{\frac{n-k}{\varphi(p^a)}, \frac{n - p^{a-1}}{\varphi(p^a)} + \frac{k}{p^{a-1}} - \left\{\frac{k - p^{a-2}}{\varphi(p^{a-1})}\right\}\right\} > \frac{n - p^{a-1}}{\varphi(p^a)}.
\end{aligned}$$

(Note that if $k \geqslant p^{a-1}$ then $k/p^{a-1} \geqslant 1 > \{(k - p^{a-2})/\varphi(p^{a-1})\}$.) Therefore, from (5.2) we get that

$$\mathrm{ord}_p(C_{p^a}(n, r)) > \frac{n - p^{a-1}}{\varphi(p^a)} - 1 \geqslant \left\lfloor \frac{n - p^{a-1}}{\varphi(p^a)} \right\rfloor - 1.$$

So $F_{p^a}(n, r) = (-p)^{-\lfloor (n - p^{a-1})/\varphi(p^a) \rfloor} C_{p^a}(n, r) \in \mathbb{Z}$ as desired. $\square$

Proof of Theorem 1.4. (i) Write $n + d = p^{a-1} - 1 + m\varphi(p^a)$ with $m \in \mathbb{N}$. Then, for any $k = 0, \ldots, d$ we have

$$\left\lfloor \frac{n + k - p^{a-1}}{\varphi(p^a)} \right\rfloor = \left\lfloor m - \frac{d - k + 1}{\varphi(p^a)} \right\rfloor = m - 1.$$

Below we use induction on d to show the desired congruence (1.21).

In the case $d=0$ (i.e., $n-p^{a-1}\equiv -1 \pmod{\varphi(p^a)}$), we have $F_{p^a}(n,r)\equiv F_{p^a}(n,0) \pmod p$ because

$$F_{p^a}(n,i)-F_{p^a}(n,i-1)=(-p)^{-m+1}C_{p^a}(n+1,i)=-pF_{p^a}(n+1,i)$$

for all $i\in\mathbb{Z}$. Furthermore, by a result of Weisman [We] (see also [SW, Theorem 1.5]), $F_{p^a}(n,r)\equiv 1 \pmod p$ if $d=0$.

Now let $d>0$ and assume that the desired result holds for smaller values of d. Clearly, $(n+1)+(d-1)=p^{a-1}-1+m\varphi(p^a)$ and

$$\left\lfloor\frac{n+1+k-p^{a-1}}{\varphi(p^a)}\right\rfloor=m-1\quad\text{for}\quad k=0,\ldots,d-1.$$

If $r\geqslant 0$ then

$$C_{p^a}(n,r)-C_{p^a}(n,0)=\sum_{0<i\leqslant r}(C_{p^a}(n,i)-C_{p^a}(n,i-1))=\sum_{0<i\leqslant r}C_{p^a}(n+1,i);$$

if $r<0$ then

$$\begin{aligned}C_{p^a}(n,r)-C_{p^a}(n,0)&=\sum_{r<i\leqslant 0}(C_{p^a}(n,i-1)-C_{p^a}(n,i))\\&=-\sum_{r<i\leqslant 0}C_{p^a}(n+1,i).\end{aligned}$$

Therefore

$$F_{p^a}(n,r)-F_{p^a}(n,0)=\begin{cases}\sum_{0<i\leqslant r}F_{p^a}(n+1,i) & \text{if } r\geqslant 0,\\ -\sum_{r<i\leqslant 0}F_{p^a}(n+1,i) & \text{if } r<0.\end{cases}$$

By the induction hypothesis, whenever $i\in\mathbb{Z}$ we have

$$F_{p^a}(n+1,i)\equiv\sum_{k=0}^{d-1}\binom{i+k-1}{k}F_{p^a}(n+1+k,0) \pmod p.$$

