Archive of Applied Mechanics 63 (1993) 513–521

Archive of
Applied Mechanics
© Springer-Verlag 1993

Approximate solutions for the shear center of plates of variable thickness

C. Gu, Seattle and **F. Y. M. Wan**, Washington

Summary: In a plate-theoretical formulation of the shear center problem, the relevant boundary-value problem is for a cantilevered rectangular plate of variable thickness with two free opposite edges and with the edge opposite to the clamped end subject to a rigid vertical displacement and free of bending moment. For plates with Poisson's ratio ν equal to zero, there is an exact elementary solution for this boundary-value problem from which the exact location of the shear center can be calculated. When Poisson's ratio is not zero, an approximate elementary solution may be obtained within the framework of a Saint-Venant flexure solution for plates by satisfying the displacement boundary conditions at the clamped edge approximately. Different forms of this approximation are discussed in [7], some with rather marked Poisson's ratio effects. Among these, the minimum complementary energy approach of [6] gives a shear center location identical to the exact solution for $\nu = 0$.

A generalized beam theory developed in [6] is implemented here to delineate the effect of ν without altering the edge conditions by ad hoc approximations. The results show that the Poisson's ratio effect is rather moderate and the shear center location is nearly the same as that for zero Poisson's ratio. A finite element solution for the plate theory boundary-value problem confirms this finding.

The generalized beam equations are also used to study the effect of the aspect ratio of the plate and orthotropy on the location of the shear center.

Näherungslösungen für den Schubmittelpunkt von Platten veränderlicher Dicke

Übersicht: In der plattentheoretischen Formulierung des Schubmittelpunktproblems [6] betrifft die relevante Randwertaufgabe eine rechteckige Kragplatte mit veränderlicher Dicke in Querrichtung, mit einer Lastverteilung, welche eine gleichförmige Durchbiegung am belasteten Rand erzeugt.

Die vorliegende Arbeit implementiert eine Näherungslösung dieses Problems in der Form einer verallgemeinerten Balkentheorie zwölfter Ordnung, welche Verwölbung sowie auch anticlastische Krümmung berücksichtigt [6]. Das Ziel ist, den quantitativen Einfluß der Poissonschen Zahl und auch den der Orthotropie zu bestimmen.

Die Resultate zeigen, daß der Einfluß der Poissonschen Zahl, welcher in der bekannten näherungsweisen Bestimmung mit Hilfe des Variationsprinzips für Spannungen vollkommen verschwindet, in der Tat sehr gering ist, insbesondere im Vergleich mit den Folgen einer Anwendung der St.-Venantschen Biegetheorie zusammen mit einer klassischen Stipulierung bezüglich des Verschiebungsfeldes am Schwerpunkt der Querschnittsfläche. Darüber hinausgehend wird gezeigt, daß die numerischen Werte nur unwesentlich von Unterschieden zwischen Isotropie und Orthotropie beeinflußt sind.

Die obigen Resultate werden durch die Ableitung und Anwendung einer Balkentheorie sechzehnter Ordnung und durch Rechnungen mit Hilfe von Finiten Elementen bestätigt.

1 Introduction

The shear center problem was formulated recently as a boundary-value problem in the theory of transverse bending of flat plates of variable thickness in [6, 7]. The results obtained there include an approximate determination of the *center of shear* by the principle of minimum complementary energy with certain semi-inverse Saint-Venant assumptions for the stresses. Also included is a twelfth order system of generalized beam equations derived by a suitable process of averaging of the plate equations for a transverse displacement field which is quadratic in the chordwise coordinate [6]. The assumed three-term displacement field allows for the incorporation of end section anticlastic curvature constraints as well as end section warping constraint.

The approximate complementary energy analysis leads to an expression for the center of shear, in terms of an effective bending stiffness factor, which is independent of Poisson's ratio and the plate aspect ratio. The form of the generalized beam equations on the other hand suggests the presence of these two effects. The principal aim of the present paper is to determine their magnitude for isotropic as well as orthotropic plates. This is done here in three distinct ways: (1) by solving the twelfth order system of generalized beam equations, (2) by formulating a 16th-order system of beam equations using a four-term (chordwise cubic) displacement approximation and evaluating its consequences, and (3) by obtaining a Kirchhoff-plate finite element solution of the two dimensional boundary-value problem.

Our solutions show that both aspect ratio and Poisson's ratio effects result in modifications of the approximate solution by way of the principle of minimum complementary. These effects are however much smaller than those resulting from the various ways to approximately satisfy the clamped end condition. They include the plate analogues of imposing (on the classical Saint-Venant flexure solution) the condition of vanishing centroidal end section rotation [2, 5] or vanishing averaged end section rotation [4, 8] which have been proposed previously for an approximate three-dimension solution of the shear center problem.

