Shear Center for Plates of Variable Thickness

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ABSTRACT

In a Reissner's type formulation of the shear center problem for flat plates, the outer asymptotic solution for the plate equations satisfying the traction free edge conditions provides an exact solution of the BVP for Poisson's ratio ν equal to zero. For the non-zero ν case, the same outer solution alone (without any reference to the more complex edge effects) satisfying appropriately prescribed end conditions obtained from a reciprocal theorem for plates is sufficient for an accurate determination of the shear center location, up to an exponentially small error. An improved version of the method of projection which takes into account the stress singularities at the corners of the clamped edge is needed for the solution. Numerical results are presented for plates with a chordwise exponentially varying plate thickness.

INTRODUCTION

The shear center problem for flat bodies was recently formulated as a BVP in the theory of transverse bending of flat plates of (chordwise) variable thickness (Reissner 1989, 1991). The results obtained there include an approximate determination of the center of shear by the principle of minimum complementary energy with certain semi-inverse Saint-Venant assumptions for the stresses. A twelfth order system of generalized beam equations is also developed in (Reissner 1989) by a suitable process of averaging of the plate equations for an assumed displacement field, quadratic in the chordwise coordinates. For plate thickness symmetric about the mid-plane $x_3 = 0$, the approximate complementary energy analysis leads to an expression for the center of shear y_s^0 (equal to the ratio of the first and zeroth moment of an effective bending stiffness factor) which is independent of Poisson's ratio and the plate aspect ratio. The solution of the generalized beam equations on the other hand suggests the presence of these two effects (Gu and Wan 1993).

Other methods for approximate solutions and for determining the effects of Poisson's ratio and aspect ratio for isotropic (and orthotropic) plates with a chordwise variable thickness distribution can be found in (Gu and Wan, 1993). These solutions show that both aspect ratio and Poisson's

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ratio effects result in a modification of the approximate solution y_s^0. The effects are however much smaller than modifications resulting from the often proposed association of the Saint Venant flexure solution of the plate BVP with the condition of vanishing centroidal end section rotation or vanishing averaged end section rotation (Reissner 1991, Gu and Wan 1993 and references therein).

The present paper describes a new method for an accurate solution of our shear center problem and to obtain specific results for the shear center location. The known *method of projection* (Gregory and Gladwell, 1982) is modified to incorporate the stress singularities known to be present at the corners of the clamped end of the plate (see also Gu 1994, Gregory, Gu and Wan, to appear). The original projection method was developed for plane strain behavior of strips with one fixed end and is not applicable to the plate case as it stands. The new improved method is to remove the real part of the singular exponent completely; a technique is devised to treat the bounded oscillating singularities that remain. The modified method of projection is simple and efficient computationally and does not require the inversion of large linear systems. Numerical results will be reported for plates with a chordwise exponentially varying thickness distribution symmetric about the midplane not previously treated in the literature.

THE SHEAR CENTER PROBLEM IN PLATE THEORY

In the classical linear theory for the transverse bending of elastic plates, the deformation of the plate in a cartesian coordinate system (x_1, x_2, x_3) is characterized by a *mid-plane transverse displacement* $w(x_1, x_2)$ in the x_3-direction. The curvature change measures κ_{ij} are defined in terms of w, by (i) the *strain displacement relations* $\kappa_{ij} = -w_{,ij}$ $(i, j = 1, 2)$ where $(\ )_{,k}$ indicates partial differentiation of ( ) with respect to x_k. The *stress couples* M_{ij} induced by the bending deformation are given in terms of w by way of (ii) the *stress-strain relations* $M_{11} = D(\kappa_{11} + \nu\kappa_{22})$, $M_{22} = D(\kappa_{22} + \nu\kappa_{11})$ and $M_{12} = M_{21} = D_t(\kappa_{12} + \kappa_{21})$ where $D = Eh^3/12(1-\nu^2)$ and $D_t = D(1-\nu)/2$. The *transverse shear resultants* Q_j are given in terms of w through (iii) the *moment equilibrium equations* $Q_j = M_{1j,1} + M_{2j,2}$ $(j = 1, 2)$ in the plate interior. Transverse force equilibrium requires that (iv) $Q_{1,1} + Q_{2,2} = 0$ when there is no distributed interior load.

We consider here rectangular plates with a mid-plane occupying the region $R = \{0 \le x_1 \le L, -a \le x_2 \le a\}$. The two long edges are free of traction so that (v) $V_2 = M_{22} = 0$ at $x_2 = \pm a$ where the effective transverse resultants V_j are given by $V_j = Q_j + M_{ji,i} (i \ne j)$. The plate is clamped along one end so that (vi) $w = w_{,1} = 0$ along $x_1 = 0$. The end $x_1 = L$ is free of bending couples but is subject to a uniform vertical displacement W_0 requiring (vii) $w = W_0$ and $M_{11} = 0$ at $x_1 = L$.