For any $k=0,\ldots,d-1$, if $r\geqslant 0$

$$\sum_{0<i\leqslant r}\binom{i+k-1}{k}=\sum_{j=0}^{r+k-1}\binom{j}{k}=\binom{r+k}{k+1}$$

by an identity of S.-C. Chu (cf. [GKP, (5.10)]); if $r<0$ then

$$\begin{aligned}-\sum_{r<i\leqslant 0}\binom{i+k-1}{k}&=(-1)^{k+1}\sum_{r<i\leqslant 0}\binom{-i}{k}=(-1)^{k+1}\sum_{j=0}^{-r-1}\binom{j}{k}\\&=(-1)^{k+1}\binom{-r}{k+1}=\binom{r+k}{k+1}.\end{aligned}$$

Thus, by the above, $F_{p^a}(n,r)$ is congruent to

$$F_{p^a}(n,0)+\sum_{k=0}^{d-1}\binom{r+k}{k+1}F_{p^a}(n+1+k,0)=\sum_{k=0}^{d}\binom{r+k-1}{k}F_{p^a}(n+k,0)$$

modulo p. This concludes the induction proof of (1.21). $\square$

(ii) In the case $a=1$, the desired results in Theorem 1.4(ii) follow from Corollaries 1.3 and 1.7.

Now we let $a\geqslant 2$ and $r\in\mathbb{Z}$. Write $n=p^{a-2}(pn_1+n_0)+s$ and $r=p^{a-2}(pr_1+r_0)+t$, where $s,t\in\{0,\ldots,p^{a-2}-1\}$, $n_0,r_0\in\{0,\ldots,p-1\}$ and $n_1\in\mathbb{N}$ and $r_1\in\mathbb{Z}$.

If $p^{a-1}\leqslant n<p^a$, then

$$F_{p^a}(n,r)=C_{p^a}(n,r)=\binom{n}{\{r\}_{p^a}}(-1)^{\{r\}_{p^a}},$$

and in particular $\mathrm{ord}_p(C_{p^a}(n,0))=0=\lfloor (n-p^{a-1})/\varphi(p^a)\rfloor$.

Below we assume that $n\geqslant 2p^{a-1}$ (i.e., $n_1\geqslant 2$). By [SD, Theorem 1.7],

$$F_{p^a}(n,r)\equiv(-1)^t\binom{s}{t}F_{p^2}(pn_1+n_0,pr_1+r_0)\ (\mathrm{mod}\ p).$$

If $p\mid n_1$, or $p-1\nmid n_1-1$, or $n_0=r_0=p-1$, then by [SW, Theorem 1.2] in the case $l=0$, we have

$$F_{p^2}(pn_1+n_0,pr_1+r_0)\equiv(-1)^{r_0}\binom{n_0}{r_0}F_p(n_1,r_1)\ (\mathrm{mod}\ p)$$

and hence $F_{p^a}(n,r)\equiv b_{n,r}F_p(n_1,r_1)\ (\mathrm{mod}\ p)$, where

$$\begin{aligned}b_{n,r}:=&(-1)^{\{r\}_{p^{a-1}}}\binom{\{n\}_{p^{a-1}}}{\{r\}_{p^{a-1}}}=(-1)^{p^{a-2}r_0+t}\binom{p^{a-2}n_0+s}{p^{a-2}r_0+t}\\ \equiv&(-1)^t\binom{s}{t}(-1)^{r_0}\binom{n_0}{r_0}\ (\mathrm{mod}\ p)\quad\text{(by Lucas' theorem (cf. [HS]))}.\end{aligned}$$

By Corollary 1.3, there is an $r_1'\in\mathbb{Z}$ such that $F_p(n_1,r_1')\not\equiv 0\ (\mathrm{mod}\ p)$. Thus, if $p\mid n_1$ or $p-1\nmid n_1-1$, then

$$F_{p^a}(n,p^{a-1}r_1')\equiv F_p(n_1,r_1')\not\equiv 0\ (\mathrm{mod}\ p).$$

If $n_0=p-1$, then

$$F_{p^a}(n,p^{a-2}(pr_1'+p-1))\equiv(-1)^{p-1}\binom{p-1}{p-1}F_p(n_1,r_1')\not\equiv 0\ (\mathrm{mod}\ p).$$