2 The shear center problem in plate theory

In the classical linear theory for the transverse bending of elastic plates, the deformation of the plate in a cartesian coordinate system (x_1, x_2, x_3) is characterized by a mid-plane transverse displacement $w(x_1, x_2)$ in the x_3-direction. In terms of w, the curvature change measures $\varkappa_{ij}$ are defined by the strain-displacement relations

$$\varkappa_{ij} = -w_{,ij} \qquad (i, j = 1, 2). \tag{1}$$

The stress couples M_{ij} induced by the bending deformation are given in terms of the $\varkappa_{ij}$ by a system of stress-strain relations, taken here (as in [6]) in the form

$$M_{11} = D_1\varkappa_{11} + D_\nu\varkappa_{22}, \qquad M_{22} = D_2\varkappa_{22} + D_\nu\varkappa_{11}, \qquad M_{12} = \frac{1}{2} D_t(\varkappa_{12} + \varkappa_{21}) = M_{21}, \tag{2}$$

(Note that D_t in [7] corresponds to $D_t/2$ of [6] and (2)). We will be concerned with homogeneous plates of thickness h which varies only with x_2 so that the bending stiffness factors D_α are functions of x_2 only. Since the plate is subject to no interior loading, the stress couples, together with the transverse shear resultants Q_i, satisfy the following three differential equations of equilibrium,

$$Q_{1,1} + Q_{2,2} = 0, \qquad M_{1k,1} + M_{2k,2} - Q_k = 0 \qquad (k = 1, 2), \tag{3}$$

in the plate interior.

We consider here rectangular plates with a mid-plane occupying the region $R = \{0 \leqq x_1 \leqq L, 0 \leqq x_2 \leqq a\}$. The edges $x_2 = 0$ and $x_2 = a$ free of traction so that

$$x_2 = 0, a: \quad V_2 = M_{22} = 0 \qquad (0 \leqq x_1 \leqq L), \tag{4}$$

where the effective transverse resultant is given by $V_2 = Q_2 + M_{21,1}$. The plate is clamped along $x_1 = 0$ so that

$$x_1 = 0: \quad w = w_{,1} = 0 \qquad (0 \leqq x_2 \leqq a). \tag{5}$$

The end $x_1 = L$ is free of bending couples but is subject to a uniform vertical displacement W_0:

$$x_1 = L: \quad w = W_0, \qquad M_{11} = 0 \qquad (0 \leqq x_2 \leqq a). \tag{6a, b}$$

The resultant transverse force F_3 and axial torque T_1 at $x_1 = L$ are given by

$$F_3 = \int_0^a V_1 \, dx_2 - [2M_{12}]_0^a, \qquad T_1 = \int_0^a V_1 x_2 \, dx_2 - [2x_2 M_{12}]_0^a \tag{7a, b}$$

with $V_1 = Q_1 + M_{12,2}$. These expressions may be rewritten as

$$F_3 = \int_0^a M_{11,1} \, dx_2, \qquad T_1 = \int_0^a (x_2 M_{11,1} - 2M_{12}) \, dx_2 \tag{8}$$

with the help of the moment equilibrium equation $Q_1 = M_{11,1} + M_{21,2}$. In the context of plate theory, the location $(x_1^s, x_2^s) = (L, y_s)$ of the shear center has been shown to be [6]

$$y_s = \frac{T_1}{F_3}. \tag{9}$$

Of interest is how y_s varies with the material parameters, the dependence of h on x_2 and the aspect ratio a/L. Note that for the purpose of determining the shear center, the plate BVP of [6] and the one of this section are equivalent.

3 Saint-Venant stress assumptions and minimum complementary energy

As in [6], we consider as a nontrivial exact solution of the equilibrium equations (3) which also satisfies the stress edge conditions (4) and (6 b)

$$Q_2 = 0, \quad M_{22} = 0 \tag{10}$$

$$Q_1 = Q(x_2), \quad M_{12} = M_t(x_2), \quad M_{11} = (Q - M_t^{\cdot})(x_1 - L), \tag{11}$$

with $(\ )^{\cdot} \equiv d(\ )/dx_2$. The introduction of (10) and (11) into the variational equation

$$\delta\left\{\frac{1}{2}\int_0^L\int_0^a \left[\frac{D_2 M_{11}^2 + D_1 M_{22}^2 - 2D_\nu M_{11} M_{22}}{D_1 D_2 - D_\nu^2} + \frac{M_{12}^2}{D_t}\right] dx_2\, dx_1 - \int_0^a V_1(L, x_2)\, W\, dx_2 + [2M_{12}(L, x_2)\, W]_0^a\right\} = 0 \tag{12 a}$$

reduces it to an equation for Q and $M_t^{\cdot}$:

$$\delta \int_0^a \left[\frac{L^3 (Q - M_t^{\cdot})^2}{6D_b} + \frac{LM_t^2}{2D_t} - W(Q - M_t^{\cdot})\right] dx_2 = 0 \tag{12 b}$$

with $D_b = D_1(1 - D_\nu^2/D_1 D_2)$. The Euler differential equations and Euler boundary conditions corresponding to (12 b) require

$$M_t = 0, \quad M_{11} = \frac{3D_b}{L^3} W(x_1 - L), \quad Q = \frac{3D_b}{L^3} W. \tag{13}$$

