On Torsion of Prismatic Bodies

By Frederic Y. M. Wan

The method of decaying residual solution is applied to obtain an approximate *interior* solution for the torsion of slender prismatic elastic bodies under different end conditions. The approximate solution is generally accurate up to terms that are exponentially small in the length-to-cross-sectional-width ratio. For stress end conditions, the result is identical to the classical Saint-Venant torsion solution. Similar types of simple solutions, not known previously, are obtained for different types of mixed end conditions. For displacement conditions at both ends, the corresponding Saint-Venant type result requires an accurate solution of a canonical problem for a semi-infinite prismatic body that is to be obtained once and for all. The solution of the canonical problem is elementary for a circular cross section. The approximate interior solution in that case is identical to the known exact interior solution.

1. Introduction

The small deformation of a slender cylindrical body in torsion is known to consist of an *interior* component that is significant throughout the entire slender body and *boundary layer* components that decay rapidly (exponentially) away from the ends (see Horgan and Knowles [7] and references therein). In the context of the method of matched asymptotic expansions, the interior solution component corresponds to the *outer* asymptotic solution. It is the part of the actual solution of primary interest in many applications. When stress data are prescribed at both ends of the prismatical body, the Saint-Venant torsion solution [11], [12] is expected to provide an "accurate" approximation of the *interior solution* without any reference to the *inner* asymptotic solutions and matching.

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Concerns for the effect of end constraints (such as no end warping) on the interior solution have generated a great deal of interest in torsion associated with prescribed displacement and mixed end data. Saint-Venant's principle is not useful for the separate determination of the interior (outer) solution for these kinds of end conditions when the resultant axial torque is not known. Various direct and semidirect methods of the calculus of variations for a Rayleigh-Ritz type solution have been developed for these problems (see Reissner [9] and references therein). These and other simple approximate solutions in terms of elementary functions are generally more informative and easier to analyze than direct numerical solutions (such as finite element solutions) for the actual three-dimensional elasticity problem.

In this paper, we describe a new method for obtaining approximate interior solutions for the torsion of prismatic bodies with a uniform cross section induced by prescribed end stresses and/or end displacements. These solutions are accurate up to exponentially small terms for large values of a dimensionless length parameter L/b, where L is the length of body and b is a representative width of the uniform cross section. The method is based on the fact that the difference between the exact solution and the interior solution, called the *residual solution*, should consist of only boundary layers that decay exponentially away from the ends of the (homogeneous, isotropic, elastic) prismatical body. It makes use of the reciprocal theorem of three dimensional elasticity theory similar to what was done first for flat plate problems in Gregory and Wan [3–6] and Lin and Wan [8]. The technique has been modified for beams [13], [14], [15] and shells [2], [10].

The key results from the *method of decaying residual solution* developed in this paper are two integral conditions for the two unknown constants in the interior solution for the torsion problem (one of these being the *angle of twist*). Each condition takes the form of a vanishing weighted average of the residual data at one of the two ends. This type of requirement is analogous to Saint-Venant's principle (which requires the residual end torque to vanish) and reduces to this principle when all end data are given in terms of stresses.

When tangential displacements and normal stress are prescribed at one end of the slender body, the relevant integral condition gives a simple expression for the two unknown constants. As in the case of plates and shells, pure displacement end data do not lead to similarly elementary relations (except for the case of a circular cross section). An accurate solution of a canonical problem is required and it may have to be obtained numerically. The method of decaying residual solution still determines the unknown torque and angle of twist, and hence a simple interior solution analogous to the Saint-Venant torsion solution for stress end data. In most cases, this simple interior solution relieves us of the need to examine the massive amount of numerical output generated by a direct numerical solution of the actual displacement end value problem for small deformations of prismatic elastic bodies.

For circular cylinders, the torsion problem admits an exact solution for any type of axisymmetric end data [12]. In Section 2 of this paper, we give the explicit, exact solutions for a typical problem. Other exact solutions are also given in the Appendix for different combinations of end data. While it is straightforward to obtain these solutions, they will be useful for comparison with the interior solution obtained by the method of decaying residual solution for the same torsion problems. The same explicit results also serve to affirm the possibility of a separate determination of the interior solution without matching for the general case of a noncircular cross section.

2. Axisymmetric torsion of circular cylinders

2.1. Axisymmetric circumferential displacements in an elastic cylinder

In cylindrical coordinates (r, θ, z), the only nonvanishing strain components of an elastic body that undergoes only axisymmetric circumferential deformation characterized by the circumferential displacement distribution $v(r, z)$ are given by [12]

$$e_{r\theta} = e_{\theta r} = v_{,r} - \frac{1}{r} v, \quad e_{z\theta} = e_{\theta z} = v_{,z}. \tag{2.1}$$

For an isotropic elastic body, the only nonvanishing stress components induced by these strain components are [12]

$$\sigma_{\theta r} = \sigma_{r\theta} = G e_{r\theta}, \quad \sigma_{\theta z} = \sigma_{z\theta} = G e_{z\theta} \tag{2.2}$$

with the shear modulus G given in terms of Young's modulus E and Poisson's ratio ν by $G = E/2(1+\nu)$. When the internal stresses are in static equilibrium with the circumferentially directed axisymmetric body load distribution $p(r, z)$, these quantities satisfy a single differential equation of equilibrium [12]:

$$\sigma_{r\theta,r} + \sigma_{z\theta,z} + \frac{2}{r}\sigma_{r\theta} + p = 0. \tag{2.3}$$

The stress components may be expressed in terms of v by using (2.1) in (2.2) to get

$$\sigma_{r\theta} = Gr\left(\frac{1}{r} v\right)_{,r}, \quad \sigma_{z\theta} = G v_{,z}. \tag{2.4}$$

Upon introducing the resulting expressions for $\sigma_{r\theta}$ and $\sigma_{z\theta}$ into (2.3), we obtain a single second order PDE for $v(r, z)$:

$$v_{,rr} + \frac{1}{r} v_{,r} - \frac{1}{r^2} v + v_{,zz} + p = 0. \tag{2.5}$$

We are particularly interested in axisymmetric circumferential deformations of solid and hollow circular cylinders spanning the region $\{a \le r \le b, 0 \le \theta \le 2\pi, 0 \le z \le L\}$ in space with $a = 0$ for a solid cylinder. The cylindrical surfaces of the cylinders are free of external loads so that

$$r = a, b: \quad \sigma_{r\theta} = Gr\left(\frac{1}{r}v\right)' = 0 \tag{2.6}$$

where $()' \equiv ()_{,r}$. For a solid cylinder (with $a = 0$), the traction-free condition (2.6) at $r = a$ is replaced by the condition of bounded stresses at $r = 0$. At the two $z =$ constant ends, either the circumferential displacement v or the shear stress component $\sigma_{z\theta}$ (or a linear combination of both) may be prescribed. For example, we may have

$$z = 0: \quad v = f(r) \tag{2.7}$$

and

$$z = L: \quad \sigma_{z\theta} = Gv_{,z} = \frac{G}{b}g(r) \tag{2.8}$$

for some prescribed function $f(r)$ and $g(r)$.

For the purpose of illustrating the *method of decaying residual state* for determining the outer solution, it suffices to consider only a solid cylinder ($a = 0$) with no interior loading so that $p \equiv 0$. The boundary value problem (BVP) can, of course, be solved exactly by the method of eigenfunction expansions. The usual method of separation of variables gives

$$v(r,z) = \left[a_0\frac{z}{L} + b_0\left(1 - \frac{z}{L}\right)\right]r + \sum_{n=1}^{\infty}\left[a_n\frac{\sinh(\lambda_n z)}{\sinh(\lambda_n L)} + b_n\frac{\sinh(\lambda_n\{L - z\})}{\sinh(\lambda_n L)}\right]J_1(\lambda_n r) \tag{2.9}$$

where we have from the traction-free condition (2.6) at $r = b$ that $\lambda_n b = t_n$ is the n-th nonzero root of [1]

$$\left[\frac{dJ_1(t)}{dt} - \frac{1}{t}J_1(t)\right]_{t=t_n} = \left[t\frac{d}{dt}\left\{\frac{1}{t}J_1(t)\right\}\right]_{t=t_n} = J_2(t_n) = 0. \tag{2.10}$$

Hence, t_n is the n-th nonzero root of $J_2(t)$. The coefficients $\{a_n\}$ and $\{b_n\}$ are determined by the end conditions at $z = 0$ and $z = L$ with the help of the relevant orthogonality conditions among $J_k(\lambda_n r)$ for $k=1,2$ and $n=1,2,3,\ldots$ (see Appendix) to be

$$b_0 = \frac{4}{b^4}\int_0^b f(r)r^2\,dr, \quad b_m = \frac{1}{\xi_m^2}\int_0^b f(r)J_1(\lambda_m r)r\,dr \tag{2.11a,b}$$

$$a_0 = \frac{4}{b^4}\left\{\frac{L}{b}\int_0^b g(r)r^2\,dr + \int_0^b f(r)r^2\,dr\right\} \tag{2.11c}$$

$$a_m = \frac{1}{\xi_m^2}\frac{\tanh(\lambda_m L)}{\lambda_m b}$$

$$\times\left\{\frac{b}{L}\frac{\lambda_m L}{\sinh(\lambda_m L)}\int_0^b f(r)J_1(\lambda_m r)r\,dr + \int_0^b g(r)J_1(\lambda_m r)r\,dr\right\} \tag{2.11d}$$

with

$$\xi_n^2 = \int_0^b J_1^2(\lambda_n r)r\,dr = \frac{1}{2\lambda_n^2}\left\{[J_1'(t_n)]^2 + \left(1-\frac{1}{t_n^2}\right)[J_1(t_n)]^2\right\}. \tag{2.12}$$

It is evident from (2.9), (2.11b), (2.11d) and (2.12) that terms other than those multiplied by a_0 and b_0 are exponentially small away from the ends and are therefore *boundary layer* phenomena. For many problems, these layer solutions are of little or no interest. In these cases, it would be desirable to be able to determine the coefficients a_0 and b_0 in the *outer solution*

$$v^0(r,z) = \left[a_0\frac{z}{L} + b_0\left(1-\frac{z}{L}\right)\right]r \tag{2.13}$$

without any reference to the layer components. This is shown to be possible for a circular cross section with the help of a reciprocity relation in the subsection (2.2) and subsequently for prismatic bodies with any cross sectional shape for which the torsion problem may not have an exact solution.

