106

Journal of Elasticity **36:** 155–182, 1994.

Boundary Conditions at the Edge of a Thin or Thick Plate Bonded to an Elastic Support

R.D. GREGORY
Department of Mathematics, University of Manchester, Oxford Rd., Manchester M13 9PL, England

and

F.Y.M. WAN
Department of Applied Mathematics, University of Washington, Seattle, WA 98195, USA

Received 14 January 1994

Abstract. At the clamped edge of a thin plate, the interior transverse deflection $w(x_1, x_2)$ of the mid-plane $x_3 = 0$ is required to satisfy the boundary conditions $w = \partial w/\partial n = 0$. But suppose that the plate is not held fixed at the edge but is supported by being bonded to another elastic body; what now are the boundary conditions which should be applied to the *interior* solution in the plate? For the case in which the plate and its support are in two-dimensional plane strain, we show that the correct boundary conditions for w must always have the form

$$w - \frac{4W^B}{3(1-\nu)}h^2\frac{d^2w}{dx_1^2} + \frac{4W^F}{3(1-\nu)}h^3\frac{d^3w}{dx_1^3} = 0,$$

$$\frac{dw}{dx_1} - \frac{4\Theta^B}{3(1-\nu)}h\frac{d^2w}{dx_1^2} + \frac{4\Theta^F}{3(1-\nu)}h^2\frac{d^3w}{dx_1^3} = 0,$$

with *exponentially* small error as $L/h \to \infty$, where $2h$ is the plate thickness and L is the length scale of w in the x_1-direction. The four coefficients W^B, W^F, Θ^B, Θ^F are computable constants which depend upon the geometry of the support and the elastic properties of the support and the plate, but are independent of the length of the plate and the loading applied to it. The leading terms in these boundary conditions as $L/h \to \infty$ (with all elastic moduli remaining *fixed*) are the same as those for a thin plate with a clamped edge. However by obtaining asymptotic formulae and general inequalities for Θ^B, W^F, we prove that these constants take large values when the support is 'soft' and so may still have a strong influence even when h/L is small. The coefficient W^F is also shown to become large as the size of the support becomes large but this effect is unlikely to be significant except for very thick plates. When h/L is small, the first order corrected boundary conditions are

$$w = 0,$$

$$\frac{dw}{dx_1} - \frac{4\Theta^B}{3(1-\nu)}h\frac{d^2w}{dx_1^2} = 0,$$

which correspond to a hinged edge with a restoring couple proportional to the angular deflection of the plate at the edge.

1. Introduction

The determination of the correct boundary conditions satisfied by the interior interest for a century and a half. In the case of a *thin* plate with a clamped edge, the classical conditions

$$w = \frac{\partial w}{\partial n} = 0, \tag{1}$$

(where w is the transverse displacement of the mid-plane and $\mathbf{n}$ is perpendicular to the edge) were first suggested by intuition. However, for a free edge, the same intuition suggests *three* boundary conditions on w whereas the governing equation of thin plate theory admits only *two*. The resolution of this difficulty was first deduced variationally by Kirchhoff [12] and confirmed asymptotically by Gol'denveizer [3] and Friedrichs and Dressler [2]. They showed that, in the limit as the plate thickness $2h$ tends to zero, there is a boundary layer of thickness $O(h)$ near the edge of the plate and that the classical thin plate solution is the leading term (as $h \to 0$) of the *interior solution**, which excludes the edge zone contribution. Thus the correct boundary conditions for thin plate theory are the conditions satsified, at the edge of the plate, by the leading term of this *interior* solution. This approach ([3], [2]) justifies the use of Kirchhoff's two 'reduced' boundary conditions at the traction free edge of a thin plate, and also the use of conditions (1) at an edge of a thin plate at which the (three-dimensional) displacement $\mathbf{u} = \mathbf{0}$.

For moderately thick plates, the interior solution may not be represented with sufficient accuracy by its leading term as $h \to 0$. In this case it is necessary to include one or more of the higher order terms in its asymptotic expansion, or perhaps even to use the full interior solution. Gregory and Wan [7]–[11] and Lin and Wan [13], [14] have developed a general method for determining the boundary conditions which should be applied to the full interior solution (or its approximations) for various kinds of imposed data at the edge of a plate or shell. In [7], [8], [9], examples are given in which the correct boundary conditions satisfied by the full interior solution in an isotropic plate can be found exactly; these results are extended to the orthotropic case in [13], [14]. In [11], a two term asymptotic theory is developed for the axisymmetric deformation of a circular cylindrical shell. A feature of such higher order theories is that the edge data must be known pointwise (and not just its stress and couple resultants given, for instance). This is because the higher order terms of the interior solution are sensitive to changes in the edge data which leave the leading term unaffected.

In the present paper we develop the theory for a model of a clamped edge, which is more general, and more physically realistic, than the imposition of the data $\mathbf{u} = \mathbf{0}$ over the edge. The condition $\mathbf{u} = \mathbf{0}$ corresponds to the edge of the plate being perfectly bonded to a fixed *rigid* surface with the same contour as the edge; in the particular case of a straight edge, it corresponds to the edge being butt-jointed to a flat rigid wall. However, this is not the only, nor even the most

* In the language of matched asymptotic expansions, this is the *outer asymptotic solution* (as $h \to 0$); the corresponding asymptotic solution valid in the neighborhood of the plate edge is called the *inner asymptotic solution*.

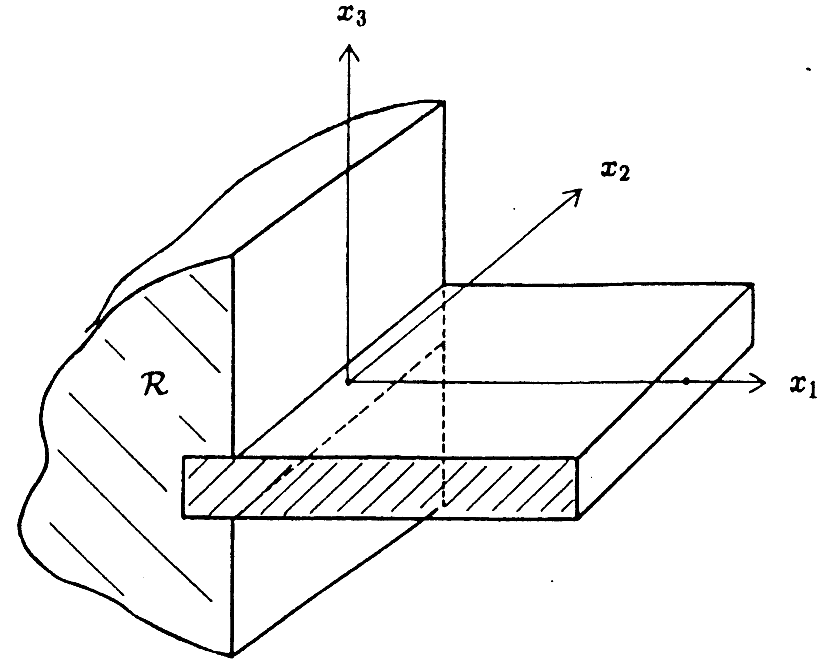

Fig. 1. The geometry of the plate and its support.

usual, method of clamping. Instead, the plate might have its edge built into an elastic wall (or other structure), or have its faces near the edge gripped by a vise; a typical configuration** for a plate‡ and its support (in two-dimensional plane strain) is shown in Figure 1. Quite apart from the fact that these alternative forms of support act on a band on the faces of the plate, as well as on the edge itself, they differ from the condition $\mathbf{u} = \mathbf{0}$ in that *the support is itself elastic*.

The question now arises as to what boundary conditions should be applied to the interior plate solution at such an edge. Even for thin plates, one would not expect the conditions (1) to continue to hold in general, since the material of the support could be relatively soft and thus allow the edge of the plate to translate and rotate. This question is investigated for the case in which the plate has a straight edge, and the plate and its support are in two-dimensional plane strain deformation; the plate is not restricted to be thin. By use of the elastic reciprocal theorem, we prove that,

** The elastically clamped edge that is the subject of the present paper should not be confused with the 'elastically built in edge' described by Timoshenko and Woinowski-Krieger [18], p. 86, which is essentially an edge strengthened by a rib that is otherwise free to move. Our support is elastic but is restrained by having part of its *own* boundary clamped.

‡ We restrict ourselves to the case in which the plate is homogeneous and isotropic; μ, ν are its shear modulus and Poisson's ratio respectively.

for *any* form of support, the correct boundary conditions satisfied by the mid-plane displacement $w(x_1)$ of the interior solution have the form

$$w - \frac{4W^B}{3(1-\nu)}h^2\frac{d^2w}{dx_1^2} + \frac{4W^F}{3(1-\nu)}h^3\frac{d^3w}{dx_1^3} = 0, \tag{2}$$

$$\frac{dw}{dx_1} - \frac{4\Theta^B}{3(1-\nu)}h\frac{d^2w}{dx_1^2} + \frac{4\Theta^F}{3(1-\nu)}h^2\frac{d^3w}{dx_1^3} = 0, \tag{3}$$

with *exponentially* small error as $L/h \to \infty$, where $2h$ is the plate thickness and L is the length scale of w in the x_1-direction. The four dimensionless coefficients W^B, W^F, Θ^B, Θ^F are computable constants which depend on the geometry of the support and the elastic properties of both the support and the plate, but are independent of the length of the plate and the loading applied to it.

One immediate, and interesting, consequence of (2) and (3) is that their leading terms as $L/h \to \infty$ (with all elastic moduli remaining fixed) are the same as (1). Thus, in this 'thin plate' limit, any form of support, however soft, will clamp the edge in the sense that (1) holds. However, for a soft support, Θ^B and W^F are large constants (see below) so that h/L must be taken correspondingly small for (1) to approximate (2), (3) accurately.

We show that the terms in (2), (3) involving W^F, Θ^B have an appealing physical interpretation. This is that the action of the support is (i) to oppose the rotation of the plate at $x_1 = 0$ as if it were a (coiled) spring of modulus $\mu h^2/\Theta^B$, and (ii) to oppose the transverse displacement of the plate at $x_1 = 0$ as if it were a (linear) spring of modulus μ/W^B. The terms in (2), (3) involving W^B, Θ^F have no such interpretation, but are further 'thickness effects' of order $O(h^2/L^2)$ as $L/h \to \infty$.