We then have from (8) and (9)

$$y_s^0 \equiv \frac{\int_0^a x_2 D_b(x_2)\, dx_2}{\int_0^a D_b(x_2)\, dx_2} \tag{14}$$

as an approximation for the shear center coordinate y_s. For homogeneous plates, y_s^0 is independent of the material parameters and is in fact identical to the exact location of the shear center for the special case $D_\nu/D_2 = 0$.

4 A generalized beam theory solution

The form of the end conditions (5) and (6) and of the Saint-Venant solution for the case of a constant $\nu_2 = D_\nu/D_2$ suggests that an adequate approximate solution of the boundary-value problem for the determination of the center of shear may be obtained by assuming [6]

$$w(x_1, x_2) = \bar{w}_0(x_1) + \bar{w}_1(x_1)\, x_2 + \bar{w}_2(x_1)\, x_2^2 \tag{15}$$

and by working with one-dimensional stress measures

$$\{M_{xn}, M_{yn}, M_{tn}, Q_{xn}, Q_{yn}\} = \int_0^a \{M_{11}, M_{22}, M_{12} = M_{21}, Q_1, Q_2\}\, x_2^n \,dx_2 . \tag{16}$$

We take moments of the three plate equilibrium equations (3) and use the free edge conditions (4) to obtain the following system of one dimensional equilibrium equations

$$Q'_{xk} - kQ_{y(k-1)} = 0, \qquad M'_{xk} - kM_{t(k-1)} = Q_{xk}, \qquad M'_{tj} - jM_{y(j-1)} = Q_{yj}, \tag{17}$$

$k = 0, 1, 2$ and $j = 0, 1$. In (17), primes denote differentiation with respect to x_1 and terms with a negative subscript are to be set equal to zero.

Use of the last two equations to eliminate Q_{xk} and $Q_{y(k-1)}$ from the first of (17) leaves the system

$$M''_{xi} - iM'_{t(i-1)} + 2M_{y(i-2)} = 0 \qquad (i = 0, 1, 2). \tag{18}$$

The three equations in (18) contain six unknowns $\{M_{x0}, M_{x1}, M_{x2}, M_{t0}, M_{t1}, M_{y0}\}$. The first two equations of (18) can be integrated with the help of (6) and (8) to give

$$M_{x0} = F_3(x_1 - L), \qquad M'_{x1} - 2M_{t0} = T_1, \qquad M''_{x2} - 4M'_{t1} + 2M_{y0} = 0 \tag{19, 20, 21}$$

where F_3 and T_1 are two constants of integration which are readily identified with the force and torque as calculated in (8).

In order to transform (19)–(21) into three equations for $\bar{w}_0$, $\bar{w}_1$ and $\bar{w}_2$, we deduce from (2), (1) and (15) the following system of one-dimensional constitutive relations:

$$M_{xk} = D_{xk}\bar{w}''_0 + D_{x(k+1)}\bar{w}''_1 + D_{x(k+2)}\bar{w}''_2 + 2D_{vk}\bar{w}_2 \qquad (k = 0, 1, 2) \tag{22}$$

$$M_{y0} = D_{v0}\bar{w}''_0 + D_{v1}\bar{w}''_1 + D_{v2}\bar{w}''_2 + 2D_{y0}\bar{w}_2 \tag{23}$$

$$M_{tn} = D_{tn}\bar{w}'_1 + 2D_{t(n+1)}\bar{w}'_2 \qquad (n = 0, 1) \tag{24}$$

where

$$\{D_{xk}, D_{yk}, D_{vk}, D_{tk}\} = \int_0^a \{D_1, D_2, D_v, D_t\}\, x_2^k \,dx_2 . \tag{25}$$