2.2. *Boundary conditions for the outer solution*

Let $v^{(1)}$ and $v^{(2)}$ be two axisymmetric displacement fields of the cylinder induced by the external load system $\{p^{(1)}, \sigma_{n\theta}^{(1)}\}$ and $\{p^{(2)}, \sigma_{n\theta}^{(2)}\}$, respectively. We have from the reciprocal theorem of elasticity theory [11] for the case of vanishing radial and axial displacement components

$$\iiint_R p^{(1)}v^{(2)}\,dV + \oint\!\!\oint_S \sigma_{n\theta}^{(1)}v^{(2)}\,dS = \iiint_R p^{(2)}v^{(1)}\,dV + \oint\!\!\oint_S \sigma_{n\theta}^{(2)}v^{(1)}\,dS \tag{2.14}$$

for any portion $R = \{\underline{z} \le z \le \bar{z}, 0 \le a \le r \le b\}$ of the cylinder enclosed by the surface S with unit normal $\mathbf{n}$ (positive outward), where $\sigma_{n\theta}$ is the circumferential component of the stress vector $\boldsymbol{\sigma}_n$ associated with $\mathbf{n}$.

We apply the reciprocal relation (2.14) to a section of a solid cylinder ($a = 0$) that extends from $z = 0$ to $z = \bar{z}$, for $b < \bar{z} < L - b$, with $p \equiv 0$ and with the cylindrical surface $r = b$ being stress-free for both elastostatic states. We further require the stresses and displacements to be bounded in all cases. Let the (1) state be the (boundary layer) residual state

$$\begin{aligned} v^R(r,z) &\equiv v(r,z) - v^0(r,z) \\ &= v(r,z) - \left[a_0\frac{z}{L} + b_0\left(1 - \frac{z}{L}\right)\right]r. \end{aligned} \tag{2.15}$$

In that case, (2.14) reduces to

$$\int_0^{2\pi}\int_0^b \left[\sigma_{z\theta}^R v^{(2)} - \sigma_{z\theta}^{(2)} v^R\right]_{z=0}^{\bar{z}} r\,dr\,d\theta = 0. \tag{2.16}$$

since $\sigma_{n\theta}^{(i)} = 0$ on the cylindrical surface for $i = 1, 2$.

Suppose the (2) state has no worse than an algebraic growth as $z \to \infty$. We call such an elastostatic state a *regular* state. Then for $\bar{z} \gg b$, the condition (2.16) simplifies to

$$\int_0^b \sigma_{z\theta}^R(r,0) v^{(2)}(r,0) r\,dr = \int_0^b \sigma_{z\theta}^{(2)}(r,0) v^R(r,0) r\,dr \tag{2.17}$$

except for exponentially small terms.

To eliminate terms involving the unknown end stresses at $z = 0$ that appear on the left side of (2.17), we further require

$$v^{(2)}(r,0) = 0. \tag{2.18}$$

In that case, the left side of (2.17) vanishes leaving us with

$$\int_0^b \sigma_{z\theta}^{(2)}(r,0) v^R(r,0) r\,dr = 0, \tag{2.19}$$

again except for exponentially small terms. We summarize the results obtained above as

PROPOSITION 2.1. *For the residual state v^R to be an exponentially decaying elasto-static state, the residual displacement data at $z = 0$ must satisfy the condition* (2.19) *for any regular* (2) *state that is stress free on $r = b$ and distortion free at $z = 0$.*

A solution of (2.5) that has all the required properties for the (2) state needed for the displacement end condition at $z = 0$ is

$$v^{(2)}(r,z) = zr. \tag{2.20}$$

The corresponding nonvanishing stress component is

$$\sigma_{z\theta}^{(2)}(r,z) = Gr. \tag{2.21}$$

Upon substituting (2.21) and

$$v^R(r,0) = f(r) - b_0 r \tag{2.22}$$

into (2.19), we obtain

$$b_0 = \frac{4}{b^4}\int_0^b f(r)r^2\,dr, \tag{2.23}$$

which is identical to the exact solution we found earlier in (2.11a).

For $z = L$, a change of variable $\tilde{z} = L - z$ transforms the problem to one for the $\tilde{z} = 0$ end. For the end condition (2.8), we do not know $v^R(r, \tilde{z} = 0)$; hence we want to choose a (2) state which satisfies the end condition

$$\sigma_{z\theta}^{(2)}(r,\tilde{z}=0) = Gv_{,z}^{(2)}(r,\tilde{z}=0) = 0 \tag{2.24}$$

instead. For such a (2) state, (2.17) reduces to

$$\int_0^b \left[\sigma_{z\theta}^R v^{(2)}\right]_{\tilde{z}=0} r\,dr = 0 \tag{2.25a}$$

except for exponentially small terms. Thus, we have

PROPOSITION 2.2. *If the residual solution v^R is an exponentially decaying elasto-static state, the residual stress end data at $x_3 = L$ must satisfy the condition (2.25a) for any regular (2) state that is stress free on $r = b$ and stress free at $z = L$.*

A regular solution of (2.5) that satisfies the condition (2.24) (as well as the traction-free condition at $r = b$) is

$$v^{(2)}(r,\tilde{z}) = r. \tag{2.26}$$

For this (2) state, (2.25a) becomes

$$\int_0^b \sigma_{z\theta}^R(r,\tilde{z}=0)r^2\,dr = 0 \tag{2.25b}$$

with

$$\sigma_{z\theta}^R(r,\tilde{z}=0) = Gv_{,z}^R(r,L) = \frac{G}{b}g(r) - \left[\frac{a_0 - b_0}{L}\right]Gr, \tag{2.25c}$$

which follows from (2.8) and (2.15). Hence, we have from (2.25b)

$$\int_0^b g(r)r^2\,dr = \frac{b}{L}\int_0^b (a_0 - b_0)r^2\,dr, \tag{2.25d}$$

so that

$$\begin{aligned} a_0\frac{b^4}{4} &= \frac{L}{b}\int_0^b g(r)r^2\,dr + b_0\int_0^b r^3\,dr \\ &= \frac{L}{b}\int_0^b g(r)r^2\,dr + \int_0^b f(r)r^2\,dr \end{aligned}$$

or

$$a_0 = \frac{4L}{b^5}\left\{\int_0^b g(r)r^2\,dr + \frac{b}{L}\int_0^b f(r)r^2\,dr\right\}. \tag{2.27}$$

The expression (2.27) is identical to the exact solution given by (2.11c).

It should be evident that the method of a decaying residual solution developed above can be modified in a straightforward way for problems with stress or displacement data (or a linear combination of both) prescribed at both ends of the cylinder (see Appendix for the corresponding exact solutions). The detail of the modification needs not be given here.

3. Torsion of general slender cylindrical bodies

3.1. Torsion of prismatic bodies with a uniform noncircular cross section

For prismatical bodies with a uniform noncircular cross section, tractable exact solutions are generally not available when these bodies are in torsion. In this section, we describe the relevant boundary value problems that give rise to torsion. When stresses are prescribed at the two ends of the slender body, the classical Saint-Venant torsion solution [12] offers an adequate approximate solution away from the two ends. To derive accurate interior solution of torsion problems with prescribed end displacements or mixed displacement and stress data, it will be useful to have a brief summary of the Saint-Venant torsion solution to facilitate the development of the new method.

In cartesian coordinates (x_1, x_2, x_3) with x_3 in the direction along the length of the cylindrical body, the relevant equations of linear elasticity for an interior solution of torsion problems are the strain-displacement relations

$$e_{3j} = e_{j3} = u_{j,3} + u_{3,j} \quad (j=1,2). \tag{3.1}$$

the stress-strain relations

$$e_{3j} = \frac{\sigma_{3j}}{G}, \qquad (3.2)$$

where $G = E/2(1+\nu)$ for an isotropic material, and, in the absence of interior loading, the single homogeneous equilibrium equation:

$$\sigma_{13,1} + \sigma_{23,2} = 0. \qquad (3.3)$$

(The remaining stress and strain components are expected to vanish in the interior.)