By extending energy arguments employed by Sternberg and Knowles [16], we prove a number of results concerning the changes in Θ^B, W^F which result from changes in the size or elastic properties of the support. Loosely expressed, these are:

(i) Θ^B and W^F increase as the size of the support increases. In particular, for supports lying entirely in $x_1 < 0$, the least possible values for Θ^B, W^F are those corresponding to the edge condition $\mathbf{u} = 0$ at $x_1 = 0$ (butt-jointing to a rigid wall).

(ii) Θ^B and W^F increase as the support becomes 'softer'. In particular, if the support is homogeneous and isotropic with shear modulus μ' and Poisson's ratio ν', then Θ^B and W^F increase as μ' decreases or as ν' decreases. Moreover, as $\mu'/\mu \to 0$, Θ^B and W^F increase like μ/μ'. [W^B and Θ^F probably also behave in the same way, but this is not proved in the general case.]

We further prove that if $R/h \to \infty$, where R is a representative 'radius' of the support, then Θ^B increases to a limit, but W^F behaves like $(\mu/\mu')\ln(R/h)$; thus W^F tends to infinity as $R/h \to \infty$, albeit rather slowly. For this reason, a semi-infinite support cannot necessarily be used to model a very large support.

Collectively, these results show that, when μ/μ' is moderate or large, the conditions (2), (3) may be accurately approximated by the classical thin plate conditions (1) provided that

$$\left(\frac{\mu}{\mu'}\right)\left(\frac{h}{L}\right) \ll 1 \tag{4}$$

and

$$\left(\frac{\mu}{\mu'}\right)\left(\frac{h}{L}\right)^3 \ln\left(\frac{R}{h}\right) \ll 1. \tag{5}$$

When μ/μ' is small, Θ^B and W^F approach their 'rigid support' limits so that we then require

$$\frac{h}{L} \ll 1 \tag{6}$$

and

$$\left(\frac{\mu}{\mu'}\right)\left(\frac{h}{L}\right)^3 \ln\left(\frac{R}{h}\right) \ll 1. \tag{7}$$

Although in principle the conditions (5), (7) are independent of (4), (6) respectively, in practice it would be highly unlikely for them not to hold when (4) or (6) do. When $h/L = 0.1$, say, this would require R/h to exceed e^{100}, an unrealistically large value!

When $(h/L)^2W^B, (h/L)^2\Theta^F \ll 1$, the terms in (2), (3) involving W^B, Θ^F are negligible and the support acts (as explained above) just as if it were a pair of springs exerting a linear restoring couple and transverse force on the edge of the plate. In this case, the W^F term (which gives rise to the restoring transverse force) is most unlikely to be as significant as the Θ^B term (which gives rise to the restoring couple); this can only happen when the support 'radius' is unreallistically large. If h/L is small enough so that the W^F term is negligible (by virtue of (5) or (7)) then the boundary conditions at the edge of the plate reduce to

$$w = 0, \tag{8}$$

$$\frac{dw}{dx_1} - \frac{4\Theta^B}{3(1-\nu)}h\frac{d^2w}{dx_1^2} = 0, \tag{9}$$

which corresponds to a hinged edge with a restoring couple proportional to the angular deflection of the plate at the edge. In all practical cases (8), (9) represent the most significant correction to the 'thin plate' conditions (1).

An interesting question remains which is not addressed in the present paper. Suppose that the plate is bonded to some large support whose material properties are similar to those of the plate so that μ/μ' is neither large nor small. What are the numerical values of Θ^B etc., and how small must h/L actually be for the support to be represented accurately by the limiting conditions (1)? This question will be answered in a later communication.

2. The Fundamental Bending and Flexure Problems and the Constants Θ^B, W^B; Θ^F, W^F

Consider the elastic system shown in Figure 1 which is in plane strain deformation* parallel to the (x_1, x_3)–plane. In the region $x_1 > 0$ we have the homogeneous, isotropic plate** $x_1 > 0$, $|x_3| \leq h$ whose faces $x_3 = \pm h$ are traction free; the plate has shear modulus μ and Poisson's ratio ν. In the region $x_1 < 0$, this plate is bonded to an elastic support $\mathcal{R}$ which is not required to be either homogeneous or isotropic. Part of the boundary of this support may be traction free, but some part of its boundary is clamped (that is, $\mathbf{u} = \mathbf{0}$ there) and it is this clamping which balances any loading applied to the plate. Suppose that the semi-infinite plate is loaded at $x_1 = +\infty$ by the couple $M = \mu h^2$ (per unit length in the x_2-direction). This loading generates a unique elastic field $\{\boldsymbol{\tau}^B(x_1, x_3), \mathbf{u}^B(x_1, x_3)\}$ in the plate and its support. In particular, $\mathbf{u}^B$ can be expanded in $x_1 > 0$ (see [7], Theorem 1) in the form

$$\begin{pmatrix} u_1^B \\ u_3^B \end{pmatrix} = \mu h^2 \mathbf{u}^{VB} + \Theta^B \begin{pmatrix} -x_3 \\ x_1 \end{pmatrix} + hW^B \begin{pmatrix} 0 \\ 1 \end{pmatrix} + hO(e^{-\gamma x_1/h}), \tag{10}$$

$$\boldsymbol{\tau}^B = \mu h^2 \boldsymbol{\tau}^{VB} + \mu O(e^{-\gamma x_1/h}) \tag{11}$$

as $x_1 \to \infty$, uniformly for $|x_3| \leq h$, where Θ^B, W^B are dimensionless constants. Note that

(i) $\{\boldsymbol{\tau}^{VB}, \mathbf{u}^{VB}\}$ is the Saint-Venant bending field for unit bending couple as given in Appendix 8.

(ii) γ is a positive constant whose value is approximately 3.75 if the support is symmetrical about the plane $x_3 = 0$; in general, $\gamma = 2.1$ approximately.

Similarly if the plate is subject to flexure* at $x_1 = +\infty$ by the transverse force $Q = \mu h$ (per unit length in the x_2-direction) then the resulting elastic field $\{\boldsymbol{\tau}^F(x_1, x_3), \mathbf{u}^F(x_1, x_3)\}$ can be expanded in $x_1 > 0$ in the form

$$\begin{pmatrix} u_1^F \\ u_3^F \end{pmatrix} = \mu h \mathbf{u}^{VF} + \Theta^F \begin{pmatrix} -x_3 \\ x_1 \end{pmatrix} + hW^F \begin{pmatrix} 0 \\ 1 \end{pmatrix} + hO(e^{-\gamma x_1/h}), \tag{12}$$

$$\boldsymbol{\tau}^F = \mu h \boldsymbol{\tau}^{VF} + \mu O(e^{-\gamma x_1/h}) \tag{13}$$

as $x_1 \to \infty$, uniformly for $|x_3| \leq h$, where Θ^F, W^F are dimensionless constants; here $\{\boldsymbol{\tau}^{VF}, \mathbf{u}^{VF}\}$ is the Saint-Venant flexure field for unit transverse force as given in Appendix 8.

* The linearized theory of elasticity is assumed throughout. Where necessary, we assume the strain energy density to be positive definite.

** For the purpose of defining the fundamental bending and flexure solutions, this plate is semi-infinite; however, in more general problems it will extend to $x_1 = L$ and be loaded by tractions acting over $x_1 = L$, $|x_3| \leq h$.

* In our definition of flexure, the bending moment at $x_1 = 0$ is zero. The same applies to the Saint-Venant flexure field.

The four dimensionless constants Θ^B, W^B; Θ^F, W^F depend upon the geometry and elastic properties of the support and upon the elastic properties of the plate. In (10), (12) the terms involving these constants represent additional rigid body deflections which will be different for different supports.

Although the constants Θ^F, W^B are defined independently, they are in fact simply related:

THEOREM 1. *For any form of support,*

$$\Theta^F - W^B = \frac{3}{20}(4 + \nu). \tag{14}$$

Proof. We apply the (two-dimensional) elastic reciprocal theorem to the elastic states $\{\boldsymbol{\tau}^B, \mathbf{u}^B\}$, $\{\boldsymbol{\tau}^F, \mathbf{u}^F\}$, that is

$$\int_{C_X} \left[u_i^B \tau_{ij}^F - u_i^F \tau_{ij}^B\right] n_j ds = 0. \tag{15}$$

In (15), C_X is the boundary of V_X, which is the region occupied by the support together with that part of the plate lying in $x_1 \leq X$; here X is an arbitrarily chosen positive constant. The contributions to (15) from the support boundary and from the free surfaces of the plate vanish so that (15) becomes*

$$\int_{-h}^{h} \left[u_1^B \tau_{11}^F + u_3^B \tau_{31}^F - u_1^F \tau_{11}^B - u_3^F \tau_{31}^B\right]_{x_1 = X} dx_3 = 0 \tag{16}$$

for all $X > 0$.

LEMMA 1. *If $\{\boldsymbol{\tau}^{(1)}, \mathbf{u}^{(1)}\}$, $\{\boldsymbol{\tau}^{(2)}, \mathbf{u}^{(2)}\}$ are any plane strain elastic fields defined in $A < x_1 < B$, $|x_3| \leq h$ and satisfying traction free conditions on $A < x_1 < B$, $|x_3| = h$, then the 'reciprocal product'*

$$\int_{-h}^{h} \left[u_1^{(1)} \tau_{11}^{(2)} + u_3^{(1)} \tau_{31}^{(2)} - u_1^{(2)} \tau_{11}^{(1)} - u_3^{(2)} \tau_{31}^{(1)}\right]_{x_1 = X} dx_2 \tag{17}$$

is independent of X for $A < X < B$.

Proof. This follows immediately by applying the reciprocal theorem to $\{\boldsymbol{\tau}^{(1)}, \mathbf{u}^{(1)}\}$, $\{\boldsymbol{\tau}^{(2)}, \mathbf{u}^{(2)}\}$ around the perimeter of the rectangle $A_1 \leq x_1 \leq B_1$, $|x_3| \leq h$, $(A < A_1 < B_1 < B)$.

Now substitute the expansions (10)–(13) into (16). The exponentially decaying parts of $\{\boldsymbol{\tau}^B, \mathbf{u}^B\}$, $\{\boldsymbol{\tau}^F, \mathbf{u}^F\}$, and also the non-decaying parts, are elastic fields defined in $x_1 > 0$, $|x_3| \leq h$ which satisfy traction free conditions on $x_1 > 0$, $|x_3| = h$; thus Lemma 1 applies to their *separate* reciprocal products. For any such reciprocal product involving an exponentially decaying field, let $X \to \infty$;

* The elastic fields may have certain points of singularity (at $(0, \pm h)$ for example). In such cases the contour C_X must be indented into a small circular arc around each singular point (see [7], Figure 1). We assume that the contributions to (15) from such arcs vanish as their radii tend

clearly the value of the product tends to zero, and hence by Lemma 1 *is* zero. The one remaining reciprocal product, involving the non-decaying parts of $\{\tau^B, \mathbf{u}^B\}$, $\{\tau^F, \mathbf{u}^F\}$ is most easily evaluated by setting $X = 0$; this is permissible since these non-decaying parts are defined (and satisfy traction free conditions) in the *infinite* plate $|x_3| \le h$. The integrations are elementary and lead immediately to (14).