Introduction of (22)–(24) into (19)–(21) leads to the following system of ordinary differential equations for $\bar{w}_0$, $\bar{w}_1$ and $\bar{w}_2$ [6]:

$$D_{x0}\bar{w}''_0 + D_{x1}\bar{w}''_1 + D_{x2}\bar{w}''_2 + 2D_{v0}\bar{w}_2 = F_3(x_1 - L) \tag{26a}$$

$$D_{x1}\bar{w}''_0 + D_{x2}\bar{w}''_1 + D_{x3}\bar{w}''_2 + 2D_{v1}\bar{w}_2 - 2D_{t0}\bar{w}_1 - 4D_{t1}\bar{w}_2 = T_1(x_1 - L) \tag{26b}$$

$$\begin{aligned}[D_{x2}\bar{w}''_0 + D_{x3}\bar{w}''_1 + D_{x4}\bar{w}''_2 + 2D_{v2}\bar{w}_2]'' - 4[D_{t1}\bar{w}''_1 + 2D_{t2}\bar{w}''_2]'' \\ + 2[D_{v0}\bar{w}''_0 + D_{v1}\bar{w}''_1 + D_{v2}\bar{w}''_2 + 2D_{y0}\bar{w}_2] = 0\end{aligned} \tag{26c}$$

where the second equation has been integrated once to reduce its order.

The boundary conditions (5) require

$$\bar{w}_k(0) = \bar{w}'_k(0) = 0 \qquad (k = 0, 1, 2) \tag{27}$$

while the boundary conditions (6) require

$$\bar{w}_0(L) = W_0, \qquad \bar{w}_1(L) = 0 \tag{28a, b}$$

$$\bar{w}_2(L) = 0, \qquad \bar{w}''_2(L) = 0. \tag{28c, d}$$

By our choice of constants of integration in (19)–(21) and (26), $\bar{w}''_0(L)$ and $\bar{w}''_1(L)$ both vanish if (28 b), (28 c) and (28 d) are satisfied.

Only $\bar{w}''_0$ (and no lower derivative of $\bar{w}_0$) appears in (26). Hence, we may use (26 a) to eliminate $\bar{w}''_0$ from (26 b) and (26 c) to obtain a sixth order system of two differential equations for $\bar{w}_1$ and $\bar{w}_2$ with two unknown parameters F_3 and T_1. The system will be stated in dimensionless form in section (7). The six unknown constants of integrations for this system and the unknown ratio T_1/F_3 are determined by the boundary conditions $\bar{w}_1(0) = \bar{w}_1'(0) = \bar{w}_2(0) = \bar{w}_2'(0) = \bar{w}_1(L) = \bar{w}_2(L) = \bar{w}_2''(L) = 0$. Equation (26 a) together with the three boundary conditions $\bar{w}_0(0) = \bar{w}_0'(0) = 0$ and $\bar{w}_0(L) = W_0$ can then be used to determine $\bar{w}_0(x_1)$ and F_3. The sixth order system for $\bar{w}_1$ and $\bar{w}_2$ is of constant coefficients; an exact solution of the corresponding boundary-value problem has been obtained with the characteristic equation (cubic in the square of the exponent) solved numerically.

An accurate numerical solution for this sixth order system or the eighth order system (26) can also be obtained by COLSYS [1] for validation. For this purpose, it is convenient to work with $\bar{w}_k/F_3$ and set $\bar{w}_3(x_1) = T_1(x_1 - L)$ so that

$$(\bar{w}_3/F_3)'' = 0, \qquad \bar{w}_3(L)/F_3 = 0. \tag{29 a, b}$$

We may now apply COLSYS to the eighth order system (26 b), (26 c) and (29 a) for $\bar{w}_1$, $\bar{w}_2$ and $\bar{w}_3$ (with $\bar{w}_0$ eliminated by (26 a)) using (28 c, d), (29 b), $\bar{w}_1(L) = \bar{w}_k(0) = \bar{w}_k'(0) = 0$, k = 1, 2, as the needed (eight) boundary conditions. The solution for $\bar{w}_3'/F_3$ (which is a constant) gives y_s:

$$y_s = \frac{\bar{w}_3'}{F_3}. \tag{30}$$

For comparison, the numerical results of y_s for homogeneous, isotropic plates with linearly and exponentially varying thickness are given in Tables 1 and 2 for $\varepsilon \equiv a/L = 1/10$ and 1 and for several typical Poisson's ratios. Variations of y_s/a with a/L are shown in Figures 1 and 2 with $h/a \ll 1$ and $h/L \ll 1$ in all cases. The results indicate that Poisson's ratio has a very moderate effect on y_s for the entire range of aspect ratios. In fact, the location of y_s deviates by less than 2% from the approximate solution y_s^0 (which is also the exact solution for the $v_2 \equiv D_v/D_2 = 0$ case) in the range $0 < a/L \leqq 10$ and $0 < v \leqq 1/2$.