The governing equations for torsion above are supplemented by appropriate boundary conditions on the boundary surface of the body. For simplicity, we limit further consideration to cylinders with a uniform simply-connected cross section A bounded by the curve C. We characterize this edge curve of the solid cross section by $f(x_1, x_2) = 0$. The cylindrical surface $S_c = \{f(x_1, x_2) = 0,\ 0 \le x_3 \le L\}$ with an outward unit normal $\boldsymbol{\nu}$ is stress free so that

$$\nu_1 \sigma_{31} + \nu_2 \sigma_{32} = 0 \quad \text{on } S_c \qquad (3.4)$$

where $\nu_k = \boldsymbol{\nu} \cdot \boldsymbol{i}_k$. At a constant x_3 end, say $x_3 = l$, we limit our discussion to the following four sets of end conditions:

(A) $$\sigma_{3k} = \tau_k^l(x_1, x_2) \quad (k = 1,2,3) \qquad (3.5a)$$

(B) $$u_3 = v_3^l(x_1, x_2), \quad \sigma_{3j} = \tau_j^l(x_1, x_2) \quad (j = 1,2) \qquad (3.5b)$$

(C) $$\sigma_{33} = \tau_3^l(x_1, x_2), \quad u_j = v_j^l(x_1, x_2) \quad (j = 1,2) \qquad (3.5c)$$

(D) $$u_k = v_k^l(x_1, x_2) \quad (k = 1,2,3) \qquad (3.5d)$$

It will be evident that other types of end data can be similarly treated by the method described in the next few sections.

For stress end data (Case (A)), we require

$$\iint_A \tau_k^l \, dx_1 \, dx_2 = 0, \quad \iint_A \left(x_3 \tau_j^l - x_j \tau_3^l\right) dx_1 \, dx_2 = 0, \qquad (3.6a)$$

for $k = 1,2,3$, $j = 1,2$ and $l = 0, L$. For global equilibrium, we should have the same resultant axial torque at the two ends $l = 0$ and $l = L$:

$$\iint_A \left(x_1 \tau_2^0 - x_2 \tau_1^0\right) dx_1 \, dx_2 = \iint_A \left(x_1 \tau_2^L - x_2 \tau_1^L\right) dx_1 \, dx_2 = M_t. \qquad (3.6b)$$

The conditions (3.6a) and (3.6b) must also be satisfied by the solution for other sets of end data to have pure torsion away from the ends.

3.2. *Saint-Venant torsion*

An exact solution of the PDE (3.3) and the traction free condition (3.4) on S_c can be obtained by setting

$$u_1 = -\theta x_2(x_3+\alpha_0), \quad u_2 = \theta x_1(x_3+\alpha_0), \quad u_3 = \theta\psi(x_1,x_2), \tag{3.7}$$

where $\psi(x_1,x_2)$ is a *warping function* to be specified. The corresponding stress components are

$$\sigma_{31} = \sigma_{13} = G\theta(\psi_{,1} - x_2), \quad \sigma_{32} = \sigma_{23} = G\theta(\psi_{,2} + x_1) \tag{3.8}$$

(and $\sigma_{11}=\sigma_{22}=\sigma_{33}=\sigma_{21}=\sigma_{12}=0$). Note that the terms multiplied by α_0 give rise only to a rigid body rotation about the x_3-axis and do not contribute to stress components. We see from the exact solution of Section 2 that this rigid body component should play a role in the solution for problems with displacement or mixed end data.

Upon substituting (3.8) into the equilibrium equation (3.3) and the boundary condition (3.4), the warping function $\psi(x_1,x_2)$ is seen to be a harmonic function in the interior of the cross section A, i.e.,

$$\psi_{,11} + \psi_{,22} = 0, \quad (x_1,x_2) \text{ in } A \tag{3.9}$$

and to satisfy the Neumann boundary conditions

$$\begin{aligned}\psi_{,\nu} \equiv \nu_1\psi_{,1} + \nu_2\psi_{,2} &= -\nu_1 x_2 + \nu_2 x_1 \\ &= -\nu_1 x_2 + \nu_2 x_1 = -\frac{1}{2}\frac{d}{ds}\left(x_1^2 + x_2^2\right)\end{aligned} \tag{3.10}$$

where s is an arc length variable along the edge curve C of the uniform simply-connected cross section.

It is well-known that the BVP (3.9)–(3.10) determines ψ up to an arbitrary constant (which is effectively a rigid body translation in the x_3 direction). Evidently, the corresponding stress components σ_{31}, σ_{32} and $\sigma_{33}\equiv 0$ cannot be made to fit an arbitrary set of admissible stress data at the ends $x_3=0$ and $x_3=L$ (including those with $\tau_3^0=\tau_3^L=0$). For slender cylindrical bodies, Saint-Venant's principle [7] allows us to use the simple solution (3.7)–(3.10) as an accurate approximation of the exact solution away from the two ends. This principle requires that the only unknown constant θ in σ_{3j} be chosen to satisfy

$$\iint_A \{x_1\sigma_{32} - x_2\sigma_{31}\}\,dx_1\,dx_2 = M_t \tag{3.11}$$

or, in terms of ψ,

$$\theta \iint_A \{(\psi_{,2} + x_1)x_1 - (\psi_{,1} - x_2)x_2\} dx_1\, dx_2 = M_t. \tag{3.12}$$

Saint-Venant's principle asserts that the difference between the exact solution of the original problem and the interior (Saint-Venant torsion) solution obtained above consists of boundary layer solution components that are significant only in a narrow region adjacent to each of the two ends of the slender body (and perhaps also a rigid rotation since the parameter α_0 in (3.7) is not specified by our solution process).

3.3. Stress function formulation

To obtain the actual Saint-Venant torsion solution for specific cross sectional geometries, it is more convenient to work with an alternative stress function formulation. In this formulation, we satisfy the single equilibrium equation (3.3) by setting $\sigma_{31} = \theta\Phi_{,2}$ and $\sigma_{32} = -\theta\Phi_{,1}$. Expressions for the stress components σ_{31} and σ_{32} in (3.8) can then be written as

$$\Phi_{,2} = G(\psi_{,1} - x_2), \quad -\Phi_{,1} = G(\psi_{,2} + x_1). \tag{3.8$'$}$$

We may now eliminate ψ from (3.8′) to get

$$\left(\frac{\Phi_{,1}}{G}\right)_{,1} + \left(\frac{\Phi_{,2}}{G}\right)_{,2} = -2 \tag{3.13}$$

or, for a homogeneous material,

$$\Phi_{,11} + \Phi_{,22} = -2G. \tag{3.13$'$}$$

The boundary conditions (3.4) on the cylindrical surface may also be expressed in terms of Φ:

$$\nu_1\Phi_{,2} - \nu_2\Phi_{,1} = \frac{d\Phi}{ds} = 0 \quad (\text{on } C). \tag{3.14}$$

For a simply-connected cross section, we may take (3.14) in the form

$$\Phi = 0 \quad (\text{on } C). \tag{3.14$'$}$$

The resultant torque condition (3.11) now becomes

$$-\theta\iint_A (x_1\Phi_{,1} + x_2\Phi_{,2})\,dx_1\,dx_2 = M_t \tag{3.15}$$

with

$$\iint_A (x_1\Phi_{,1} + x_2\Phi_{,2})\,dx_1\,dx_2 = \iint_A [(x_1\Phi)_{,1} + (x_2\Phi)_{,2} - 2\Phi]\,dx_1\,dx_2$$

$$= \oint_C (\nu_1 x_1 + \nu_2 x_2)\Phi\,ds - 2\iint_A \Phi\,dx_1\,dx_2$$

and $\Phi = 0$ on C, the condition (3.15) simplifies to

$$2\theta\iint_A \Phi\,dx_1\,dx_2 = M_t. \tag{3.15'}$$

Instead of solving a Neumann problem for ψ, we are now to solve the Dirichlet problem (3.13′) and (3.14′) for Φ and use it in (3.15′) to determine θ to complete the Saint-Venant solution.

For the special case of a *circular cross section*, the boundary curve C may be taken in the form

$$f(x_1, x_2) = b^2 - (x_1^2 + x_2^2) = 0 \tag{3.16}$$

where b is the cross section radius. Evidently, the function f itself satisfies the homogeneous Dirichlet condition (3.14′); a constant multiple of f, $\Phi = \Phi_0 G f$, can also be made to satisfy the PDE (3.13′) by choosing $\Phi_0 = 1/2$ so that

$$\Phi = \frac{1}{2} G f(x_1, x_2). \tag{3.17}$$

The condition (3.15′) then gives the torque-twist relation

$$\theta G \iint_A \left[b^2 - (x_1^2 + x_2^2)\right] dx_1\,dx_2 = M_t \tag{3.18}$$

or

$$\theta = \frac{2M_t}{\pi b^4 G}. \tag{3.18'}$$

The expression (3.18′) is identical to the exact solution of Section 2 when stress data are prescribed at both ends with no resultant force and only a resultant axial torque.

The results for the circular cross section case suggests that $\Phi_0 f$ for some choice of the constant Φ_0 may be the appropriate solution of the Dirichlet problem (3.13′)–(3.14′) for some other cross section geometries. This is certainly the case for elliptical and triangular cross sections discussed in [12].