THEOREM 2. The constants $\Theta^B, W^B; \Theta^F, W^F$ can be expressed as integrals along the arc $x_1 = 0, |x_3| \le h$ as follows:

$$\Theta^B = -\frac{3}{8h^3}\int_{-h}^{h}\{4x_3u_1^B(0,x_3) + \nu x_3^2\mu^{-1}\tau_{13}^B(0,x_3)\}dx_3, \tag{18}$$

$$W^B = -\frac{3}{8h^3}\int_{-h}^{h}\{4x_3u_1^F(0,x_3) + \nu x_3^2\mu^{-1}\tau_{13}^F(,x_3)\}dx_3, \tag{19}$$

$$\Theta^F = \frac{1}{8h^4}\int_{-h}^{h}\{6(h^2-x_3^2)u_3^B(0,x_3)+(2-\nu)x_3^3\mu^{-1}\tau_{11}^B(0,x_3)\}dx_3+\frac{3}{4}, \tag{20}$$

$$W^F = \frac{1}{8h^4}\int_{-h}^{h}\{6(h^2-x_3^2)u_3^F(0,x_3) + (2-\nu)x_3^3\mu^{-1}\tau_{11}^F(0,x_3)\}dx_3. \tag{21}$$

Proof. These formulae are also proved by using reciprocity. For example, by applying Lemma 1 to the elastic states $\{\tau^B, \mathbf{u}^B\}$, $\{\tau^{VB}, \mathbf{u}^{VB}\}$ we obtain that

$$J(X) \equiv \int_{-h}^{h}\left[u_1^B\tau_{11}^{VB} + u_3^B\tau_{31}^{VB} - u_1^{VB}\tau_{11}^B - u_3^{VB}\tau_{31}^B\right]_{x_1=X} dx_3 \tag{22}$$

is independent of $X (X > 0)$. If we replace $\{\tau^B, \mathbf{u}^B\}$ by the expansions (10), (11) and proceed as in the proof of Theorem 1, we find that

$$J = \mu h^2\Theta^B. \tag{23}$$

However we may instead let $X \to 0+$ in (22) and replace $\{\tau^{VB}, \mathbf{u}^{VB}\}$ by the explicit expressions in Appendix 8. This yields (18), and the other formulae are obtained in a similar manner.

Note: For the particular case in which the plate is bonded to a rigid wall at $x_1 = 0$, the constants $\Theta^B \ldots W^F$ have already been determined and we shall denote their values by $\Theta_0^B \ldots W_0^F$. In this case, the displacements appearing in (18)–(21) are zero and the tractions at $x_1 = 0$ have been calculated numerically by Gregory and Gladwell [6]. The necessary weighted integrals of these tractions were calculated in [6] and are tabulated there and also by Gregory and Wan [7], p. 44. In terms of the notation* used in [7],

$$\Theta_0^B = -\frac{3\nu}{8}t_2^B,\quad W_0^B = -\frac{3\nu}{8}t_2^F, \tag{24}$$

* In [7], the symbols $\Theta^{BB} \ldots W^{BF}$ are used to denote $\mu^{-1}\Theta_0^B \ldots \mu^{-1}W_0^F$, but this notation is less convenient in the present paper.

$$\Theta_0^F = \frac{2-\nu}{8}n_3^B + \frac{3}{4},\quad W_0^F = \frac{2-\nu}{8}n_3^F, \tag{25}$$

where the dimensionless constants $t_j^X(\nu), n_j^X(\nu)$ are defined in [7] and are tabulated there for $\nu = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.

3. Boundary Conditions Satisfied by the Interior Plate Solution at $x_1 = 0$

Let the plate and its support be as shown in Figure 1, but suppose now that the plate terminates at $x_1 = L\ (L > 0)$ and that tractions* act on the edge $x_1 = L, |x_3| \le h$. [These tractions may be prescribed, or may arise indirectly from e.g. prescribed displacements on $x_1 = L, |x_3| \le h$.] Let the resulting elastic field in the plate and its support be denoted by $\{\tau, \mathbf{u}\}$. Then in the region $0 < x_1 < L, |x_3| \le h$, $\{\tau, \mathbf{u}\}$ can be decomposed in the form (see Gregory [4], Gregory and Wan [7])

$$\{\tau, \mathbf{u}\} = \{\tau^I, \mathbf{u}^I\} + \{\tau^{PF+}, \mathbf{u}^{PF+}\} + \{\tau^{PF-}, \mathbf{u}^{PF-}\}, \tag{26}$$

where $\{\tau^I, \mathbf{u}^I\}$ is the *interior field* derived from a transverse mid-plane displacement w (see Appendix 9; $w = w(x_1)$ in our case) and the $PF\pm$ fields are exponentially decaying with exponent γ as $x_1/h \to \pm\infty$. [We note that while the decomposition (26) is known to hold only for $0 < x_1 < L$, the three terms on the right side have analytic continuations into the regions $-\infty < x_1 < \infty$, $x_1 > 0$ and $x_1 < L$ respectively, and satisfy traction free conditions on $|x_3| = h$ in these regions.]

THEOREM 3. *The mid-plane transverse displacement $w(x_1; L)$ of the interior field $\{\tau^I, \mathbf{u}^I\}$ satisfies the following boundary conditions at $x_1 = 0$:*

$$\left[\frac{dw}{dx_1} - \frac{4\Theta^B}{3(1-\nu)}h\frac{d^2w}{dx_1^2} + \frac{4\Theta^F}{3(1-\nu)}h^2\frac{d^3w}{dx_1^3}\right]_{x_1=0} = O(e^{-\gamma L/h}), \tag{27}$$

$$\left[w - \frac{4W^B}{3(1-\nu)}h^2\frac{d^2w}{dx_1^2} + \frac{4W^F}{3(1-\nu)}h^3\frac{d^3w}{dx_1^3}\right]_{x_1=0} = hO(e^{-\gamma L/h}), \tag{28}$$

as $L/h \to \infty$.

Proof. On applying the reciprocal theorem to $\{\tau, \mathbf{u}\}$ and $\{\tau^B, \mathbf{u}^B\}$ as in Theorem 1, we obtain

$$\int_{-h}^{h}\left[u_1\tau_{11}^B + u_3\tau_{31}^B - u_1^B\tau_{11} - u_3^B\tau_{31}\right]_{x_1=X} = 0 \tag{29}$$

for $0 < X < L$. In (29) we may replace $\{\tau, \mathbf{u}\}$ by the decomposition (26) and $\{\tau^B, \mathbf{u}^B\}$ by (10), (11); Lemma 1 can then be applied to each of the various

* These tractions are supposed to be consistent with a plane strain deformation. We shall also suppose that they have zero resultant in the x_1-direction; however, it is not necessary to restrict the tractions to have anti-symmetry about $x_3 = 0$.

reciprocal products which now appear. The reciprocal product of $\{\tau^I, \mathbf{u}^I\}$ with the nondecaying parts of $\{\tau^B, \mathbf{u}^B\}$ is most easily evaluated by setting $X = 0$ (this is permissible) and yields

$$\frac{1}{2}\mu h^2\left[(1-\nu)\frac{dw}{dx_1} - \frac{4\Theta^B}{3}h\frac{d^2w}{dx_1^2} + \frac{1}{5}(4+\nu+\frac{20}{3}W^B)h^2\frac{d^3w}{dx_1^3}\right]_{x_1=0}. \quad (30)$$

All but one of the other reciprocal products vanish by letting $X \to +\infty$ or $-\infty$. The one which survives is the reciprocal product of $\{\tau^{PF-}, \mathbf{u}^{PF-}\}$ with the exponentially decaying part of $\{\tau^B, \mathbf{u}^B\}$. If the loading at $x_1 = L$ is such that $\{\tau^{PF-}, \mathbf{u}^{PF-}\}|_{x_1=L-}$ exists and*

$$\int_{-h}^{h}\left\{|\tau^{PF-}|, |\mathbf{u}^{PF-}|\right\}|_{x_1=L-}dx_3 < Ah\{\mu, h\}, \quad (31)$$

where A is a constant *independent of* L, then this last reciprocal product is of order $\mu h^2 O(\exp^{-\gamma L/h})$ as $L/h \to \infty$; also A may be absorbed into the $O(\cdot)$. On using this result (and (30)) in (29), and using Theorem 1 to replace W^B by Θ^F, we obtain the boundary condition (27). The condition (28) is obtained in a similar manner by using $\{\tau^F, \mathbf{u}^F\}$ instead of $\{\tau^B, \mathbf{u}^B\}$.

From now on, we shall assume that L/h is large enough so that the right sides of (27), (28) are negligible. Since $\gamma \doteq 2.1$ (at least), we expect this to be true when $L/h > 4$, say. This corresponds to a plate whose lateral dimension is at least twice its thickness; thus the plate is certainly *not* restricted to be *thin*, in the engineering sense. In this approximation, **the boundary conditions satisfied by w at $x_1 = 0$ become**

$$\frac{dw}{dx_1} - \frac{4\Theta^B}{3(1-\nu)}h\frac{d^2w}{dx_1^2} + \frac{4\Theta^F}{3(1-\nu)}h^2\frac{d^3w}{dx_1^3} = 0, \quad (32)$$

$$w - \frac{4W^B}{3(1-\nu)}h^2\frac{d^2w}{dx_1^2} + \frac{4W^F}{3(1-\nu)}h^3\frac{d^3w}{dx_1^3} = 0. \quad (33)$$

The thin plate limit

If the lateral length scale (for the variation of $w(x_1; L)$ with x_1) is L, in the sense that

$$h\frac{d^{n+1}w}{dx_1^{n+1}} = (L/h)^{-1}O\left\{\max_{x_1\in[0,L]}\left|\frac{d^n w}{dx_1^n}\right|\right\} \quad (34)$$

* Although it is possible to find (L dependent) loadings for which (31) is false, most practical loadings will satisfy this condition. For instance, if tractions, independent of L, are prescribed on $x_1 = L$, we expect $\{\tau^{PF-}, \mathbf{u}^{PF-}\}|_{x_1=L-}$ to tend to a limit, independent of L, as $L/h \to \infty$.

as $L/h \to \infty$, uniformly for $x_1 \in [0, L]$ and for $n = 0, 1, 2$, then (32), (33) have the limiting forms

$$\left.\frac{dw}{dx_1}\right|_{x_1=0} = \frac{h}{L}O\left\{\max_{x_1\in[0,L]}\left|\frac{dw}{dx_1}\right|\right\}, \quad (35)$$

$$w\Big|_{x_1=0} = \frac{h}{L}O\left\{\max_{x_1\in[0,L]}|w|\right\}, \quad (36)$$

as $L/h \to \infty$. [Note that we are supposing that $\Theta^B \ldots W^F$ remain constant in this limit so that the elastic constants must not change and the only geometrical parameter which is changing is L.] Thus, in this thin plate limit, the action of *any* support is to 'clamp' the plate at $x_1 = 0$. It seems at first surprising that if, say, a steel plate is bonded to a support made of jello, then the jello will clamp the steel plate if L/h is large enough; but this is so. However we will later show (Theorems 7,8) that, in such a case, the constants Θ^B, W^F would have extremely large values, and so L/h would have to be taken correspondingly large before (32), (33) are accurately approximated by (35), (36) .