The resultant transverse force F_3 and axial torque T_1 at $x_1 = L$ may be written as (see Gu and Wan 1993)

$$(vi) \qquad F_3 = \int_{-a}^{a} M_{11,1}\, dx_2, \qquad T_1 = \int_{-a}^{a} (x_2 M_{11,1} - 2M_{12}) dx_2$$

with the integrand evaluated at the end $x_1 = L$. In the context of plate theory, the location $(x_1, x_2) = (L, y_s)$ of the shear center has been shown to be (vii) $y_s = T_1/F_3$ (Reissner 1989, Gu and Wan 1993). Of interest is how y_s varies with the material parameters, the dependence of the half-thickness h on x_2 and the aspect ratio $2a/L$.

THE INTERIOR SOLUTION

Methods used in (Gu and Wan 1993) for the approximate determination of the shear center are based on different geometrical or physical considerations which are not direct consequences of plate theory; there is no rational way to assess their accuracy. With the help of a reciprocal theorem for plates, a new method has been developed (Gu 1994) which determines the shear center location to within an exponentially small error with respect to the small thickness-to-span ratio. The new method is based on the observation that overall equilibrium permits the evaluation of the expressions for F_3 and T_1 in (vi) at any location $\bar{x}_1$ along the plate span instead of the end $x_1 = L$. For plates with a small aspect ratio, we may take $\bar{x}_1$ to be sufficiently far away from the two ends so that the end effects are negligible in the expressions for the relevant stress couples which appear in the two integrals in (vi). The *method of decaying residual states* developed in (Gregory and Wan 1984, 1985 and 1988, Lin and Wan 1988) can be employed to determine the interior (or outer asymptotic expansion) solution of the plate BVP independent of the complex edge effects of the plate.

It has been shown that the outer asymptotic expansion of the exact solution for the plate BVP defined by conditions (i)-(vi) is identical to that obtained using the Saint-Venant type assumptions $Q_2 = M_{22} = 0$ throughout the plate. The resulting outer solution for the mid-plane displacement can be written as

$$w_I(x_1, x_2) = w_0(x_2) + w_1(x_2)x_1 - c_0\frac{x_1^2}{2!} - c_1\frac{x_1^3}{3!}, \qquad (1a)$$

$$w_0(x_2) = c_5 + c_4 x_2 + c_0 \int_{-a}^{x_2}\int_{-a}^{z} \nu ds dz, \quad w_1(x_2) = c_3 - c_2 x_2 + c_1 \int_{-a}^{x_2}\int_{-a}^{z} \nu ds dz. \qquad (1b, c)$$

The interior solution (1) (also called the Saint-Venant solution or the outer asymptotic solution) for the plate BVP satisfies the free edge conditions (iv) at the two long edges $x_2 = \pm a$. For the case of $\nu = 0$, it can be made also to satisfy the clamped edge conditions at $x_1 = 0$ by setting $c_2 = c_3 = c_4 = c_5 = 0$. The remaining edge conditions at $x_1 = L$ are then satisfied by setting $c_0 = -3W_0/L^2$ and $c_1 = 3W_0/L^3$. We then have as the exact solution of the BVP

$$w(x_1, x_2) = w_I(x_1, x_2) = \frac{3W_0}{L^3}\left(\frac{1}{2}x_1^2 - \frac{1}{6}x_1^3\right). \qquad (2)$$

For non-zero Poissons ratios, the boundary conditions at $x_1 = L$ are satisfied by

$$c_0 + c_1 L = 0, \qquad c_4 - c_2 L = 0, \qquad c_5 - \frac{1}{3}c_0 L^2 + c_3 L = W_0. \qquad (3)$$

But it is not possible to choose the remaining constants of integration $\{c_i\}$ to satisfy the boundary conditions at $x_1 = 0$. On the other hand, it is still possible, as we shall see below, to determine the interior solution w_I away from the two ends of the plate to within an exponentially small error. With F_3 and T_1 independent of x_1 by overall equilibrium, the shear center location is still expected to be determined essentially by the interior solution (1) with a suitably chosen set of constants of integration.

To take advantage of the observation above, we take $w = w_I$ and rewrite the expression (vii) for y_s in a form that depends only on the unknown ratio c_2/c_{11}:

$$y_s = \frac{\int_{-a}^{a}[x_2 D_b + D_t \int_{-a}^{x_2} \nu dz]dx_s}{\int_{-a}^{a} D_b dx_2} - \frac{c_2}{c_1}\frac{\int_{-a}^{a} D_t dx_2}{\int_{-a}^{a} D_b dx_2} \qquad (4)$$

where $D_b = D(1-\nu^2)$. The derivation of (4) can be found in (Reissner 1989, Gu 1994). We need only to determine the appropriate value for c_2/c_1 to complete the solution of the problem. The method of decaying *residual states* first introduced in (Gregory and Wan 1984 and 1985) for elastic strips and plates offers an approach to determine accurately this quantity which appears in the interior solution for the plate boundary value problem without any need to determine the corresponding boundary layer effects for the problem.