3.4. The method of decaying residual solution

The Saint-Venant solution is not useful for torsion problems when end data at $x_3 = 0, L$ are prescribed in terms of displacements since we do not know the axial torque M_t in this case. The results in Section 2 for circular cylinders suggest that the method of decaying residual solution should still be useful for these problems. For this method, we need a reciprocal relation analogous to (2.14), but now for noncircular cross sections. We again obtain this relation as a special case of the reciprocity theorem of linear elasticity theory [11]. In the absence of interior loading and surface tractions on the cylindrical surface of the beam S_c, this reciprocity theorem, applied to a portion of the cylindrical body extending from $x_3 = 0$ to $x_3 = z < L$, becomes

$$\iint_A \left[\sum_{k=1}^{3} \sigma_{3k}^{(1)} u_k^{(2)} \right]_{x_3=0}^{z} dx_1\, dx_2 = \iint_A \left[\sum_{k=1}^{3} \sigma_{3k}^{(2)} u_k^{(1)} \right]_{x_3=0}^{z} dx_1\, dx_2 . \quad (3.19)$$

We take the (1) state to be the difference between the exact solution and the Saint-Venant type (interior) solution (3.7)–(3.10). (The latter can be shown to correspond to the outer asymptotic solution of the BVP for the torsion of slender bodies.) As before, we will refer to this difference as the *residual solution* $\{u_j^R, \sigma_{ij}^R\}$. The (2) state is again taken to be a solution of the equations of elasticity that is stress free on the cylindrical surface S_c. For the prescribed end data at $x_3 = 0$, we further require this (2) state to be *regular*, i.e., it has at most an algebraic growth* as $x_3 \to \infty$. If the residual state is, in fact, an exponential by decaying state, then except for exponentially small terms, the relation (3.19) simplifies to

$$\iint_A \left[\sigma_{31}^R u_1^{(2)} + \sigma_{32}^R u_2^{(2)} + \sigma_{33}^R u_3^{(2)} \right]_{x_3=0} dx_1\, dx_2$$

$$= \iint_A \left[\sigma_{31}^{(2)} u_1^R + \cdots + \sigma_{33}^{(2)} u_3^R \right]_{x_3=0} dx_1\, dx_2 . \quad (3.19')$$

* It is actually sufficient for the (2) state to grow at rate slower than the decay rate of the residual solution. This relaxed condition may be needed for problems shell.

For stress end data, we have

$$x_3=0:\ \sigma_{31}^R = \tau_1^0 - G\theta(\psi_{,1} - x_2) = \tau_1^0 - \theta\Phi_{,2}, \tag{3.20}$$

$$\sigma_{32}^R = \cdots = \tau_2^0 + \theta\Phi_{,1}, \quad \sigma_{33}^R = \tau_3^0$$

and we do not know the end displacement components that appear on the right side of (3.19′). As in the case of a circular cross section discussed in subsection (2.2), the unknown quantities can be eliminated by an appropriate choice of the (2) state. This observation leads to the following result for problems with prescribed end stresses:

PROPOSITION 3.1. *For the residual solution to be an exponentially decaying elasto-static state, the residual end stresses at $x_3 = 0$ must satisfy the integral condition*

$$\iint_A \left[\sigma_{31}^R u_1^{(2)} + \sigma_{32}^R u_2^{(2)} + \sigma_{33}^R u_3^{(2)}\right]_{x_3=0} dx_1\, dx_2 = 0 \tag{3.21}$$

for any regular elastostatic (2) *state that is stress free on the cylindrical surface S_c and on the end $x_3 = 0$.*

One (2) state that meets all the requirements of Proposition 3.1 is the rigid body rotation state

$$u_1^{(2)} = -x_2, \quad u_2^{(2)} = x_1, \quad u_3^{(2)} = 0. \tag{3.22}$$

For this (2) state, (3.21) becomes

$$\iint_A \left[x_1\sigma_{32}^R - x_2\sigma_{31}^R\right]_{x_3=0} dx_1\, dx_2 = 0 \tag{3.23}$$

except for exponentially small terms. The condition (3.23) is identical to (3.11) (or (3.12)) obtained from invoking Saint-Venant's principle with σ_{31} and σ_{32} being the stress components associated with the interior solution (3.7). This condition determines the *angle of twist* θ for the Saint-Venant torsion solution.

Note that the prescribed axial end stress τ_3^0 does not contribute to the value of θ. It contributes only to boundary layer solution component near $x_3 = 0$ as τ_3^0 is required to be self-equilibrating. In fact, the overall equilibrium conditions (3.6) are now direct consequences of taking the six available rigid body displacement and rotations as the (2) state for (3.21).

4. Mixed end data

4.1. Case (B) mixed data

Suppose we have at $x_3 = 0$ prescribed mixed end data of the Case (B) type:

$$x_3 = 0: \quad u_3 = v_3^0(x_1, x_2), \quad \sigma_{3j} = \tau_j^0(x_1, x_2) \quad (j=1,2), \qquad (4.1)$$

and we expect the interior solution of the new boundary value problem to be given by (3.7) for an appropriate pair of α_0 and θ. The residual data at $x_3 = 0$ now consist of

$$x_3 = 0: \sigma_{31}^R = \tau_1^0 - G\theta(\psi_{,1} - x_2) = \tau_1^0 - \theta\Phi_{,2}, \qquad (4.1')$$

$$\sigma_{32}^R = \ldots = \tau_2^0 + \theta\Phi_{,1}, \quad u_3^R = v_3^0 - \theta\psi.$$

To eliminate terms involving unknown end quantities in the necessary condition (3.19′) for (3.7) to be the interior solution, we need to choose a (2) state that satisfies the homogeneous conditions

$$x_3 = 0: \quad u_3^{(2)} = 0, \quad \sigma_{3j}^{(2)} = 0, \quad (j=1,2). \qquad (4.2)$$

With (4.2), the general necessary condition (3.19′) for the residual solution to be a decaying state simplifies to

$$\iint_A \left[\sigma_{31}^R u_1^{(2)} + \sigma_{32}^R u_2^{(2)} - u_3^R \sigma_{33}^{(2)}\right]_{x_3=0} dx_1\, dx_2 = 0, \qquad (4.3)$$

except for exponentially small terms. We have then the following counterpart of Proposition 3.1:

PROPOSITION 4.1. *For the residual mixed end data* σ_{31}^R, σ_{32}^R *and* u_3^R *at* $x_3 = 0$ *to induce only a decaying residual state, they must satisfy the integral condition* (4.3) *for any regular elastostatic* (2) *state that is stress free on the cylindrical surface S and satisfies the homogeneous conditions* (4.2) *at* $x_3 = 0$.

One (2) state that meets all the requirements of Proposition 4.1 is the rigid body rotation (3.22). For this (2) state, (4.3) further reduces to (3.22). Thus, the set of Case (B) type mixed end data is indistinguishable from pure stress data as far as the interior (torsion) solution is concerned. If the set of prescribed data at the other end are also of the Case (B) (or Case (A)) type, then the interior solution is again given by the Saint-Venant solution of subsection (3.3), again with the constant α_0 in (3.7) unspecified. Hence, we have

PROPOSITION 4.2. *The constraint of no end section warping,* $u_3(x_1, x_2, 0) = 0$, *has no effect on the interior* (*torsion*) *solution as long as the shear stresses* σ_{31} *and* σ_{32} *are prescribed at both ends.*

4.2. *Case (C) mixed data at one end*

Suppose we have at $x_3 = 0$ prescribed mixed end data of the Case (C) type:

$$x_3 = 0: \quad \sigma_{33} = \tau_3^0, \quad u_j = v_j^0 \quad (j=1,2), \tag{4.4}$$

and we continue to expect the interior solution to be given by (3.7) for a suitable pair of α_0, θ. In that case, we have as the residual data at the end section $x_3 = 0$,

$$x_3 = 0: \quad \sigma_{33}^R = \tau_3^0, \quad u_1^R = v_1^0 + \theta\alpha_0 x_2, \quad u_2^R = v_2^0 - \theta\alpha_0 x_1. \tag{4.5}$$

To apply the necessary condition (3.19′) for the residual solution to be a decaying state, we need to eliminate terms involving unknown end quantities at $x_3 = 0$. This is accomplished by choosing a *regular* elastostatic (2) state (stress free on S_c) to satisfy

$$x_3 = 0: \quad \sigma_{33}^{(2)} = 0, \quad u_j^{(2)} = 0 \quad (j=1,2) \tag{4.6}$$

instead. For such a (2) state, the condition (3.19′) simplifies to

$$\iint_A \left[u_1^R\sigma_{31}^{(2)} + u_2^R\sigma_{32}^{(2)} - u_3^{(2)}\sigma_{33}^R\right]_{x_3=0} dx_1\, dx_2 = 0 \tag{4.7}$$

except for exponentially small terms. Then, we have for Case (C) type end data the following counterpart of Propositions 3.1 and 4.1:

PROPOSITION 4.3. *For the Case* (*C*) *type mixed residual data at* $x_3 = 0$ *to induce only a decaying residual solution, it is necessary that they satisfy* (4.7) *for any regular elastostatic* (2) *state that is stress free on the cylindrical surface* S_c *and satisfies the homogeneous mixed end conditions* (4.6) *at* $x_3 = 0$.

One (2) state that meets all the requirements of Proposition 4.3 is a special case of the interior solution (3.7) itself:

$$u_1^{(2)} = -x_3 x_2, \quad u_2^{(2)} = x_3 x_1, \quad u_3^{(2)} = \psi(x_1, x_2) \tag{4.8}$$

where $\psi(x_1, x_2)$ is the warping function of Saint Venant torsion of Section (3.2) for the given uniform cross section* and the corresponding axial torque $M_t^{(2)}$ is determined by (3.12) or (3.15′) for $\theta = 1$. For this (2) state and the residual end data (4.5), (4.7) becomes

$$\theta\alpha_0 \iint_A [x_2\Phi_{,2} + x_1\Phi_{,1}]\, dx_1\, dx_2 = -\iint_A \left[v_1^0\Phi_{,2} - v_2^0\Phi_{,1} - \psi\tau_3^0\right] dx_1\, dx_2 \tag{4.9}$$

* Note that the stress function and the warping function for Saint-Venant torsion are already normalized to correspond to $\theta = 1$.

where Φ is the stress function for the Saint Venant torsion solution for the given uniform cross section.