'Thickness corrections' and 'spring effects'

Although it is legitimate to regard all the terms in (32), (33) involving $\Theta^B \ldots W^F$ as 'thickness corrections' which vanish in the limit as $L/h \to \infty$, there is a more physically appealing interpretation. Since the resultant transverse shear force Q and couple $M(x_1)$ (per unit length in the x_2-direction) are given* by

$$\begin{aligned} Q &= \int_{-h}^{h}\tau_{13}(x_1, x_3)dx_3 \\ &= \int_{-h}^{h}\tau_{13}^I(x_1, x_3)dx_3 = -\frac{4\mu h^3}{3(1-\nu)}\frac{d^3w}{dx_1^3}, \end{aligned} \quad (37)$$

and

$$\begin{aligned} M &= -\int_{-h}^{h}x_3\tau_{11}(x_1, x_3)dx_3 \\ &= -\int_{-h}^{h}x_3\tau_{11}^I(x_1, x_3)dx_3 = \frac{4\mu h^3}{3(1-\nu)}\frac{d^2w}{dx_1^2}, \end{aligned} \quad (38)$$

it follows that (32), (33) may be rewritten in the form

$$\left[\frac{dw}{dx_1} + \frac{4\Theta^F}{3(1-\nu)}h^2\frac{d^3w}{dx_1^3}\right]_{x_1=0} = \left(\frac{\Theta^B}{\mu h^2}\right)M(0), \quad (39)$$

* See Appendix B

$$\left[w - \frac{4W^B}{3(1-\nu)}h^2\frac{d^2w}{dx_1^2}\right]_{x_1=0} = \left(\frac{W^F}{\mu}\right)Q. \tag{40}$$

If, further, $(h/L)^2\Theta^F$ and $(h/L)^2W^B$ are negligible**, then (39), (40) reduce to the form

$$\left.\frac{dw}{dx_1}\right|_{x_1=0} = \frac{M(0)}{\lambda_M}, \tag{41}$$

$$w\left.\right|_{x_1=0} = \frac{Q}{\lambda_Q}, \tag{42}$$

where λ_M, λ_Q are constants independent of the loading. Equations (41), (42) have the simple interpretation that the action of the elastic support is (i) to oppose the rotation of the plate at $x_1 = 0$, as if the support were a (coiled) spring with modulus $\lambda_M(= \mu h^2/\Theta^B)$, and (ii) to oppose the x_3-displacement of the plate at $x_1 = 0$, as if the support were a (linear) spring with modulus $\lambda_Q(= \mu/W^F)$. Thus in (32), (33) the terms involving Θ^B, W^F have the role of representing the effects of these two 'springs'; such an action of the support might have been anticipated on physical grounds. The terms involving Θ^F, W^B may be regarded as additional 'thickness corrections' order $O(h^2/L^2)$. We note that unless the value of the constant W^F is larger than the values of Θ^F, W^B by a factor L/h, then its effect will be *less* significant than that of the Θ^F and W^B terms which has already been supposed negligible. In this case equation (42) reduces to

$$w = 0 \tag{43}$$

so that the linear spring effect is negligible and any x_3–displacement of the plate at $x_1 = 0$ is prohibited by the support.

4. Dependence of Θ^B, W^F on the Shape of the Support

We recall that Θ^B, W^F are defined via the basic bending and flexure problems for a *semi-infinite* plate. Figure 2 shows a semi-infinite plate with two different supports:

(1) the support $ABCDEF$, of which the arcs AB, EF are traction free while the arc $BCDE$ is clamped (that is, $\mathbf{u} = 0$ there).

(2) the support $ABCGHDEF$, of which the arcs $ABCG$, $HDEF$ are traction free while the arc GH is clamped.

We note that support (2) has been obtained from support (1) by 'releasing' the arc $BCDE$, adding the material in the region V_+, and then clamping the arc GH.

** This does not imply that $(h/L)^3W^F$ is negligible, necessarily, since the value of W^F may be large compared to the the other constants when the dimensions of the support are very large compared to the plate thickness; see Section 6.

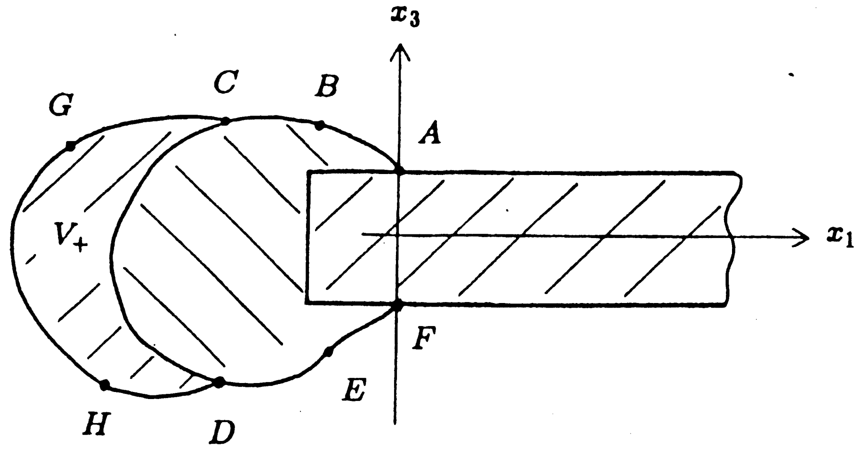

Fig. 2. The semi-infinite plate with two different supports

THEOREM 4. *Let the constants Θ^B, W^F with the supports* (1), (2) *above be denoted by Θ_1^B, W_1^F and Θ_2^B, W_2^F respectively. Then*

$$\Theta_2^B > \Theta_1^B, \tag{44}$$

$$W_2^F > W_1^F. \tag{45}$$

Proof. Let the basic *bending* solutions (with $M = \mu h^2$) for the supports (1), (2) be denoted by $\{\tau^{(1)}, \mathbf{u}^{(1)}\}$, $\{\tau^{(2)}, \mathbf{u}^{(2)}\}$ respectively. Let X be a positive constant and let V_X denote the region of the support (1) together with that part of the plate lying in $x_1 \le X$. Now apply reciprocity to $\{\tau^{(2)}, \mathbf{u}^{(2)}\}$, $\{\tau^{(2)} - \tau^{(1)}, \mathbf{u}^{(2)} - \mathbf{u}^{(1)}\}$ around ∂V_X, the boundary of V_X. Then

$$\int_{\partial V_X}(u_i^{(2)} - u_i^{(1)})\tau_{ij}^{(2)}n_j ds = \int_{\partial V_X} u_i^{(2)}(\tau_{ij}^{(2)} - \tau_{ij}^{(1)})n_j ds$$

$$= \int_{\partial V_X}(u_i^{(2)} - u_i^{(1)})(\tau_{ij}^{(2)} - \tau_{ij}^{(1)})n_j ds + \int_{\partial V_X} u_i^{(1)}(\tau_{ij}^{(2)} - \tau_{ij}^{(1)})n_j ds. \tag{46}$$

On taking account of the boundary conditions, (46) reduces to

$$\int_{CD} u_i^{(2)}\tau_{ij}^{(2)}n_j ds + \int_{-h}^{h}\left[(u_1^{(2)} - u_1^{(1)})\tau_{11}^{(2)} + (u_3^{(2)} - u_3^{(1)})\tau_{31}^{(2)}\right]_{x_1=X} dx_3$$

$$= \int_{\partial V_X}(u_i^{(2)} - u_i^{(1)})(\tau_{ij}^{(2)} - \tau_{ij}^{(1)})n_j ds$$

$$+ \int_{-h}^{h}\left[u_1^{(1)}(\tau_{11}^{(2)} - \tau_{11}^{(1)}) + u_3^{(1)}(\tau_{31}^{(2)} - \tau_{31}^{(1)})\right]_{x_1=X} dx_3. \tag{47}$$

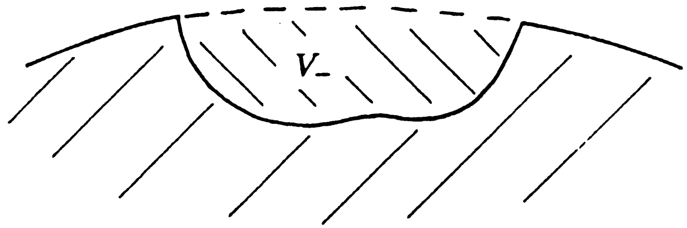

Fig. 3. The region V_- is removed from the free boundary (solid line) of support (1) to create the new free boundary of support (2).