As seen from Tables (1) and (2), this is much less than that predicted by a Saint-Venant type solution with the conditions of no end section rotation satisfied only at the centroid y_c or only by its

Table 1. Location of y_s for homogeneous isotropic plates with a linearly varying thickness, $h = h_0 x_2/a$ ($y_c/a = 2/3$)

v → Method ↓		0	1/4	1/3	1/2
y_s^0 [from Eq. (14)]		4/5	4/5	4/5	4/5
y_s^c [from Eq. (31 a)]		4/5	0.8533	0.8667	0.8889
Generalized Beam	$a/L = 0.1$	0.8000	0.8080	0.8106	0.8159
Theory (Quadratic)	$a/L = 1.0$	0.8000	0.8055	0.8073	0.8109
Generalized Beam	$a/L = 0.1$	0.8000	0.8084	0.8111	0.8168
Theory (Cubic)	$a/L = 1.0$	0.8000	0.8059	0.8079	0.8118
Finite Element	$a/L = 0.1$	0.8000	0.8068	0.8081	0.8095
Solution	$a/L = 1.0$	0.8000	0.8047	0.8055	0.8056

Table 2. Location of y_s for homogeneous isotropic plates with an exponentially varying thickness, $h = h_0 e^{x_2/a}$ ($y_c/a = 0.581976707$)

v → Method ↓		0	1/4	1/3	1/2
y_s^0 [from Eq. (14)]		0.7191	0.7191	0.7191	0.7191
y_s^c [from Eq. (31)]		0.7191	0.7739	0.7876	0.8105
Generalized Beam	$a/L = 0.1$	0.7191	0.7303	0.7340	0.7416
Theory (Quadratic)	$a/L = 1.0$	0.7191	0.7251	0.7270	0.7308
Generalized Beam	$a/L = 0.1$	0.7191	0.7312	0.7351	0.7435
Theory (Cubic)	$a/L = 1.0$	0.7191	0.7259	0.7282	0.7328
Finite Element	$a/L = 0.1$	0.7191	0.7287	0.7308	0.7329
Solution	$a/L = 1.0$	0.7191	0.7241	0.7247	0.7238

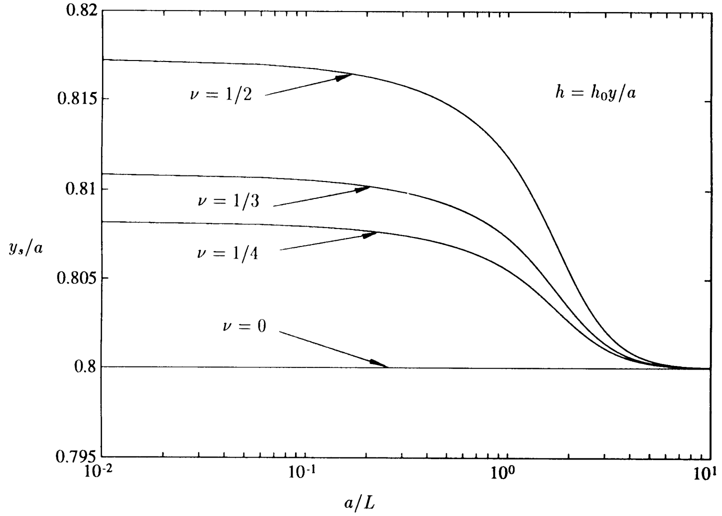

Fig. 1. Shear center location vs. the aspect ratio of an isotropic plate strip with a thickness linearly varying in the spanwise coordinate

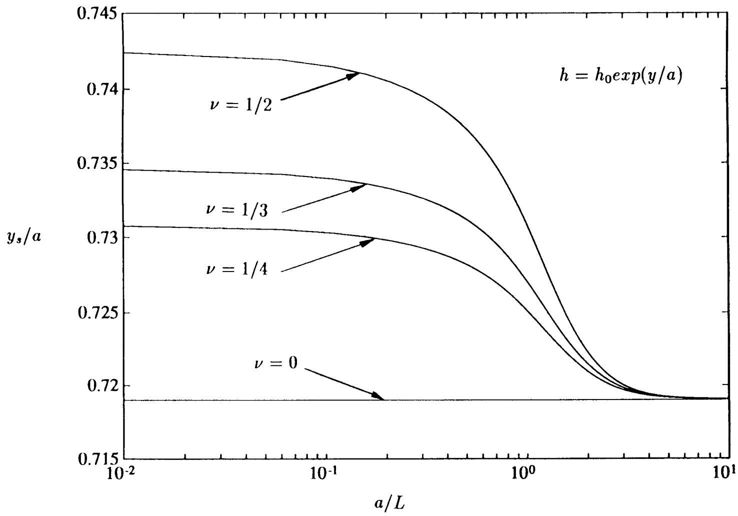

Fig. 2. Shear center location vs. the aspect ratio of an isotropic plate strip with a thickness exponentially varying in the spanwise coordinate

cross section average (obtained for the first time herein). For an isotropic plate with a constant Poisson's ratio v, both of these approximate solutions give

$$y_s \cong \frac{1}{1+v}\left[(1+3v)\, y_s^0 - 2v y_c\right] \equiv y_s^c \tag{31 a}$$

where y_s^0 is defined by (14). The expression y_s^c is a special case of the more general expression [7]

$$y_s \int_0^a D_b \, dx_2 \cong \int_0^a \left[x_2 + \frac{2v}{1+v}(x_2 - y_0)\right] D_b \, dx_2 \tag{31 b}$$

(obtained from $w_{,12}(x_1, y_0) = 0$) upon setting $y_0 = y_c$ and v equal to a constant.