If stress data or Case (B) mixed data are prescribed at the other end $x_3 = L$ with a resultant axial torque M_t, the left-hand side is seen to be $-\alpha_0 M_t$. Therefore, (4.9) may be written as

$$\alpha_0 = \frac{1}{M_t} \iint_A \left[v_1^0 \Phi_{,2} - v_2^0 \Phi_{,1} - \psi \tau_3^0 \right] dx_1\, dx_2 \tag{4.10}$$

which determines α_0. For a *circular cross section* of radius b and centered at $(0,0)$, we have

$$\Phi = \frac{G}{2}\left(b^2 - x_1^2 - x_2^2\right), \quad \psi = 0, \tag{4.11}$$

and the expression (4.10) for α_0 becomes

$$\alpha_0 = \frac{G}{M_t} \iint_A \left[x_1 v_2^0 - x_2 v_1^0 \right] dx_1\, dx_2 . \tag{4.12}$$

Similar to the case of stress end data, the prescribed normal stress τ_3^0 plays no role in the solution of the present problem. It is expected to contribute only to the layer solution provided that it gives rise to no axial force or bending moments (which we stipulated in (3.6)).

To illustrate the method of decaying residual solution for Case (A) and Case (B) data at $x_3 = L$, we apply the results for *circular cross sections* to three sets of in-plane displacement end data:

(i) $v_j^0 = 0$: The expression (4.12) gives $\alpha_0 = 0$ for this case so that the interior solution is

$$u_1 = -\theta x_2 x_3, \quad u_2 = \theta x_1 x_3, \quad u_3 = 0 \tag{4.13}$$

with θ given in terms of the known resultant axial torque M_t, assuming Case (A) or Case (B) data are prescribed at $x_3 = L$. The conditions on the tangential displacements at $x_3 = 0$ are themselves satisfied exactly.

(ii) $v_1^0 = -\beta^0 x_2$ and $v_2^0 = \beta^0 x_1$: For this case, the condition (4.12) gives

$$\alpha_0 = \frac{G\beta^0}{M_t} \iint_A \left(x_1^2 + x_2^2\right) dx_1\, dx_2 = \frac{\beta^0}{\theta} \tag{4.14}$$

where we have made use of (3.17′) to express the right-hand side in terms of θ. The interior solution is then found to be

$$u_1 = -\theta x_2 x_3 - \beta^0 x_2, \quad u_2 = \theta x_1 x_3 + \beta^0 x_1, \quad u_3 = 0. \tag{4.15}$$

Again the interior solution found by the method of decaying residual solution satisfies the prescribed tangential displacement conditions at $x_3 = 0$ exactly.

(iii) $v_1^0 = -\beta^0 x_2(1 + r^2/b^2)$ and $v_2^0 = \beta^0 x_1(1 + r^2/b^2)$ with $r^2 = x_1^2 + x_2^2$: For this case, we have from (4.9)

$$\alpha_0 = \frac{\beta^0 G}{M_t} \iint_A \left[r^2 \left(1 + \frac{r^2}{b^2} \right) \right] dx_1\, dx_2 = \frac{5\beta^0 G \pi b^4}{6 M_t} = \frac{5\beta^0}{3\theta}. \quad (4.16)$$

The interior solution is

$$u_1 = -\theta x_2 x_3 - \frac{5}{3}\beta^0 x_2, \quad u_2 = \theta x_1 x_3 + \frac{5}{3}\beta^0 x_1, \quad u_3 = 0. \quad (4.17)$$

A boundary layer component is now needed for the exact solution to satisfy the displacement end conditions at $x_3 = 0$ exactly.

For an *elliptic cross section*,

$$f(x_1, x_2) \equiv 1 - \left[\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \right] = 0, \quad (4.18)$$

we take

$$\Phi = \Phi_0 G f = \Phi_0 G \left[1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} \right] \quad (4.19)$$

for some constant Φ_0. This choice of Φ trivially satisfies the Dirichlet boundary condition (3.14′) on C. It also satisfies the PDE (3.13′) if we set $\Phi_0 = a^2 b^2/(a^2 + b^2)$. The corresponding warping function is

$$\psi = \frac{b^2 - a^2}{a^2 + b^2} x_1 x_2 + \psi_0 \quad (4.20)$$

where ψ_0 is a constant of integration corresponding to a rigid translation that may be set to zero (since τ_3^0 is to have no resultant axial force).

With (4.19) and (4.20), the expression (4.10) for α_0 becomes

$$\alpha_0 = \frac{2G}{M_t(a^2 + b^2)} \iint_A \left[b^2 x_1 v_2^0 - a^2 x_2 v_1^0 - \frac{b^2 - a^2}{2G} x_1 x_2 \tau_3^0 \right] dx_1\, dx_2 \quad (4.21)$$

assuming M_t is known from the Case (A) or Case (B) end data prescribed at $x_3 = L$. Now, the prescribed axial end stress does play a role in the interior solution, unlike the situation for the circular cross section. The integral in

(4.21) can be evaluated for specific sets of end data just as in the case of a circular cross section. It reduces to (4.12) when $b = a$.

Another cross section that admits an elementary solution for the stress function is the *triangular cross section*

$$f(x_1, x_2) \equiv (x_1^2 + x_2^2) - \frac{x_1}{a}(x_1^2 - 3x_2^2) + 2b = 0. \tag{4.22}$$

For simplicity, we limit our consideration to the special case of an equilateral triangle for which $b = -2a^2/27$. For this case, (4.22) may be written as

$$\begin{aligned} f(x_1, x_2) &= x_1^2 + x_2^2 - \frac{x_1}{a}(x_1^2 - 3x_2^2) - \frac{4}{27}a^2 \\ &= -\frac{1}{a}\left(x_1 - \sqrt{3}x_2 - \frac{2}{3}a\right)\left(x_1 + \sqrt{3}x_2 - \frac{2}{3}a\right)\left(x_1 + \frac{1}{3}a\right) = 0. \end{aligned} \tag{4.22'}$$

The stress function

$$\Phi = \Phi_0 G f = \Phi_0 G\left[(x_1^2 + x_2^2) - \frac{x_1}{a}(x_1^2 - 3x_2^2) - \frac{4}{27}a^2\right] \tag{4.23}$$

trivially satisfies the Dirichlet condition (3.14′) on C. It also satisfies the PDE (3.13′) if we set $\Phi_0 = -1/2$. The corresponding warping function is

$$\psi = \frac{1}{2a}(x_2^3 - 3x_1^2 x_2). \tag{4.24}$$

With (4.23) and (4.24), the expression (4.10) for α_0 becomes

$$\begin{aligned} \alpha_0 = \frac{G}{M_t}\iint_A \Bigg\{&\left[x_1 - \frac{3}{2a}(x_1^2 - x_2^2)\right]v_2^0 - x_2\left(1 + \frac{3}{a}x_1\right)v_1^0 \\ &- \frac{x_2}{2aG}(x_2^2 - 3x_1^2)\tau_3^0\Bigg\}\, dx_1\, dx_2. \end{aligned} \tag{4.25}$$

The angle of twist θ in the Saint-Venant type interior solution can always be expressed in terms of the axial torque by overall equilibrium (see (3.15) and (3.15′)). It is therefore known when Case (A) or (B) data are prescribed at the other end $x_3 = L$. The role of the Case (C) mixed end data at $x_3 = 0$ for these two types of data at $x_3 = L$ is merely to determine the rigid body component in the interior solution. The contribution of the method of decaying residual solution is therefore not particularly important in most applications of this class of problems.

4.3. Case (C) mixed data at both ends

A more interesting and significant development occurs when Case (C) mixed data are prescribed at both ends. The resultant axial torque M_t is no longer a known quantity for these problems. Still, the condition (4.9) determines the product $\alpha_0 \theta$ to be

$$\theta\alpha_0 = \frac{\iint_A \left[v_1^0 \Phi_{,2} - v_2^0 \Phi_{,1} - \psi \tau_3^0 \right] dx_1\, dx_2}{2\iint \Phi\, dx_1\, dx_2}. \tag{4.26}$$

At the other end, the residual end data calculated from (3.7) and

$$x_3 = L: \quad \sigma_{33} = \tau_3^L(x_1, x_2), \quad u_j = v_j^L(x_1, x_2) \quad (j = 1, 2) \tag{4.27}$$

are now

$$x_3 = L: \sigma_{33}^R = \tau_3^L, \quad u_1^R = v_1^L + \theta x_2(L + \alpha_0),$$
$$u_2^R = v_2^L - \theta x_1(L + \alpha_0). \tag{4.28}$$