Since $\{\tau^{(1)}, \mathbf{u}^{(1)}\}$, $\{\tau^{(2)}, \mathbf{u}^{(2)}\}$ each have expansions of the form (10), (11) for $x_1 > 0$, it follows on substitution that as $X \to \infty$

$$\int_{-h}^{h} \left[(u_1^{(2)} - u_1^{(1)})\tau_{11}^{(2)} + (u_3^{(2)} - u_3^{(1)})\tau_{31}^{(2)}\right]_{x_1=X} dx_3 \to \mu h^2(\Theta_2^B - \Theta_1^B) \quad (48)$$

and

$$\int_{-h}^{h} \left[u_1^{(1)}(\tau_{11}^{(2)} - \tau_{11}^{(1)}) + u_3^{(1)}(\tau_{31}^{(2)} - \tau_{31}^{(1)})\right]_{x_1=X} dx_3 \to 0. \quad (49)$$

The remaining two terms in (47) can be written as energies. Let $\mathcal{U}_{\mathcal{R}}(\tau)$ denote the total strain energy of the field τ in a region $\mathcal{R}$. Then letting $X \to \infty$ in (47) gives

$$\mu h^2(\Theta_2^B - \Theta_1^B) = 2\mathcal{U}_V(\tau^{(2)} - \tau^{(1)}) + 2\mathcal{U}_{V+}(\tau^{(2)}), \quad (50)$$

where V means $V_X|_{X=\infty}$. [Note that the unit vector $\mathbf{n}$ in (47) is outward to V_X and so inward to V_+; thus the first integral in (47) is $-2\mathcal{U}_{V_+}(\tau^{(2)})$.] We assume that the materials of the plate and support have positive definite strain energy densities so that these energies (and μ) are positive quantities. Then (50) implies that

$$\Theta_2^B > \Theta_1^B. \quad (51)$$

The inequality (45) is proved in a similar manner by using the two flexure fields instead of bending fields.

Additional results

(a) Suppose that, in addition to the changes made in support (1) (shown in Figure 2), a region V_- of material is removed; this region may be either entirely internal to support (1), or include part of its *free* boundary. Then a simple extension of the previous argument shows that

$$\mu h^2(\Theta_2^B - \Theta_1^B) = 2\mathcal{U}_V(\tau^{(2)} - \tau^{(1)}) + 2\mathcal{U}_{V_+}(\tau^{(2)}) + 2\mathcal{U}_{V_-}(\tau^{(1)}), \quad (52)$$

with a similar expression for $\mu h(W_2^F - W_1^F)$; thus (44), (45) still hold.

(b) The inequalities (44), (45) also continue to hold if part of the clamped boundary of support (1) is replaced by a rigid-lubricated boundary in support (2); also when a rigid-lubricated boundary in support (1) becomes traction free.

COROLLARY *For any form of support which lies entirely in $x_1 < 0$, the constants Θ^B, W^F must satisfy*

$$\Theta^B > \Theta_0^B, \quad (53)$$

$$W^F > W_0^F, \quad (54)$$

where Θ_0^B, W_0^F are the constants for the case in which the plate is bonded to the rigid wall $x_1 = 0$; see the note after Theorem 2.

In this sense, the rigid wall is the 'stiffest' of all supports.

Examples

(i) Consider the three supports S_1, S_2, S shown in Figure 4. All are traction free on the parts of their boundaries which lie on $x_1 = 0$, $|x_3| > h$, and are clamped on the parts of their boundaries which lie in $x_1 < 0$. Then Theorem 4 shows that

$$\Theta_1^B < \Theta^B < \Theta_2^B, \quad (55)$$

$$W_1^F < W^F < W_2^F. \quad (56)$$

Thus the Θ^B and W^F values for an irregular support can be bounded by the corresponding values for geometrically simpler supports.

(ii) Figure 5 shows four different modes of support. Theorem 4 (and its extensions) apply so that Θ^B, W^F both increase from (a) to (b) to (c) to (d). A particular case of this result, in which the clamped surfaces extend to $x_1 = -\infty$, has been obtained by Austin [1].

5. Dependence of Θ^B, W^F on the Elastic Moduli of the Support

Consider now a support $\mathcal{R}$ whose elastic moduli may be smoothly varied by changing the value of a real parameter α; the shape of the support and its boundary conditions do not change. Let the basic bending solution (for the semi-infinite plate) be denoted by $\{\tau^B(\mathbf{r}, \alpha), \mathbf{u}^B(\mathbf{r}, \alpha)\}$; in particular, the constants Θ^B, W^F are smooth functions of α. The corresponding basic flexure solution carries the suffix F.

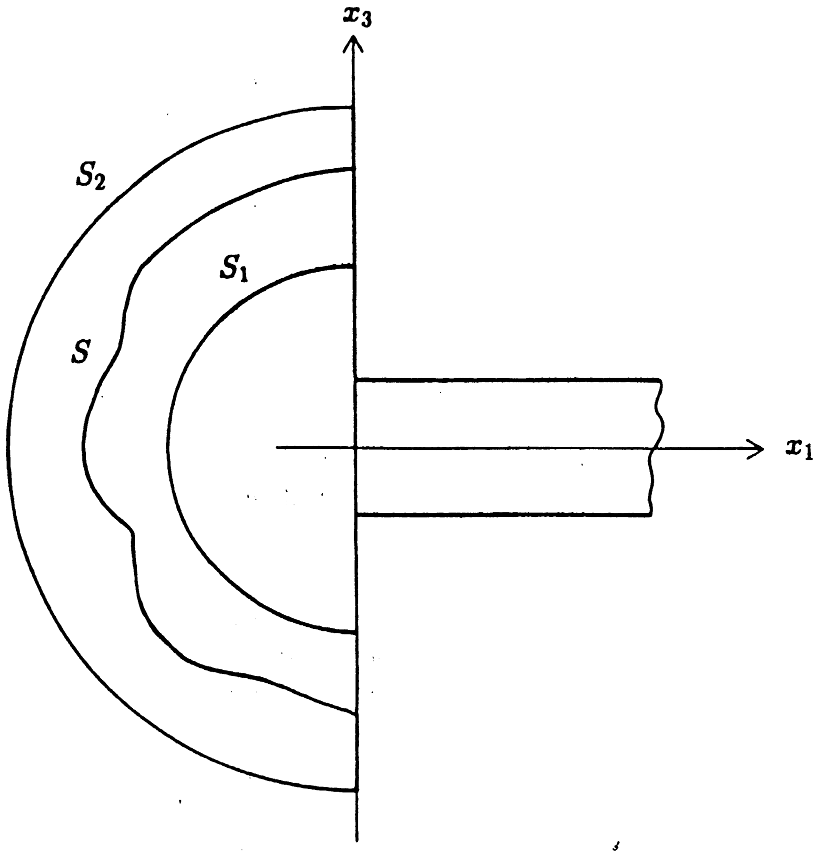

Fig. 4. The semi-circular supports S_1, S_2 and the irregular support S.

THEOREM 5.

$$\mu h^2 \frac{d}{d\alpha}(\Theta^B) = \int_{\mathcal{R}} \left(\frac{\partial}{\partial\alpha} c_{ijkl}\right) \tau_{ij}^B \tau_{kl}^B dx_1\, dx_3, \tag{57}$$

$$\mu h^2 \frac{d}{d\alpha}(W^F) = \int_{\mathcal{R}} \left(\frac{\partial}{\partial\alpha} c_{ijkl}\right) \tau_{ij}^F \tau_{kl}^F dx_1\, dx_3, \tag{58}$$

where $c_{ijkl}(\mathrm{r}, \alpha)$ are the elastic compliances of the support (that is $e_{ij} = c_{ijkl}\tau_{kl}$) and $\mathcal{R}$ is the region occupied by the support.

Moreover, if $\partial \mathrm{c}/\partial\alpha$ is positive definite, then Θ^B, W^F are increasing functions of α.

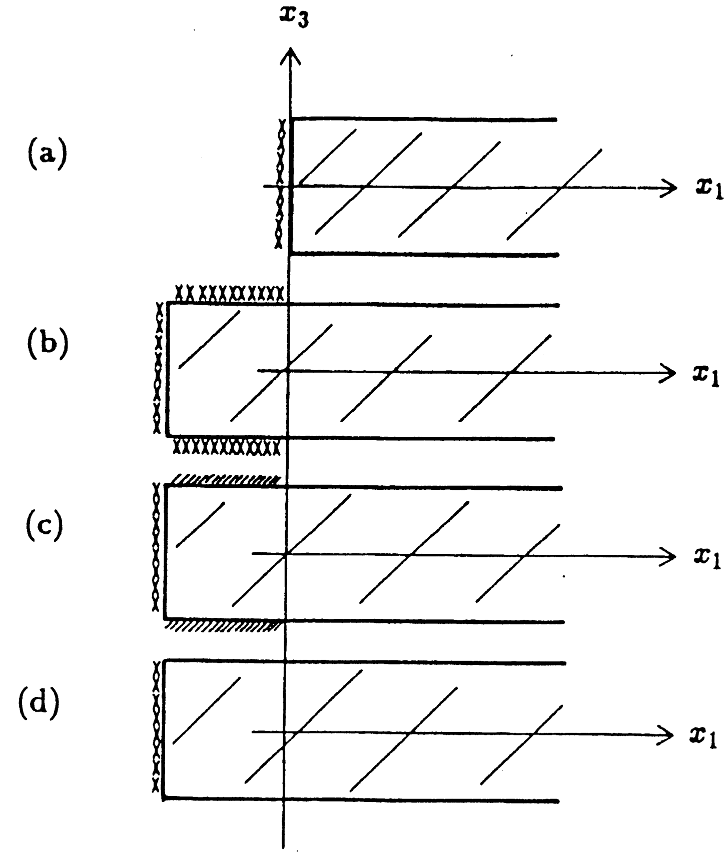

Fig. 5. Four different 'supports'. $XXXX$ means clamping, while //// means a rigid lubricated restraint.