5 The chordwise cubic approximation

Evidently, it is possible to use a higher degree polynomial approximation in x_2 for $w(x_1, x_2)$ instead of the quadratic approximation (15). A higher order system of ODE will then be obtained instead of the eighth order system (26) and hopefully will give a more accurate solution of the shear center problem. We implemented this solution process for a cubic approximation:

$$w(x_1, x_2) = \bar{w}_0(x_1) + \bar{w}_1(x_1)\, x_2 + \bar{w}_2(x_1)\, x_2^2 + \bar{w}_3(x_1)\, x_2^3 \tag{32}$$

The derivation of the governing equations for $\{\bar{w}_0, \ldots, \bar{w}_3\}$ is similar to that for the chordwise quadratic approximation (15); it gives the following four equations for $\{\bar{w}_i\}$:

$$D_{x0}\bar{w}_0'' + D_{x1}\bar{w}_1'' + D_{x2}\bar{w}_2'' + 2D_{v0}\bar{w}_2 + 6D_{v1}\bar{w}_3 = F_3(x_1 - L) \tag{33a}$$

$$D_{x1}\bar{w}_0''' + D_{x2}\bar{w}_1''' + D_{x3}\bar{w}_2''' + D_{x4}\bar{w}_3''' + 2D_{v1}\bar{w}_2' + 6D_{v2}\bar{w}_3' \\ - 2D_{t0}\bar{w}_1' - 4D_{t1}\bar{w}_2' - 6D_{t2}\bar{w}_3' = T_1 \tag{33b}$$

$$[D_{x2}\bar{w}_0'' + D_{x3}\bar{w}_1'' + D_{x4}\bar{w}_2'' + D_{x5}\bar{w}_3'' + 2D_{v0}\bar{w}_0 + 2D_{v1}\bar{w}_1 + 4D_{v2}\bar{w}_2 + 8D_{v3}\bar{w}_3]'' \\ - 4[D_{t1}\bar{w}_1'' + 2D_{t2}\bar{w}_2'' + 3D_{t3}\bar{w}_3''] + 2[2D_{y0}\bar{w}_2 + 6D_{y1}\bar{w}_3] = 0 \tag{33c}$$

$$[D_{x3}\bar{w}_0'' + D_{x4}\bar{w}_1'' + D_{x5}\bar{w}_2'' + D_{x6}\bar{w}_3'' + 6D_{v1}\bar{w}_0 + 6D_{v2}\bar{w}_1 + 8D_{v3}\bar{w}_2 + 12D_{v4}\bar{w}_3]'' \\ - [6D_{t2}\bar{w}_1'' + 12D_{t3}\bar{w}_2'' + 18D_{t4}\bar{w}_3''] + [12D_{y1}\bar{w}_2 + 36D_{y2}\bar{w}_3] = 0. \tag{33d}$$

The first three equations of (33) reduce to (26) upon setting $\bar{w}_3 = 0$. The boundary conditions (5) and (6) require

$$\bar{w}_k(0) = \bar{w}_k'(0) = 0, \qquad \bar{w}_0(L) = W_0, \qquad \bar{w}_j(L) = 0, \qquad \bar{w}_k''(L) = 0 \tag{34}$$

for $k = 0, \ldots, 3$ and $j = 1, 2, 3$. The condition $\bar{w}_0''(L) = 0$ is automatically satisfied if we have $\bar{w}_j''(L) = \bar{w}_2(L) = \bar{w}_3(L) = 0$, $j = 1, 2, 3$. The remaining 15 conditions in (34) completely determine the solution of the 13th order system (33) and the two unknown constants F_3 and T_1.

An exact solution of (33) and (34) is again possible; however, a COLSYS solution is more efficient. The corresponding approximate shear center locations for a linear thickness profile and an exponential profile are also given in Tables 1 and 2, respectively. For each value of Poisson's ratio, the approximate value for y_s/a of the *cubic theory* is within 0.2% of the corresponding value for quadratic theory.