After a change of variable $\bar{x}_3 = L - x_3$ in the reciprocal theorem, the analysis of Section (3.4) gives again the necessary condition, Proposition 4.3, for a decaying residual solution now applied at $\bar{x}_3 = 0$ (instead of $x_3 = 0$). Again, the (2) state needed is available in the form of a Saint-Venant torsion solution. We take it to be

$$u_1^{(2)} = (L - x_3)x_2, \quad u_2^{(2)} = -(L - x_3)x_1, \quad u_3^{(2)} = \psi(x_1, x_2), \tag{4.29}$$

so that the associated stress components are identical to those for (4.8). With (4.29), we obtain from (4.7)

$$\iint_A \left[v_1^L \Phi_{,2} - v_2^L \Phi_{,1} - \psi \tau_3^L \right] dx_1\, dx_2$$
$$= -\theta(L + \alpha_0) \iint_A \left[x_2 \Phi_{,2} + x_1 \Phi_{,1} \right] dx_1\, dx_2 \tag{4.30}$$

or

$$\theta(L + \alpha_0) = \frac{\iint_A \left[v_1^L \Phi_{,2} - v_2^L \Phi_{,1} - \psi \tau_3^L \right] dx_1\, dx_2}{2\iint_A \Phi\, dx_1\, dx_2} \tag{4.30'}$$

For a homogeneous body, the stress function Φ is again the solution of the Dirichlet problem (3.13′)–(3.14′) for the given uniform cross section and

ψ is the associated warping function. We may use (4.26) to eliminate $\theta\alpha_0$ from (4.30′) leaving us with

$$2\theta L \iint_A \Phi \, dx_1 \, dx_2 = M_t L$$

$$= \iint_A \left[(v_1^L - v_1^0)\Phi_{,2} - (v_2^L - v_2^0)\Phi_{,1} - (\tau_3^L - \tau_3^0)\psi \right] dx_1 \, dx_2. \tag{4.31}$$

The condition (4.31) completely determines θ (and hence M_t) and (4.26) is then used to specify α_0:

$$\frac{\alpha_0}{L} = \frac{\iint_A \left[v_1^0 \Phi_{,2} - v_2^0 \Phi_{,1} - \psi\tau_3^0 \right] dx_1 \, dx_2}{2L\theta \iint_A \Phi \, dx_1 \, dx_2}$$

$$= \frac{\iint_A \left[v_1^0 \Phi_{,2} - v_2^0 \Phi_{,1} - \psi\tau_3^0 \right] dx_1 \, dx_2}{\iint_A \left[(v_1^L - v_1^0)\Phi_{,2} - (v_2^L - v_2^0)\Phi_{,1} - (\tau_3^L - \tau_3^0)\psi \right] dx_1 \, dx_2}. \tag{4.32}$$

Note that the expression (4.32) for α_0 would be more symmetric in the two sets of prescribed end data if we should relocate the origin of our cartesian coordinate system at the midpoint along the length of the slender body.

Unlike the results of subsections (4.1) and (4.2), the method of decaying residual solution succeeds in determining the twist angle θ as well as the shear stresses in the slender body for a class of torsion problems for which Saint-Venant's principle is not useful. In fact, the conditions (4.9) (or (4.26)) and (4.30) are the analogues of the resultant torque condition (3.15) for the stress data case. We summarize the main result as

PROPOSITION 4.4. *When prescribed at both ends,* $x_3 = 0$ *and* $x_3 = L$, *the Case* (C) *mixed data determine* θ *and* α_0 *by* (4.31) *and* (4.32), *respectively.*

If $v_1^0 = v_2^0 = \tau_3^0 = 0$ for the entire cross section, we have from (4.26) $\theta\alpha_0 = 0$. In that case, θL is given explicitly by the right-hand side of (4.30′), which is generally nonzero. It follows then $\alpha_0 = 0$ so that the end conditions at $x_3 = 0$ are satisfied exactly. On the other hand, if we have $v_1^L = v_2^L = \tau_3^L = 0$ instead, then (4.30′) requires $\theta(L + \alpha_0) = 0$ while $\theta\alpha_0$ is given by the right-hand side of (4.26), which is generally nonzero. In that case, we have $L + \alpha_0 = 0$ and the end conditions at $x_3 = L$ are satisfied exactly. These two special cases show that the general method of decaying residual solution does provide us (somewhat indirectly) with an appropriate choice of θ and α_0 so that end conditions that can be satisfied exactly are so satisfied.

For the case of a *circular cross section*, Φ and ψ are given by (4.11). The two conditions (4.26) and (4.30′) for θ and α_0 simplify to

$$\theta\alpha_0 = -\frac{2}{\pi b^4}\iint_A \left[x_2 v_1^0 - x_1 v_2^0\right] dx_1\, dx_2 \tag{4.33a}$$

$$\theta(L+\alpha_0) = -\frac{2}{\pi b^4}\iint_A \left[x_2 v_1^L - x_1 v_2^L\right] dx_1\, dx_2. \tag{4.33b}$$

Correspondingly, the axial torque M_t experienced by the slender body (see (4.31)) now becomes

$$M_t = \frac{G}{L}\iint_A \left[(v_1^0 - v_1^L)x_2 - (v_2^0 - v_2^L)x_1\right] dx_1\, dx_2. \tag{4.33c}$$

With $\alpha_0\theta = b_0$ and $\theta(\alpha_0 + L) = a_0$, the results given by (4.33a) and (4.33b) are identical to the exact solution given by (2.11a) and (A.10b), respectively. It is also straightforward to verify that M_t as given by (4.33c) is identical to that given by the exact interior solution.

If the end section $x_3 = 0$ is restrained from any in-plane displacement, we have immediately from (4.33a) $\theta\alpha_0 = 0$ so that (4.33) becomes

$$\theta L = -\frac{2}{\pi b^4}\iint_A \left[x_2 v_1^L - x_1 v_2^L\right] dx_1\, dx_2. \tag{4.34}$$

For the end displacement $v_1^L = -x_2$ and $v_2^L = x_1$, (4.34) gives $\theta = 1/L$ as we would expect (see (3.7)).

On the other hand, if we have $v_1^L = x_1$ and $v_2^L = x_2$ instead, then (4.34) gives $\theta = 0$. The result is consistent with the fact that the prescribed end data do not induce a torsional deformation in the interior of the slender body (but may induce a different kind of interior state).

For the slightly more complex set of end data

$$v_1^L = -\frac{x_2^3}{b^2}, \quad v_2^1 = \frac{x_1^3}{b^2} \tag{4.35}$$

and any (self-equilibrating) distribution of τ_3^L, the results of this section allow us to determine for the first time a simple interior solution of the form (3.7). With (4.35) and the end $x_3 = 0$ restrained against in-plane displacement, the expression (4.34) gives $\theta = 1/2L$, while (4.33c) gives

$$M_t = \frac{\pi G b^4}{4L}.$$

There is no other known rational method for obtaining these results for such a simple problem.

The interior behavior of slender bodies with other cross section geometries and Case (C) mixed end data are more complex, but can be obtained in relatively simple form by solving a Dirichlet problem in the plane and evaluating the integrals in (4.26) and (4.30′). For elliptical and triangular cross sections, the relevant Dirichlet problems, in fact, admit elementary solutions.

5. Displacement end data

5.1. Displacement data at only one end

Suppose now only displacement data are prescribed at $x_3 = 0$ so that we have the Case (D) end data (3.5d) there. At the other end, Cases (A), (B) or (C) data are prescribed. In that case, we have from the end data at $x_3 = L$ either

$$\theta = \frac{M_t}{M_t^{(2)}} = \frac{1}{M_t^{(2)}} \iint_A \left(x_1 \tau_2^L - x_2 \tau_1^L \right) dx_1\, dx_2 \tag{5.1a}$$

for Cases (A) and (B), or

$$\theta(L + \alpha_0) = \frac{1}{M_t^{(2)}} \iint_A \left[v_1^L \Phi_{,2} - v_2^L \Phi_{,1} - \psi \tau_3^L \right] dx_1\, dx_2 \tag{5.1b}$$

for Case (C), where

$$M_t^{(2)} = 2 \iint_A \Phi\, dx_1\, dx_2 \tag{5.2}$$

and Φ and ψ are the stress function and warping function for the given cross section of the slender body. A second condition needed to determine the two unknown constants (θ and α_0) comes from the application of the reciprocal relation (3.19′) to the displacement end data at $x_3 = 0$.

For (3.19′) to be useful, we again have to eliminate terms involving unknown end quantities in the residual solution. This is now accomplished by choosing the (2) state to satisfy the homogeneous displacement boundary conditions

$$x_3 = 0: \quad u_k^{(2)} = 0 \quad (k = 1, 2, 3) \tag{5.3}$$

instead. With (5.3), the reciprocal relation (3.19′) becomes

$$\iint_A \left[u_1^R \sigma_{31}^{(2)} + u_2^R \sigma_{32}^{(2)} + u_3^R \sigma_{33}^{(2)}\right]_{x_3=0} dx_1\, dx_2 = 0 \tag{5.4}$$

or

$$\theta \iint_A \left[-\alpha_0 x_2 \sigma_{31}^{(2)} + \alpha_0 x_1 \sigma_{32}^{(2)} + \psi \sigma_{33}^{(2)}\right]_{x_3=0} dx_1\, dx_2$$

$$= \iint_A \left[v_1^0 \sigma_{31}^{(2)} + v_2^0 \sigma_{32}^{(2)} + v_3^0 \sigma_{33}^{(2)}\right]_{x_3=0} dx_1\, dx_2 \tag{5.4′}$$

except for exponentially small terms. Thus, we have

PROPOSITION 5.1. *If the residual displacement end data at* $x_3 = 0$ *are to induce only an exponentially decaying residual solution, it is necessary that they satisfy the integral condition* (5.4) (*or* (5.4′)) *for any regular elastic* (2) *state that is stress free on* S_c *so that*

$$\nu_1 \sigma_{31}^{(2)} + \nu_2 \sigma_{32}^{(2)} = 0 \quad on \quad C \tag{5.5}$$

and distortion free at $x_3 = 0$ *so that* (5.3) *is satisfied.*

One (2) state, denoted by $\{u_k^{(0)}, \sigma_{3k}^{(0)}\}$, which meets all the requirements of Proposition 5.1 is a solution of the equations of elasticity that is stress free on S_c, distortion free at $x_3 = 0$ and tends to the Saint-Venant solution with $\theta = 1$ as $x_3 \to \infty$, i.e.,

$$x_3 \to \infty: \quad u_1^{(0)} \to -x_2 x_3, \quad u_2^{(0)} \to x_1 x_3, \quad u_3^{(0)} \to \psi \tag{5.6}$$

where ψ is the warping function for the given cross section geometry. A simple solution for such a (2) state exists for a circular cross section, namely the Saint-Venant solution (3.7) with $\theta = 1$ and $\alpha_0 = 0$.