Proof. Let the region V_X be the same as in the proof of Theorem 4 (see Figure 2). Then, by the divergence theorem,

$$\begin{aligned}
&\int_{\partial V_X} \left\{ \frac{\partial u_i^B}{\partial\alpha} \tau_{ij}^B - u_i^B \frac{\partial \tau_{ij}^B}{\partial\alpha} \right\} n_j ds \\
&= \int_{V_X} \frac{\partial}{\partial x_j} \left\{ \frac{\partial u_i^B}{\partial\alpha} \tau_{ij}^B - u_i^B \frac{\partial \tau_{ij}^B}{\partial\alpha} \right\} dx_1\, dx_3 \qquad (59) \\
&= \int_{V_X} \left\{ \left(\frac{\partial}{\partial\alpha} \frac{\partial u_i^B}{\partial x_j} \right) \tau_{ij}^B + \frac{\partial u_i^B}{\partial\alpha} \frac{\partial \tau_{ij}^B}{\partial x_j} - \frac{\partial u_i^B}{\partial x_j} \frac{\partial \tau_{ij}^B}{\partial\alpha} - u_i^B \left(\frac{\partial}{\partial\alpha} \frac{\partial \tau_{ij}^B}{\partial x_j} \right) \right\} dx_1\, dx_3
\end{aligned}$$

$$= \int_{V_X} \left\{ \frac{\partial e_{ij}^B}{\partial \alpha} \tau_{ij}^B - e_{ij}^B \frac{\partial \tau_{ij}^B}{\partial \alpha} \right\} dx_1\, dx_3, \tag{60}$$

where $e_{ij}^B(\mathbf{r},\alpha)$ is the strain field of $\mathbf{u}_i^B(\mathbf{r},\alpha)$. This last step follows from the symmetry of τ_{ij}^B and its equilibrium equations. If we now replace e_{ij}^B by using the strain-stress relation

$$e_{ij} = c_{ijkl}\tau_{kl}, \tag{61}$$

where $c_{ijkl}(\mathbf{r},\alpha)$ are the elastic compliances, (60) becomes

$$\int_{V_X} \left(\frac{\partial}{\partial \alpha} c_{ijkl} \right) \tau_{ij}^B \tau_{kl}^B dx_1\, dx_3. \tag{62}$$

However, on making use of the boundary conditions, the left side of (59) becomes

$$\int_{-h}^{h} \left[\frac{\partial u_1^B}{\partial \alpha}\tau_{11}^B + \frac{\partial u_3^B}{\partial \alpha}\tau_{31}^B - u_1^B \frac{\partial \tau_{11}^B}{\partial \alpha} - u_3^B \frac{\partial \tau_{31}^B}{\partial \alpha} \right]_{x_1 = X} dx_3 \to \mu h^2 \frac{d}{d\alpha}(\Theta^B) \tag{63}$$

as $X \to \infty$, on using (10), (11). Equating (62), (63) yields (57), and (58) is obtained in a similar manner by using $\{\tau^F, \mathbf{u}^F\}$ instead of $\{\tau^B, \mathbf{u}^B\}$.

If $\frac{\partial}{\partial \alpha}\mathbf{c}$ is positive definite, then the integrals in (57), (58) are strictly positive, which implies that Θ^B, W^F increase with α.

Remark. Theorem 5 remains true if the compliances of the support vary with α only in some sub-region of $\mathcal{R}$, and remain constant elsewhere. In this case, this sub-region may replace $\mathcal{R}$ as the region of integration in (57), (58).

THEOREM 6. *Let the material of the support $\mathcal{R}$ be homogeneous and isotropic with shear modulus μ' and Poisson's ratio ν'. Then*

(i) *Θ^B, W^F increase as μ' decreases with ν' held constant (and the elastic constants elsewhere remain constant).*

(ii) *Θ^B, W^F also increase (or remain constant) as ν' decreases with μ' held constant.*

Proof. (i) Let $\mu' = \mu e^{-\alpha}$, where μ is the (constant) shear modulus of the plate and α a variable parameter. Then in the support $\mathcal{R}$

$$\begin{aligned} c_{ijkl} &= \frac{1}{4\mu'} \left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2\nu'}{1+\nu'}\delta_{ij}\delta_{kl} \right] \\ &= \frac{e^{\alpha}}{4\mu} \left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2\nu'}{1+\nu'}\delta_{ij}\delta_{kl} \right] \end{aligned} \tag{64}$$

so that

$$\frac{\partial}{\partial \alpha} c_{ijkl} = c_{ijkl}. \tag{65}$$

Thus $\partial c_{ijkl}/\partial \alpha$ is positive definite in $\mathcal{R}$. Theorem 5 then implies that Θ^B, W^F increase as α increases (that is, as μ' decreases).

(ii) Now let μ' be held constant and $\nu' = \alpha$. Then, in $\mathcal{R}$,

$$\frac{\partial}{\partial \alpha} c_{ijkl} = -\frac{1}{2\mu'(1+\alpha)^2}\delta_{ij}\delta_{kl}. \tag{66}$$

Theorem 5 then implies that

$$\begin{aligned} \mu h^2 \frac{d}{d\alpha}(\Theta^B) &= -\frac{1}{2\mu'(1+\alpha)^2} \int_F \tau_{ii}^B \tau_{kk}^B dx_1 dx_3 \\ &\leq 0, \end{aligned} \tag{67}$$

and the same result holds for W^F. Thus Θ^B, W^F increase (or remain constant) as α (that is, ν') decreases.

THEOREM 7. *Let the material of the support $\mathcal{R}$ be homogeneous and isotropic with elastic constants μ', ν'. Then there exist positive functions $C(\nu')$, $D(\nu')$ such that*

$$\Theta^B - \Theta_0^B > C(\nu')\frac{\mu}{\mu'}, \tag{68}$$

$$W^F - W_0^F > D(\nu')\frac{\mu}{\mu'}. \tag{69}$$

[Recall that μ, ν are the elastic constants of the plate and Θ_0^B, W_0^F are the values of Θ^F, W^F when the plate is bonded to the rigid wall $x_1 = 0$.]

Proof. On applying Theorem 4 (in the form (50)) with $\tau^{(2)} = \tau^B$ and $\tau^{(1)} = [\tau^B]_{\mu'=\infty}$ we obtain

$$\mu h^2(\Theta^B - [\Theta^B]_{\mu'=\infty}) > 2\mathcal{U}_{\mathcal{R}}(\tau^B). \tag{70}$$

Moreover, $[\Theta^B]_{\mu'=\infty} \geq \Theta_0^B$ by the Corollary to Theorem 4 and so

$$\mu h^2(\Theta^B - \Theta_0^B) > 2\mathcal{U}_{\mathcal{R}}(\tau^B). \tag{71}$$

Now we show in Appendix 10 that this strain energy $\mathcal{U}_{\mathcal{R}}(\tau^B)$ must exceed the strain energy which would be stored in $\mathcal{R}$ if the plate were perfectly rigid, and the same couple $M(= \mu h^2)$ were applied to it. Thus

$$\mu h^2(\Theta^B - \Theta_0^B) > (\mu h^2)\epsilon, \tag{72}$$

where ϵ is the angle turned by this *rigid* plate. However, for a support of fixed shape $\mathcal{R}$, it follows from linearity and dimensional analysis that ϵ must have the functional form

$$\epsilon = \frac{M}{\mu' h^2}C(\nu') \tag{73}$$

for some (dimensionless) positive function $C(\nu')$. Thus

$$\epsilon = \frac{\mu h^2}{\mu' h^2} C(\nu') = C(\nu') \frac{\mu}{\mu'}. \tag{74}$$

On combining (72), (74) we obtain

$$\Theta^B - \Theta_0^B > C(\nu') \frac{\mu}{\mu'} \tag{75}$$

as required; (69) is proved in a similar manner.

COROLLARY. *For the special case in which the plate with elastic constants μ, ν occupies the region $x_1 \geq 0$, $|x_3| \leq h$ and is butt-jointed to the elastic half-space $x_1 \leq 0$ (with elastic constant μ', ν'), the bound* (68) *becomes*

$$\Theta^B - \Theta_0^B > \frac{2(1-\nu')}{\pi(1+4\beta^2)} \left(\frac{\mu}{\mu'}\right), \tag{76}$$

where $\beta = (2\pi)^{-1} \ln(3 - 4\nu')$.

Proof. In this case, the angle ϵ in equation (72) has been determined exactly and explicitly by Muskhelishvili [15], p. 477; (76) is obtained from (72) by using this value for ϵ.

Note: There is no bound for W^F corresponding to (76); see Section 6.

THEOREM 8. *Let the material of the support $\mathcal{R}$ be homogeneous and isotropic with elastic constants μ', ν'. Then there exist positive functions $E(\nu, \nu')$, $F(\nu, \nu')$ such that*

$$\Theta^B - \Theta_0^B < E(\nu, \nu') \frac{\mu}{\mu'}, \tag{77}$$

$$W^F - W_0^F < F(\nu, \nu') \frac{\mu}{\mu'}, \tag{78}$$

when $\mu' < \mu$.

Proof. Let $\mu' = \mu e^{-\alpha}$, where α is a variable parameter, while ν' remains constant. Then, from Theorem 5,

$$\mu h^2 \frac{d\Theta^B}{d\alpha} = \int_{\mathcal{R}} \left(\frac{\partial}{\partial \alpha} c_{ijkl}\right) \tau_{ij}^B \tau_{kl}^B \, dx_1 dx_3 \tag{79}$$

$$= \int_{\mathcal{R}} c_{ijkl} \tau_{ij}^B \tau_{kl}^B \, dx_1 dx_3 \qquad \text{(see (64), (65))} \tag{80}$$

$$= 2\mathcal{U}_{\mathcal{R}}(\tau^B) < \mu h^2 (\Theta^B - [\Theta^B]_{\mu'=\infty}). \tag{81}$$

As in Theorem 7, this last step follows from the equality (50) in which $\tau^{(2)} = \tau^B$ and $\tau^{(1)} = [\tau^B]_{\mu'=\infty}$. Moreover, since $[\Theta^B]_{\mu'=\infty} \geq \Theta_0^B$ (see (53)) and Θ_0 is independent of α, it follows that

$$\frac{d}{d\alpha}(\Theta^B - \Theta_0^B) < \Theta^B - \Theta_0^B \tag{82}$$

$(-\infty < \alpha < \infty)$. Hence

$$\frac{d}{d\alpha}\left\{\ln\left(\frac{\Theta^B - \Theta_0^B}{e^\alpha}\right)\right\} < 0 \tag{83}$$

and so

$$(\Theta^B - \Theta_0^B) e^{-\alpha} \tag{84}$$

is a decreasing function of α. In particular, its values when $\alpha > 0$ must be less than its value when $\alpha = 0$, so that

$$\Theta^B - \Theta_0^B < \frac{\mu}{\mu'}\left([\Theta^B]_{\mu'=\mu} - \Theta_0^B\right) \tag{85}$$

for $\mu' < \mu$. However, for a support of fixed shape $\mathcal{R}$ and with the plate subjected to a couple $M (= \mu h^2)$ at $x_1 = +\infty$, the quantity $[\Theta^B]_{\mu'=\mu}$ depends only on ν, ν' and is known to exceed $\Theta_0^B(\nu)$. This proves (77) and (78) is shown similarly.

Remark on the behavior of $\Theta^B \ldots W^F$ *as* $\mu'/\mu \to 0$.

Theorems 7, 8 show that as $\mu'/\mu \to 0$ (with constant ν, ν') then Θ^B, W^F tend to infinity like μ/μ'. This substantiates our claim (made in Section 3), that Θ^B, W^F take large values when the support is soft.