6 Finite element solution

A finite element code described in [3] for plate bending by the stiffness method has also been used to solve the cantilever plate problem defined by (1)–(6). The code is based on linear curvature compatible triangular elements. Runs using 400 to 1,000 nodes for aspect ratios (a/L) of 0.1 to 10 seem to give adequate results for all cases calculated. The results for the case $a/L = 0.1$ and 1.0 are given in Tables 1 and 2 for $h = h_0 x_2/a$ and $h = h_0 e^{x_2/a}$. Evidently, the finite element results are within 2% of those from the generalized beam theory and tend toward the latter as the number of elements increases.

As expected, the behavior of the finite element solutions is more erratic at (and near) $x_1 = 0$ and $x_1 = L$; the value of y_s calculated from the resultants and couples at $x_1 = L$ is therefore not reliable. Since F_3 and T_1 are in fact independent of x_1, a more accurate y_s is obtained from the resultants and couples at an interior cross section instead.

7 Effect of orthotropy

For a homogeneous orthotropic plate of variable thickness, we may write

$$D_j = \frac{E_j h^3}{12(1 - v^2)}, \qquad D_v = \frac{v\sqrt{E_1 E_2}\, h^3}{12(1 - v^2)}, \qquad D_t = \frac{G h^3}{6} \tag{35}$$

with

$$v^2 = v_{12} v_{21}, \qquad \frac{v_{12}}{E_2} = \frac{v_{21}}{E_1} \tag{36}$$

where E_j is Young's modulus in the x_j-direction and G is the in-plane shear modulus ($= E/2(1+\nu)$ for isotropic material). The corresponding beam bending stiffness factors give

$$\frac{D_{xk}}{a^k D_{x0}} = \frac{\int_0^1 [h(z)]^3 z^k \, \mathrm{d}z}{\int_0^1 [h(z)]^3 \, \mathrm{d}z} \equiv d_k$$

$$\frac{D_{y0}}{D_{x0}} = \frac{E_2}{E_1} \equiv \lambda^2, \qquad \frac{D_{tk}}{a^k D_{x0}} = \frac{2G(1-\nu^2)}{E_1} d_k \equiv \mu d_k. \tag{37}$$

For the quadratic theory, the two ODEs for $\bar{w}_1$ and $\bar{w}_2$, obtained by using (26a) to eliminate $\bar{w}_0$ from (26b) and (26c), may now be written in the following dimensionless form:

$$\varepsilon^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} [(d_2 - d_1^2) W_1 + (d_3 - d_1 d_2) W_2] - 2\mu W_1 - 4\mu d_1 W_2 = (t - d_1)(x - 1) \tag{38a}$$

$$\varepsilon^4 \frac{\mathrm{d}^4}{\mathrm{d}x^4} [(d_3 - d_1 d_2) W_1 + (d_4 - d_2^2) W_2] - 4\mu\varepsilon^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} [d_1 W_1 + 2 d_2 W_2] + 4(1-\nu^2)\lambda^2 W_2 = -2\nu\lambda(x - 1) \tag{38b}$$

where

$$\varepsilon = \frac{a}{L}, \qquad x = \frac{x_1}{L}, \qquad f = \frac{F_3 L a}{D_{x0}},$$

$$t = \frac{T_1 L}{f D_{x0}} = \frac{T_1}{F_3 a}, \qquad W_k = \frac{a^{k-1}}{f} \bar{w}_k \qquad (k = 1, 2). \tag{39}$$

The sixth-order system (38), with an unknown parameter t, is supplemented by the seven boundary conditions

$$x = 0: \qquad W_k = \frac{\mathrm{d}W_k}{\mathrm{d}x} = 0 \qquad (k = 1, 2) \tag{40}$$

$$x = 1: \qquad W_k = \frac{\mathrm{d}^2 W_2}{\mathrm{d}x^2} = 0 \qquad (k = 1, 2) \tag{41}$$

as in the isotropic case. The location of y_s/a is given by t. The appearance of ε as a multiplicative factor for the highest derivative in the two ODEs of (38) indicates the singular perturbation nature of the BVP and the possibility of boundary layer phenomena for small aspect ratio plates.

The effects of orthotropy are captured by the two material parameters λ and μ (since the third parameter ν is essentially a Poisson's ratio effect) with $\lambda = 1$ and $\mu = 1 - \nu$ for the isotropic case. For $\nu = 0$, the exact solution of the BVP (38)–(41) is $t = d_1$ and $W_k = 0$, $k = 1, 2$, and we recover the exact solution of the original plate problem. For $\nu > 0$, the approximate values of y_s/a for a *linear* thickness profile have been obtained for different combinations of λ and μ; the results for $\nu = 1/3$ are given in Table 3. We found from this table and other cases computed (but not shown herein) that y_s/a remains close to y_s^0/a, still within 3% at maximum deviation.