When $p\nmid n_1$, $p-1\mid n_1-1$ and $n_0<r_0$, by applying the second part of [SW, Theorem 1.2] in the case $l=0$, we have

$$F_{p^2}(pn_1+n_0,pr_1+r_0)\equiv [\![n_1>1]\!]\frac{(-1)^{n_0}n_1}{r_0\binom{r_0-1}{n_0}}=\frac{(-1)^{n_0}n_1}{r_0\binom{r_0-1}{n_0}}\pmod p$$

and hence

$$F_{p^a}(n,r)\equiv(-1)^{n_0+t}\frac{n_1\binom{s}{t}}{r_0\binom{r_0-1}{n_0}}\pmod p.$$

In particular, if $p\nmid n_1$, $p-1\mid n_1-1$ and $n_0<p-1$, then

$$F_{p^a}(n,p^{a-2}(n_0+1))\equiv\frac{(-1)^{n_0}n_1}{n_0+1}\not\equiv 0\pmod p.$$

In view of the above, we already have (1.22).

To prove the congruence in (1.23), we should also consider the case $p\nmid n_1$, $p-1\mid n_1-1$ and $n_0\geqslant r_0$. By [SW, Lemmas 3.2 and 3.3],

$$\begin{aligned}
&p^{-\lfloor(pn_1+n_0-p)/\varphi(p^2)\rfloor}C_{p^2}(pn_1+n_0,pr_1+r_0)\\
&-(-1)^{r_0}\binom{n_0}{r_0}p^{-\lfloor(n_1-1)/(p-1)\rfloor}C_p(n_1,r_1)\\
\equiv&(-1)^{n_1-1}p^{-\lfloor(n_1-1-1)/(p-1)\rfloor}C_p(n_1-1,r_1)(-1)^{n_1+r_0}n_1\binom{n_0}{r_0}\frac{\sigma_{n_0,r_0}(n_1)}{p}\\
\equiv&-(-1)^{r_0}\binom{n_0}{r_0}p^{-(n_1-1)/(p-1)+1}C_p(n_1-1,r_1)n_1\frac{\sigma_{n_0,r_0}(n_1)}{p}\pmod p,
\end{aligned}$$

where

$$\sigma_{n_0,r_0}(n_1)=1+(-1)^p\frac{\prod_{1\leqslant i\leqslant p,\,i\neq p-r_0}(p(n_1-1)+r_0+i)}{\prod_{1\leqslant i\leqslant p,\,i\neq p-(n_0-r_0)}(n_0-r_0+i)}\equiv 0\pmod p.$$

Therefore

$$\begin{aligned}
&F_{p^2}(pn_1+n_0,pr_1+r_0)-(-1)^{r_0}\binom{n_0}{r_0}F_p(n_1,r_1)\\
\equiv&(-1)^{r_0}\binom{n_0}{r_0}F_p(n_1-1,r_1)n_1\frac{\sigma_{n_0,r_0}(n_1)}{p}\pmod p
\end{aligned}$$

and hence

$$F_{p^a}(n,r)\equiv b_{n,r}\left(F_p(n_1,r_1)+F_p(n_1-1,r_1)n_1\frac{\sigma_{n_0,r_0}(n_1)}{p}\right)\pmod p,$$

Observe that $n+p^a(p-1)=p^{a-2}(pn_1'+n_0)+s$ with $n_1'=n_1+p(p-1)$. Clearly $F_p(n_1',r_1)\equiv F_p(n_1,r_1)\pmod p$ by Corollary 1.7, and $\sigma_{n_0,r_0}(n_1')\equiv\sigma_{n_0,r_0}(n_1)\pmod{p^2}$ if $n_0\geqslant r_0$. Thus, by the above, $F_{p^a}(n+p^a(p-1),r)\equiv F_{p^a}(n,r)\pmod p$. This concludes the proof. $\square$

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