We give no proof that W^B, Θ^F behave in the same way in the limit as $\mu'/\mu \to 0$. When the support is soft, one might expect a large *rotation* of the plate in the bending problem, and a large *x_3-displacement* in the flexure problem (and not for example the other way round). The formulae (18)–(21) of Theorem 2 show that we should then expect large values of Θ^B, W^F (which we have *proved* to be true), but no large values for W^B, Θ^F. However, an examination of a particular problem solved by Mukhelishvili [15] p.477–481 (which corresponds to a rigid plate butt-jointed to the surface of an elastic half-space) shows that in the *bending* problem, with zero displacements at infinity, the plate does suffer a *shift* in the x_3–direction as well as the expected rotation. The value of this shift, corresponding to applied couple M is not actually derived in [15] but can be inferred to be

$$\frac{4\beta(1-\nu')M}{\pi(1+4\beta^2)\mu' h}, \tag{86}$$

where μ', ν' are the elastic constants of the half-space and $\beta = (2\pi)^{-1} \ln(3 - 4\nu')$. It follows that when a plate with elastic constants μ, ν is butt-jointed to an elastic half-space with elastic constants μ', ν' then, in the limit in which $\mu'/\mu \to 0$,

$$W^B \sim \frac{4\beta(1-\nu')}{\pi(1+4\beta^2)} \left(\frac{\mu}{\mu'}\right). \tag{87}$$

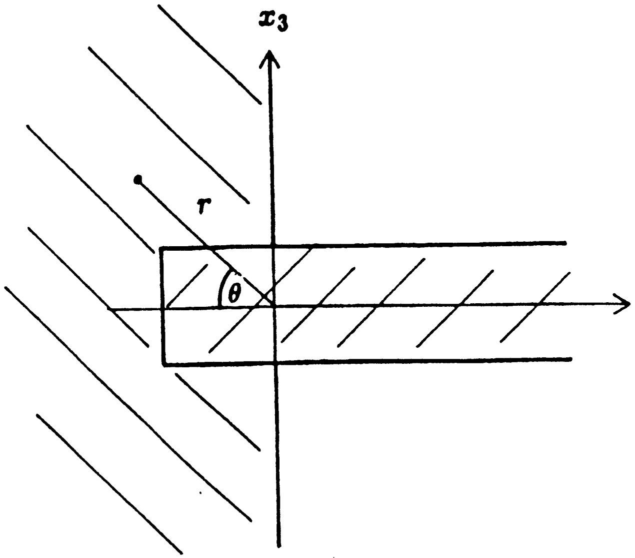

Fig. 6. An (isotropic) semi-infinite support

It further follows from Theorem 1 that Θ^F also must behave in the same way.

It is highly likely then that in general all the constants $\Theta^B \ldots W^F$ increase like μ/μ' as $\mu'/\mu \to 0$. We have *proved* this to be true for Θ^B, W^F and, where necessary, we make the reasonable assumption that this is also true for Θ^F, W^B.

6. The Semi-Infinite Support

One might suppose that, when the plate thickness $2h$ is small compared with the size of the support, such a support could be approximated by an appropriate semi-infinite support (as shown in Figure 6); the calculation of the constants $\Theta^B, \ldots, W^F$ might then be more tractable. Unfortunately, this is *never* true for the flexure problem because, as the 'radius' R of the support tends to infinity, the displacement field $\mathbf{u}^F$ does *not* tend to a limit, even though $\boldsymbol{\tau}^F$ does.

Consider, for example, a homogeneous, isotropic support with elastic constants μ', ν' which extends as far as $r = R$, ($|\theta| \leq \pi/2$), where the co-ordinates r, θ are shown in Figure 6; the arc $r = R$, $|\theta| \leq \pi/2$ is clamped. Let the elastic field for the flexure problem (with $Q = \mu h$) be denoted by $\{\boldsymbol{\tau}^F(\mathbf{x}; h, R),\ \mathbf{u}^F(\mathbf{x}; h, R)\}$, and denote the unique stress field corresponding to the limit $R/h \to \infty$ by $\boldsymbol{\tau}^F(\mathbf{x}; h, \infty)$. Then

$$\tau^F_{rr}(\mathbf{x}; h, \infty) = -\left(\frac{2\mu}{\pi}\right)\frac{h}{r}\sin\theta + \mu O(h^2/r^2) \tag{88}$$

as $r/h \to \infty$, $|\theta| \leq \pi/2$. (The right side of (88) coincides with the well known solution for a tangential line load μh acting at the surface of the half-space $x_1 \leq 0$; see Timoshenko and Goodier [17] section 33.) There is no displacement field corresponding to (88) which is bounded as $r/h \to \infty$. However we may take one for which

$$\mathbf{u}^F(\mathbf{x}; h, \infty) = -\frac{1-\nu'}{\pi}\left(\frac{\mu}{\mu'}\right) h \ln\left(\frac{r}{h}\right) \mathbf{e}_3 + O(h) \tag{89}$$

as $r/h \to \infty$, $|\theta| \leq \pi/2$, where $\mathbf{e}_3$ is the unit vector in the x_3-direction. We may now in principle construct $\mathbf{u}^F(\mathbf{x}; h, R)$ in the form

$$\begin{aligned}\mathbf{u}^F(\mathbf{x}; h, R) = {} & \mathbf{u}^F(\mathbf{x}; h, \infty) + \tfrac{1-\nu'}{\pi}\left(\tfrac{\mu}{\mu'}\right) h \ln\left(\tfrac{R}{h}\right) \mathbf{e}_3 \\ & + \mathbf{u}^{COR}(\mathbf{x}; h, R),\end{aligned} \tag{90}$$

where the correction term $\mathbf{u}^{COR}$ is the unique solution of the boundary value problem for the system, corresponding to zero loading at $x_1 = +\infty$ and prescribed displacements on $r = R$, $|\theta| \leq \pi/2$; these prescribed displacements are chosen so that $\mathbf{u}^F(\mathbf{x}; h, R) = 0$ on the arc $r = R$, $|\theta| \leq \pi/2$; they are $O(h)$ as $R/h \to \infty$. On the reasonable assumption that

$$\mathbf{u}^{COR}(\mathbf{x}; h, R) = O(h), \tag{91}$$

$$\boldsymbol{\tau}^{COR}(\mathbf{x}; h, R) = O(1) \tag{92}$$

as $R/h \to \infty$, Theorem 2 then implies that

$$W^F = \frac{1-\nu'}{\pi}\left(\frac{\mu}{\mu'}\right) \ln\left(\frac{R}{h}\right) + O(1) \tag{93}$$

as $R/h \to \infty$, while Θ^F remains bounded as $R/h \to \infty$.

Note that

(i) The result (93) is not restricted to semi-circular supports. Any large support that can be bounded by semi-circular supports (as in Figure 4) will have a similar form for W^F.

(ii) In the *bending* problem, both $\boldsymbol{\tau}^B$, $\mathbf{u}^B$ tend to limits as $R/h \to \infty$ and so it is possible to approximate Θ^B, W^B for a relatively large support by their limiting values as $R/h \to \infty$; this limiting value for Θ^B is also an upper bound. Theorem 1 then shows that Θ^F must also tend to a limit as $R/h \to \infty$; this fact is consistent with the arguments above. This leaves only W^F which cannot be so approximated, but which has the asymptotic form (93).

(iii) Since W^F appears in the term of the boundary condition (33) that is of order $O(h/L)^3$ as $L/h \to \infty$ (where L is the lateral length scale in the plate), it might appear that, even for moderately thick plates, this term is negligible. However, W^F may be large by virtue of (i) the support being relatively soft (μ'/μ small), or (ii) the support being relatively large (R/h large). As a consequence, the W^F term in (33) can possibly be more significant than the 'thickness correction' terms in (32), (33); these terms involve the constants Θ^F, W^B, which do not become large in the second limit. However, the W^F term in (33) is unlikely ever to be as significant as the Θ^B term in (32). For the W^F term to be of equal significance, it is required that R/h be so large that $(h/L)^3 \ln(R/h)$ is of similar magnitude to h/L. Thus, even for a moderately thick plate ($2h/L = 1/5$, say), R/h would have to be about e^{100}, an unrealistically large value!

7. Concluding Remarks

The preceding theorems tell us under what circumstances the various higher order terms in (32), (33) may be neglected; we restrict our discussion to the case in which the support is homogeneous and isotropic, with elastic constants μ', ν'. When μ/μ' is moderate or large, the conditions (32), (33) may be accurately approximated by the classical thin plate conditions (1) if

$$\left(\frac{\mu}{\mu'}\right)\frac{h}{L} \ll 1, \tag{94}$$

and

$$\left(\frac{\mu}{\mu'}\right)\left(\frac{h}{L}\right)^3 \ln\left(\frac{R}{h}\right) \ll 1. \tag{95}$$

When μ/μ' is small, Θ^B and W^F approach 'rigid support' limits so that we then require

$$\frac{h}{L} \ll 1, \tag{96}$$

and

$$\left(\frac{\mu}{\mu'}\right)\left(\frac{h}{L}\right)^3 \ln\left(\frac{R}{h}\right) \ll 1. \tag{97}$$

[We note that, unless R/h is unrealistically large, (95), (97) will hold as a consequence of (94), (96) respectively.]

When $(h/L)^2 W^B, (h/L)^2\Theta^F \ll 1$, the W^B and Θ^F terms in (32), (33) are negligible and the support acts as if it were a pair of springs exerting a restoring couple and restoring transverse force respectively on the edge of the plate. If h/L is also small enough so that the W^F term is negligible (by virtue of (95) or (97)), then the boundary conditions at $x_1 = 0$ reduce to

$$w = 0, \tag{98}$$

$$\frac{dw}{dx_1} - \frac{4\Theta^B}{3(1-\nu)} h \frac{d^2 w}{dx_1^2} = 0, \tag{99}$$

which corresponds to a hinged edge with a restoring couple proportional to the angular deflection of the plate at the edge. In all practical cases (99) represents the most significant correction to the 'thin plate' conditions (1).