For a fixed pair of ν and λ, y_s/a increases from 4/5 (its exact value for $\nu = 0$) with increasing μ, reaches a maximum and then decreases toward 0.8. This behavior is at variance with that predicted by

Table 3. y_s/a for orthotropic plates with $h = h_0 x_2/a$, $\nu = 1/3$ and $a/L = 0.1$

$\mu \rightarrow$ $\lambda \downarrow$	0.01	0.1	$1-\nu$	1.0	10.	100.
0.01	0.80349	0.81111	0.81305	0.81260	0.80681	0.80237
0.1	0.80145	0.80745	0.81357	0.81418	0.81316	0.80685
1.	0.80054	0.80369	0.81055	0.81200	0.81557	0.81329
10.	0.80018	0.80150	0.80613	0.80769	0.81510	0.81611
100.	0.80006	0.80150	0.80277	0.80375	0.81510	0.81651

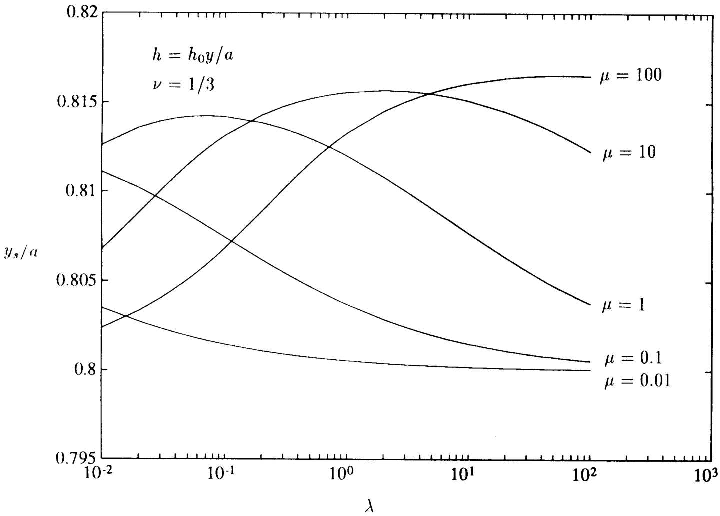

Fig. 3. Shear center location vs. the ratio of the two principal elastic moduli of an orthotropic plate strip with a thickness linearly varying in the spanwise coordinate

an application of the Duncan-Griffith-Taylor or the Goodier-Stephenson type criterion (suggested in [7] for the three-dimensional elasticity problem) which gives

$$\frac{y_s}{a} \simeq \frac{y_s^0}{a} + \frac{\mu}{\lambda} \frac{2\nu}{1-\nu^2}\left(\frac{y_s^0}{a} - \frac{y_c}{a}\right) \equiv \frac{y_s^c}{a}. \tag{42}$$

The expression for ay_s^c/a increases without bound with μ/λ. Note that (42) reduces to (31a) for an isotropic homogeneous plate.

For fixed ν and μ, y_s/a obtained from the generalized beam theory solution also exhibits a unimodel behavior as a function of λ tending to y_s^0/a for large and small λ. As μ increases, the peak value of y_s/a increases and is attained at a larger value of λ. These features can be seen from the results for the linear thickness profile case plotted in Figure 3 for $\nu = 1/3$. While the expression (42) tends to y_s^0/a as $\lambda \to \infty$, it becomes unbounded as $\lambda \to 0$ and therefore is not an appropriate approximate solution for y_s/a.

Acknowledgements

The research was supported by NSF Grant DMS-8904845. The authors are also indebted to Professor Eric Reissner for his many helpful comments on the manuscript as well as during the course of the research project.

References

1. Asher, U.; Christiansen, J.; Russell, R. D.: A collocation solver for mixed order systems of boundary value problems. Math. Comp. 33 (1979) 659–679
2. Duncan, W. T.: The flexural center or centre of shear. J. Roy. Aeron. Soc. 57 (1953) 594–597
3. Elias, Z. M.: Program PLATE, VAX and PC/AT versions. Civil Engineering, Univ. of Washington, Seattle, January, 1991
4. Goodier, J. N.: A theorem on the shearing stress in beams. J. Aeron. Sci. 11 (1944) 272–280
5. Griffith, A. A.; Taylor, G. I.: The problem of flexure. Reports and Memoranda Adv. Comm. Aeronautics 399 (9117) 1–21
6. Reissner, E.: The center of shear as a problem of the theory of plates of variable thickness. Ing. Arch. 59 (1989) 325–332
7. Reissner, E.: Approximate determinations of the center of shear by use of the St.-Venant solution for the flexure problem of plates of variable thickness. Arch. Appl. Mech. 61 (1991) 555–566
8. Stevensen, A. C.: Flexure with shear and associated torsion. Phil. Trans. Roy. Soc. London/A 237 (1939) 161–229 and 566

Received September 2, 1992

C. Gu
F. Y. M. Wan, Professor
Department of Applied Mathematics
University of Washington, FS-20
Seattle, WA 98195