For other cross section geometries, numerical methods may be needed for an accurate approximate solution for this (2) state. In the actual solution process for this (2) state, it is usually simpler to work with the residual solution $\{\bar{u}_k^{(0)}\}$

$$u_1^{(0)} = \bar{u}_1^{(0)} - x_2 x_3, \quad u_2^{(0)} = \bar{u}_2^{(0)} + x_1 x_3, \quad u_3^{(0)} = \bar{u}_3^{(0)} + \psi. \tag{5.7}$$

Note that $\{\bar{u}_k^{(0)}\}$ is a solution of the equations of elasticity theory and stress free on S_c, since both the Saint-Venant solution and $\{u_k^{(0)}\}$ are solutions of the equations and free of tractions on the cylindrical surface. At the end section $x_3 = 0$, we have from $u_k^{(0)} = 0$, $k = 1,2,3$, the following inhomogeneous end conditions for $\{\bar{u}_k^{(0)}\}$:

$$x_3 = 0: \quad \bar{u}_3^{(0)} = -\psi(x_1, x_2), \quad \bar{u}_j^{(0)} = 0 \quad (j = 1,2). \tag{5.8}$$

Given (5.6) and (5.7), we also have

$$x_3 \to \infty: \quad \bar{u}_k^{(0)} \to 0 \quad (k = 1,2,3). \tag{5.9}$$

Once we have found the (2) state $\{u_k^{(0)}, \sigma_{3k}^{(0)}\}$ for the $x_3 = 0$ end as specified by Proposition 5.1, the requirements (5.4′) and (5.1a) or (5.1b) provide the two conditions needed for determining θ and α_0. When we have Case (A) or (B) data at $x_3 = L$, we may use (5.1a) to eliminate θ from (5.4′) and obtain the following formula for α_0:

$$\begin{aligned} \alpha_0 \iint_A \left[x_1 \sigma_{32}^{(0)} - x_2 \sigma_{31}^{(0)} \right]_{x_3 = 0} dx_1\, dx_2 &= -\iint_A \left[\psi \sigma_{33}^{(0)} \right]_{x_3 = 0} dx_1\, dx_2 \\ &\quad + \frac{M_t^{(2)}}{M_t} \iint_A \left[v_1^0 \sigma_{31}^{(0)} + v_2^0 \sigma_{32}^{(0)} + v_3^0 \sigma_{33}^{(0)} \right]_{x_3 = 0} dx_1\, dx_2 \end{aligned} \tag{5.10}$$

where $M_t^{(2)}$ is given by (5.2) and M_t is the actual torque for the prescribed stress data at $x_3 = L$. The integral on the left may be simplified as it is just the axial torque for the (2) state over the cross section at $x_3 = 0$. By overall equilibrium, this must be the same torque for $\{\sigma_{3k}^{(0)}\}$ as $x_3 \to \infty$ given by (5.2). In that case, (5.10) can be written as

$$\begin{aligned} \alpha_0 &= -\frac{1}{M_t^{(2)}} \iint_A \left[\psi \sigma_{33}^{(0)} \right]_{x_3 = 0} dx_1\, dx_2 \\ &\quad + \frac{1}{M_t} \iint_A \left[v_1^0 \sigma_{31}^{(0)} + v_2^0 \sigma_{32}^{(0)} + v_3^0 \sigma_{33}^{(0)} \right]_{x_3 = 0} dx_1\, dx_2. \end{aligned} \tag{5.11}$$

For Case (C) mixed data at $x_3 = L$, (5.1b) and (5.4′) provide two linear equations for θ and $\theta\alpha_0$. Upon eliminating $\theta\alpha_0$, we obtain

$$\begin{aligned} \theta \left\{ M_t^{(2)} L - \iint_A \psi \sigma_{33}^{(0)}\, dx_1\, dx_2 \right\} &= \iint_A \left[v_1^L \Phi_{,2} - v_2^L \Phi_{,1} - \psi \tau_3^L \right] dx_1\, dx_2 \\ &\quad - \iint_A \left[v_1^0 \sigma_{31}^{(0)} + v_2^0 \sigma_{32}^{(0)} + v_3^0 \sigma_{33}^{(0)} \right]_{x_3 = 0} dx_1\, dx_2. \end{aligned} \tag{5.12}$$

The constant $\theta\alpha_0$ (which gives rise only to a rigid body rotation) can then be calculated from (5.12) and (5.1b) if we wish.

For a *circular cross section*, we have $\psi = 0$. The expression (5.11) for Cases (A) and (B) data at $x_3 = L$ simplifies to

$$\alpha_0 = \frac{1}{M_t}\iint_A \left[v_1^0\sigma_{31}^{(0)} + v_2^0\sigma_{32}^{(0)} + v_3^0\sigma_{33}^{(0)}\right]_{x_3=0} dx_1\, dx_2 \qquad (5.11')$$

where M_t is the resultant axial torque of the prescribed shear stresses at $x_3 = L$. The angle of twist θ in this case is given by (3.18′).

For Case (C) type end data at $x_3 = L$ of a *circular cross section*, θ is given by a simplified version of (5.12) instead. With $M_t^{(2)} = \pi b^4 G/2$, $\psi = 0$, and Φ given by (3.16) and (3.17), we have from (5.12)

$$\theta = \frac{2}{\pi b^4 L}\iint_A \left[x_2\left(v_1^0 - v_1^L\right) - x_1\left(v_2^0 - v_2^L\right)\right] dx_1\, dx_2 \qquad (5.12')$$

and $M_t = \pi b^4 G\theta/2$ (as in (4.33c)), whatever v_3^0 and v_3^L may be. In arriving at (5.12′), we have made use of the fact that, for a circular cross section, the Saint-Venant torsion solution with $\theta = 1$ and $\alpha_0 = 0$ satisfies the distortion free condition at $x_3 = 0$ and has the proper far field behavior as $x_3 \to \infty$ imposed on the (2) state. In other words, we have $u_1^{(0)} = -x_2x_3$, $u_2^{(0)} = x_1x_3$ and $u_3^{(0)} = 0$.

5.2. *Displacement data at both ends*

If displacement data are prescribed at both ends, we need one condition of the type (5.4) for each end of the slender body. At the $x_3 = 0$ end, this condition may be written as

$$\theta\alpha_0 M_t^{(2)} = \theta\alpha_0 \iint_A \left[x_1\sigma_{32}^{(0)} - x_2\sigma_{31}^{(0)}\right]_{x_3=0} dx_1\, dx_2$$

$$= -\theta\iint_A \left[\psi\sigma_{33}^{(0)}\right]_{x_3=0} dx_1\, dx_2$$

$$+ \iint_A \left[v_1^0\sigma_{31}^{(0)} + v_2^0\sigma_{32}^{(0)} + v_3^0\sigma_{33}^{(0)}\right]_{x_3=0} dx_1\, dx_2 \qquad (5.13)$$

where $M_t^{(2)}$ is the axial torque needed to produce a unit angle of twist, $\theta = 1$, for the given cross section (see (5.4′), (5.10) and (5.11)).

The necessary condition for the end data at $x_3 = L$ to induce only a decaying residual solution is also obtained in the form (5.4) (but now applied at $\bar{x}_3 = 0$) from the reciprocal theorem after a change of variable, $\bar{x}_3 = L -$

x_3. The (2) state needed for this necessary condition, denoted by $\{u_k^{(L)}, \sigma_{ij}^{(L)}\}$, is the solution of the equations of elasticity theory that is stress free on the cylindrical surface S_c, distortion free at $\bar{x}_3 = 0$, and has the far field behavior

$$\bar{x}_3 \to \infty: \quad u_1^{(L)} \to x_2 \bar{x}_3, \quad u_2^{(L)} \to -x_1 \bar{x}_3, \quad u_3^{(L)} \to \psi. \tag{5.14}$$

For this (2) state, we have

$$\theta(\alpha_0 + L) M_t^{(2)} = \theta(\alpha_0 + L) \iint_A \left[x_1 \sigma_{32}^{(L)} - x_2 \sigma_{31}^{(L)}\right]_{x_3 = L} dx_1\, dx_2$$

$$= -\theta \iint_A \left[\psi \sigma_{33}^{(L)}\right]_{x_3 = L} dx_1\, dx_2 + \iint_A \sum_{i=1}^{3} \left[v_i^L \sigma_{3i}^{(L)}\right]_{x_3 = L} dx_1\, dx_2 \tag{5.15}$$

where $M_t^{(2)}$ is again the axial torque needed to produce a unit angle of twist, $\theta = 1$, for the given cross section.