APPENDIX

A. Saint-Venant bending and flexure fields for the infinite plate in plane strain

For the infinite plate $|x_3| \le h$, with its faces $|x_3| = h$ traction free and in plane strain deformation parallel to the (x_1, x_3) plane, the Saint-Venant bending (VB) and flexure (VF) fields are given by:

Unit Bending (per unit length in x_2-direction).

$$\tau^{VB} \equiv \begin{pmatrix} \tau_{11}^{VB} & \tau_{13}^{VB} \\ \tau_{31}^{VB} & \tau_{33}^{VB} \end{pmatrix} = -\frac{3}{2h^3}\begin{pmatrix} x_3 & 0 \\ 0 & 0 \end{pmatrix}, \tag{100}$$

$$\mathbf{u}^{VB} \equiv \begin{pmatrix} u_1^{VB} \\ u_3^{VB} \end{pmatrix} = \frac{3}{8\mu h^3}\begin{pmatrix} -2(1-\nu)x_1x_3 \\ \nu x_3^2 + (1-\nu)x_1^2 \end{pmatrix}. \tag{101}$$

Unit Flexure (per unit length in the x_2-direction).

$$\tau^{VF} \equiv \begin{pmatrix} \tau_{11}^{VF} & \tau_{13}^{VF} \\ \tau_{31}^{VF} & \tau_{33}^{VF} \end{pmatrix} = \frac{3}{4h^3}\begin{pmatrix} 2x_1x_3 & h^2 - x_3^2 \\ h^2 - x_3^2 & 0 \end{pmatrix}, \tag{102}$$

$$\mathbf{u}^{VF} \equiv \begin{pmatrix} u_1^{VF} \\ u_3^{VF} \end{pmatrix} = \frac{1}{8\mu h^3}\begin{pmatrix} 3(1-\nu)x_1^2x_3 + 6h^2x_3 - (2-\nu)x_3^3 \\ -(1-\nu)x_1^3 - 3\nu x_1 x_3^2 \end{pmatrix}. \tag{103}$$

In the above formulae, μ, ν are the shear modulus and Poisson's ratio for the plate. Note that, in our definition of flexure, the bending moment is zero at $x_1 = 0$.

B. The interior field $\{\tau^I, \mathbf{u}^I\}$, for a plate with free faces, in terms of the mid-plane transverse displacement w

For the general case in which $w = w(x_1, x_2)$, we have

$$\tau_{11}^I = -\frac{2\mu x_3}{1-\nu}\left[\frac{\partial^2}{\partial x_1^2} + \nu\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_1^2}(h^2 - \frac{2-\nu}{6}x_3^2)\nabla^2\right] w, \tag{104}$$

$$\tau_{12}^I = -\frac{2\mu x_3}{1-\nu}\frac{\partial^2}{\partial x_1 \partial x_2}\left[(1-\nu) + (h^2 - \frac{2-\nu}{6}x_3^2)\nabla^2\right] w, \tag{105}$$

$$\tau_{13}^I = -\frac{\mu}{1-\nu}(h^2 - x_3^2)\frac{\partial}{\partial x_1}\nabla^2 w, \tag{106}$$

$$\tau_{33}^I = 0, \tag{107}$$

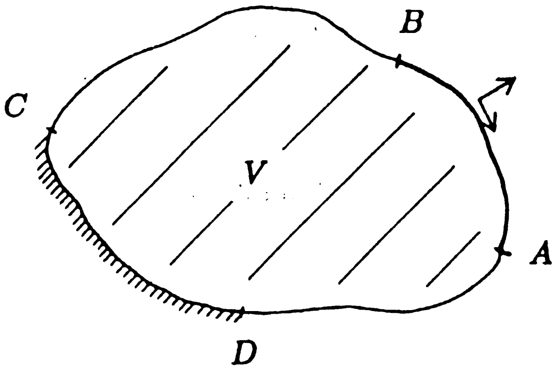

Fig. 7. The arc CD is clamped, while tractions with prescribed resultant force and couple act on the arc AB.

$$u_1^I = -\frac{x_3}{1-\nu}\frac{\partial}{\partial x_1}\left[(1-\nu) + (h^2 - \frac{2-\nu}{6}x_3^2)\nabla^2\right] w, \tag{108}$$

$$u_3^I = \left[1 + \frac{\nu}{2(1-\nu)}x_3^2\nabla^2\right] w, \tag{109}$$

where

$$\nabla^2\nabla^2 w = 0, \tag{110}$$

with similar formulae for τ_{22}^I, τ_{23}^I, u_2^I. These formulae are well known in the literature of the so-called 'exact theory of plates' (see Timoshenko and Woinowski-Krieger [18], p.103); more recently however they have been proved (Gregory [5]) to represent the general *interior* solution only.

C. Applied tractions, with prescribed resultant force and couple, which induce minimum strain energy

Solely to conform with the applications in this paper, we will present a two-dimensional (plane strain) version of this result. Consider the body shown in Figure 7 which is in plane strain deformation parallel to the plane $x_2 = 0$. The arc CD is clamped (that is, $\mathbf{u} = 0$), the arcs BC and DA are traction free, and tractions with *prescribed* resultant force and couple per unit length in the x_2-direction are applied to the arc AB. Let $\{\boldsymbol{\tau}, \mathbf{u}\}$ and $\{\boldsymbol{\tau}^{(0)}, \mathbf{u}^{(0)}\}$ be any two elastic states generated in this way, and let $\mathcal{U}_V(\boldsymbol{\tau})$ denote the total strain energy in V (per unit length in the x_2-direction) of a field $\boldsymbol{\tau}$. Then

$$\mathcal{U}_V(\boldsymbol{\tau}) - \mathcal{U}_V(\boldsymbol{\tau}^{(0)}) - \mathcal{U}_V(\boldsymbol{\tau} - \boldsymbol{\tau}^{(0)})$$
$$= \frac{1}{2}\int_{\partial V} u_i\tau_{ij}n_j ds - \frac{1}{2}\int_{\partial V} u_i^{(0)}\tau_{ij}^{(0)}n_j ds - \frac{1}{2}\int_{\partial V}(u_i - u_i^{(0)})(\tau_{ij} - \tau_{ij}^{(0)})n_j ds$$

$$= \frac{1}{2}\int_{\partial V}(u_i - u_i^{(0)})\tau_{ij}^{(0)}n_j ds + \frac{1}{2}\int_{\partial V} u_i^{(0)}(\tau_{ij} - \tau_{ij}^{(0)})n_j ds \tag{111}$$

$$= \int_{\partial V} u_i^{(0)}(\tau_{ij} - \tau_{ij}^{(0)})n_j ds \tag{112}$$

on applying reciprocity to the first integral in (111). In view of the boundary conditions on BC, CD, DA, this reduces to

$$\mathcal{U}_V(\boldsymbol{\tau}) = \mathcal{U}_V(\boldsymbol{\tau}^{(0)}) + \mathcal{U}_V(\boldsymbol{\tau} - \boldsymbol{\tau}^{(0)}) + \int_{AB} u_i^{(0)}(\tau_{ij} - \tau_{ij}^{(0)})n_j ds. \tag{113}$$

(113) holds for any two fields in V satisfying the boundary conditions on BC, CD, DA. Suppose now that $\{\boldsymbol{\tau}^{(0)}, \mathbf{u}^{(0)}\}$ is generated by tractions on AB which, in addition to having the prescribed force and couple resultants, *cause the arc AB to suffer a rigid displacement*. Then the integral in (113) is zero since it reduces to the force and couple resultants on AB of $\boldsymbol{\tau} - \boldsymbol{\tau}^{(0)}$, and these are known to be zero. In this case then

$$\mathcal{U}_V(\boldsymbol{\tau}) = \mathcal{U}_V(\boldsymbol{\tau}^{(0)}) + \mathcal{U}_V(\boldsymbol{\tau} - \boldsymbol{\tau}^{(0)}), \tag{114}$$

so that

$$\mathcal{U}_V(\boldsymbol{\tau}) > \mathcal{U}_V(\boldsymbol{\tau}^{(0)}), \qquad (\boldsymbol{\tau} \neq \boldsymbol{\tau}^{(0)}). \tag{115}$$

Thus, *out of all tractions having the same prescribed force and couple resultants which could act on AB, the tractions which induce the least strain energy in V are the ones which cause the arc AB to be rigidly displaced.*

References

1. D.M. Austin, Ph.D. Thesis, University of Manchester (1987).
2. K.O. Friedrichs and R.F. Dressler, A boundary layer theory for the bending of plates. *Comm. Pure Appl. Math.* 14 (1961) 1–33.
3. A.L. Gol'denviezer, Derivation of an approximate theory of bending of a plate by the method of asymptotic integration of the equations of the theory of elasticity. *PMM* 26 (1962) 668–686.
4. R.D. Gregory, Green's functions, bi-linear forms, and completeness of the eigenfunctions for the elastostatic strip and wedge. *J. Elasticity* 9 (1979) 283–309.
5. R.D. Gregory, The general form of the three-dimensional elastic field inside an isotropic plate with free faces. *J. Elasticity* 28 (1992) 1–28.
6. R.D. Gregory and I. Gladwell, The cantilever beam under tension, bending or flexure at infinity. *J. Elasticity* 12 (1982) 317–343.
7. R.D. Gregory and F.Y.M. Wan, Decaying states of plane strain in a semi-infinite strip and boundary conditions for plate theory. *J. Elasticity* 14 (1984) 27–64.
8. R.D. Gregory and F.Y.M. Wan, On plate theories and Saint-Venant's principle. *Int. J. Solids Structures* 21 (1985) 1005–1024.
9. R.D. Gregory and F.Y.M. Wan, Edge effects in the stretching of plates. In: P. Ladeveze (ed.), *Local Effects in the Analysis of Structures*. Amsterdam: Elsevir Science Publishers (1985), 35–54.
10. R.D. Gregory and F.Y.M. Wan, The interior solution for linear problems of elastic plates. *J. Appl. Mech.* 55 (1988) 551–559.
11. R.D. Gregory and F.Y.M. Wan, Correct asymptotic theories for the axisymmetric deformation of thin and moderately thick cylindrical shells. *Int. J. Solids Structures* 14 (1993) 1957–1981.

12. G. Kirchhoff, Über das gleichgewicht und die bewegung einer elastichen scheibe, *J. Reine Ange. Math. (Crelle)* 40 (1850) 51.
13. Y. Lin and F.Y.M. Wan, First integrals and the residual solution for orthotropic plates in plane strain or axisymmetric deformations. *Studies in Appl. Math.* 79 (1988) 93–125.
14. Y. Lin and F.Y.M. Wan, Circular elastic plates with prescribed axisymmetric edge displacements. *Canadian Appl. Math. Quart.* 1 (1993) 311–342.
15. N.I. Muskhelishvili, *Some Basic Problems of the Theory of Elasticity* (Translated into English by J.R.M. Radok) Gröningen, Netherlands: Noordhoff (1953).
16. E. Sternberg and J.K. Knowles, Minimum energy characterizations of Saint-Venant's solution to the relaxed Saint-Venant problem. *Arch. Rat. Mech. Anal.* 21 (1966) 89–107.
17. S. Timoshenko and N. Goodier, *Theory of Elasticity*, 2nd edn. New York: McGraw-Hill (1951).
18. S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, 2nd edn. New York: McGraw-Hill (1959).