The unknown α_0 may be eliminated from the two relations (5.13) and (5.15) to give an expression for θ alone. The resultant axial torque experienced by the slender body can then be calculated from (3.15′) or

$$M_t = \theta M_t^{(2)}. \tag{5.16}$$

We may also use (5.13) to calculate α_0 if we wish.

For a *circular cross section*, we have $\psi = 0$ for all points in the cross section. The Saint-Venant torsion solutions

$$u_1^{(0)} = -x_2 x_3, \qquad u_2^{(0)} = x_1 x_3, \qquad u_3^{(0)} = 0 \tag{5.17a}$$

$$u_1^{(L)} = -x_2(x_3 - L), \quad u_2^{(L)} = x_1(x_3 - L), \quad u_3^{(L)} = 0 \tag{5.17b}$$

give the desired (2) state for the $x_3 = 0$ end and $x_3 = L$ end, respectively. Both give the following shear stresses

The conditions (5.13) and (5.15) now simplify to

$$\theta\alpha_0 M_t^{(2)} = \iint_A G\left[x_1 v_2^0 - x_2 v_1^0\right] dx_1\, dx_2 \tag{5.18a}$$

$$\theta(\alpha_0 + L) M_t^{(2)} = \iint_A G\left[x_1 v_2^L - x_2 v_1^L\right] dx_1\, dx_2 \tag{5.18b}$$

so that

$$M_t^{(2)}\theta = M_t = \frac{G}{L}\iint\left[x_1\left(v_2^L - v_2^0\right) - x_2\left(v_1^L - v_1^0\right)\right] dx_1\, dx_2. \tag{5.18c}$$

The remaining unknown α_0 can be calculated from (5.18a). With a_0 and b_0 in (A.10b) and (2.11a) related to θ and α_0 by

$$\frac{a_0 - b_0}{L} = \theta, \quad \frac{b_0 L}{a_0 - b_0} = \alpha_0, \tag{5.19}$$

the approximate solution (5.17) and (5.18) for a circular cross section obtained by the method of decaying residual solution is identical to the exact interior solution for prescribed displacement end data given in Section 2.1 and the Appendix of this paper.

For other cross section geometries, simple (2) states are generally not available. But accurate solutions for $\sigma_{3k}^{(0)}(x_1, x_2, 0)$ and $\sigma_{3j}^{(L)}(x_1, x_2, L)$ may be obtained by numerial methods to complete the solution process for obtaining the interior behavior of the slender bodies in torsion.

Appendix. Exact solutions for torsion of circular cylinders

A.1. The boundary value problem (2.5)–(2.8)

With the exact solution of (2.5) and (2.6) given by (2.9) and (2.10) (where $t_n = \lambda_n b$), the condition (2.7) requires

$$b_0 r + \sum_{n=1}^{\infty} b_n J_1(\lambda_n r) = f(r). \tag{A.1}$$

By the identity [1]

$$\int_0^b J_1(\lambda_n r) r^2\, dr = \frac{b^2}{\lambda_n} J_2(\lambda_n b) = 0, \tag{A.2}$$

we obtain from (A.1)

$$b_0 = \frac{4}{b^4} \int_0^b f(r) r^2 \, dr. \tag{A.3}$$

Similarly, with the orthogonality condition [1]

$$\int_0^b J_1(\lambda_m r) J_1(\lambda_n r) r \, dr = 0 \quad (m \neq n) \tag{A.4}$$

and by the identity (A.2), we obtain from (A.1)

$$b_m = \frac{1}{\xi_m^2} \int_0^b f(r) J_1(\lambda_m r) r \, dr \tag{A.5}$$

where [1]

$$\xi_n^2 = \int_0^b J_1^2(\lambda_n r) r \, dr = \frac{1}{2\lambda_n^2} \left\{ [J_1^{\cdot}(t_n)]^2 + \left(1 - \frac{1}{t_n^2}\right) [J_1(t_n)]^2 \right\} \tag{A.6}$$

with $()^{\cdot}$ indicating differentiation of () with respect to its argument.

On the other hand, the condition (2.8) requires

$$(a_0 - b_0) \frac{r}{L} + \sum_{n=1}^{\infty} \lambda_n \left[a_n \frac{\cosh(\lambda_n L)}{\sinh(\lambda_n L)} - b_n \frac{1}{\sinh(\lambda_n L)} \right] J_1(\lambda_n r) = \frac{1}{b} g(r) \tag{A.7}$$

or

$$a_0 r + \sum_{n=1}^{\infty} (\lambda_n L) a_n \coth(\lambda_n L) J_1(\lambda_n r)$$

$$= \frac{L}{b} g(r) + b_0 r + \sum_{n=1}^{\infty} \frac{b_n \lambda_n L}{\sinh(\lambda_n L)} J_1(\lambda_n r). \tag{A.7$'$}$$

The condition (A.2) allows us to obtain from (A.7′)

$$a_0 \int_0^b r^3 \, dr = \frac{L}{b} \int_0^b g(r) r^2 \, dr + b_0 \int_0^b r^3 \, dr$$

$$= \frac{L}{b} \int_0^b g(r) r^2 \, dr + \int_0^b f(r) r^2 \, dr \tag{A.8}$$

or

$$a_0 = \frac{4}{b^4}\left\{\frac{L}{b}\int_0^b g(r)r^2\,dr + \int_0^b f(r)r^2\,dr\right\}. \tag{A.8$'$}$$

Similarly, the orthogonality condition (A.4) enables us to obtain from (A.7′)

$$\xi_m^2\lambda_m\left[a_m\coth(\lambda_m L) - \frac{b_m}{\sinh(\lambda_m L)}\right] = \frac{1}{b}\int_0^b g(r)J_1(\lambda_m r)\,r\,dr.$$

The expression (A.5) may then be used to eliminate b_m to obtain

$$a_m = \frac{1}{\xi_m^2}\frac{\tanh(\lambda_m L)}{\lambda_m b}$$
$$\cdot\left\{\frac{b}{L}\frac{\lambda_m L}{\sinh(\lambda_m L)}\int_0^b f(r)J_1(\lambda_m r)\,r\,dr + \int_0^b g(r)J_1(\lambda_m r)\,r\,dr\right\}. \tag{A.9}$$

A.2. Other combinations of end conditions

If displacement data are prescribed at both ends, we have instead of (A.7)

$$a_0 r + \sum_{n=1}^{\infty} a_n J_1(\lambda_n r) = f_L(r). \tag{A.10a}$$

With the orthogonality relations (A.2) and (A.4), we obtain from $v(r, L) = f_L(r)$

$$a_0 = \frac{4}{b^4}\int_0^b f_L(r)r^2\,dr \tag{A.10b}$$

and

$$a_m = \frac{1}{\xi_m^2}\int_0^b f_L(r)J_1(\lambda_m r)\,r\,dr. \tag{A.10c}$$

On the other hand, if stress data are prescribed at both ends, then instead of (A.2), we have from $\sigma_{z\theta}(r,0) = Gg_0(r)/b$

$$(a_0 - b_0)\frac{r}{L} + \sum_{n=1}^{\infty}\left[a_n \operatorname{csch}(\lambda_n L) - b_n\coth(\lambda_n L)\right]\lambda_n J_1(\lambda_n r) = \frac{1}{b}g_L(r).$$

With the orthogonality relation (A.4), we deduce from (A.11a)

$$\lambda_m \xi_m^2 [a_m \operatorname{csch}(\lambda_n L) - b_m \coth(\lambda_m L)] = \frac{1}{b}\int_0^b g_0(r) J_1(\lambda_m r) r\,dr. \quad \text{(A.11b)}$$

A similar condition resulting from $\sigma_{z\theta}(r, L) = G g_L(r)/b$ can be written as

$$\lambda_m \xi_m^2 [a_m \coth(\lambda_m L) - b_m \operatorname{csch}(\lambda_m L)] = \frac{1}{b}\int_0^b g_L(r) J_1(\lambda_m r) r\,dr. \quad \text{(A.11c)}$$

(A.11b) and (A.11c) determine the pair (a_m, b_m), $m = 1, 2, \ldots$

For a_0 and b_0, we use (A.2) to obtain from (A.11a)

$$\frac{a_0 - b_0}{L}\int_0^b r^3\,dr = \frac{1}{b}\int_0^b r^2 g_L(r)\,dr. \quad \text{(A.12a)}$$

A similar result is obtained from (A.7) to be

$$\frac{a_0 - b_0}{L}\int_0^b r^3\,dr = \frac{1}{b}\int_0^b r^2 g_0(r)\,dr. \quad \text{(A.12b)}$$

The conditions (A.12a) and (A.12b) together require

$$\int_0^b g_0(r) r^2\,dr = \int_0^b g_L(r) r^2\,dr, \quad \text{(A.12c)}$$

which is just a condition of moment equilibrium in the x_3 direction. With (A.12c) satisfied by $g_0(r)$ and $g_L(r)$, the prescribed stress data determine only $a_0 - b_0$ and not a_0 and b_0 by themselves. A closer examination of (2.9) shows that the elastostatics of the beam is determined up to a rigid motion as we might have expected.

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