THE METHOD OF DECAYING RESIDUAL STATES

To obtain three appropriate additional conditions to supplement the three in (3) for the determination of the six constants in the interior solution (1), we define as the *residual state* for our plate BVP the difference between the exact solution and the interior solution: $w_R(x_1, x_2) = w(x_1, x_2) - w_I(x_1, x_2)$. We expect the residual solution to be a boundary layer phenomenon decaying exponentially away from the two ends. We deduce three necessary conditions for a *decaying residual state* by means of a plate theory version of the reciprocal theorem of elasticity theory (Wan 1989, 1994). These conditionst may be taken in the form

$$c_5 + c_0 \nu v_2^F + c_2 m_1^F - c_1 \nu m_2^F = 0, \quad c_4 + c_0 \nu v_2^T + \cdots\cdots =, \quad -c_3 + c_0 \nu v_2^B + \cdots\cdots = 0 \tag{5}$$

$$v_2^Y = \frac{1}{2} \int_{-1}^{1} x_2^2 V_1^Y(0, x_2) dx - [x_2^2 M_{12}^Y(0, x_2)]_{-1}^{1}, \quad m_k^Y = \frac{1}{k!} \int_{-1}^{1} M_{11}^Y(0, x_2) x_2^k dx_2 \tag{6}$$

for $k = 1, 2$ and $Y = F$, T and B. The superscripts F, T and B denote solutions for semi-infinite plate strips free along the two long edges, clamped at the end and loaded by a unit flexural force, axial torque and bending moment at infinity, respectively. The solutions for these three canonical problems will be used to calculate v_1^Y, m_1^Y and m_2^Y which appear in (5) - (6) and in the corresponding conditions for the interior solution of problems involving other prescribed displacement boundary conditions.

THREE CANONICAL PROBLEMS FOR CANTILEVER STRIP PLATES

The canonical problems are effectively solved by an improved version of the method of projection. In this method, a reciprocal formula is used to show that the principal unknown quantities in each problem are the effective transverse shear resultant $V_1(0, x_2)$ and bending couple $M_{11}(0, x_2)$ induced at the clamped end. The required vector function $\mathbf{u}(x_2) = (V_1(0, x_2), M_{11}(0, x_2))^t$ considered as a member of the Hilbert space $H = \mathcal{L}^2[-1, 1]$, is orthogonal to a certain subspace of H. If the orthogonal complement of this subspace is one-dimensional, this provides a method for determining $\mathbf{u}$.

In the improved method, we define new unknowns $\hat{V}_1(x_2)$ and $\hat{M}_{11}(x_2)$ by

$$V_1(0, x_2) = (1 - x_2^2)^\alpha \hat{V}_1(x_2), \qquad M_{11}(0, x_2) = (1 - x_2^2)^{1+\alpha} \hat{M}_{11}(x_2) \tag{7}$$

where α is the real part of the complex exponent of the corner singularities of $\hat{V}_1$, and we construct the new unknown vector $\hat{\mathbf{u}}(x_2) = (\hat{V}_1(x_2), \hat{M}_{11}(x_2))^t$. The original reciprocity relations are now reinterpreted as orthogonality relations on $\mathbf{u}$, considered as a member of a new Hilbert space H_α. This procedure effectively removes the infinities in the corner singularities, the new unknown $\mathbf{u}$ having (infinitely many) oscillations of bounded amplitude near $x_2 = \pm a$. In (Gu 1994, Gregory, Gu and Wan: 1996), we give a fuller description of this method including the quite subtle procedure for handling the remaining oscillatory singularities of $\hat{\mathbf{u}}$. We are not aware of any other general method which is appropriate for the handling of this kind of oscillating singularity.

PLATES WITH CHORDWISE EXPONENTIALLY VARYING THICKNESS

For plates with a chordwise variable thickness distribution, the governing system of equations (i)-(iv) can still be reduced to a fourth order partial differential equation (PDE) for w but now no longer to the biharmonic equation. For $h = h_0 e^{\rho x_2}$, this equation is of constant coefficients: (viii) $\nabla^4 w + 6\rho(w_{,112} + w_{,222}) + 9\rho^2(w_{,22} + \nu w_{,11}) = 0$. The free edge conditions along the two long edges may be reduced to the same form as the constant thickness case: (ix) $w_{,22} + \nu w_{,11} = 0$, and $w_{,222} + (2-\nu)w_{,112} = 0$. Papkovich-Fadle type eigen-pairs can be obtained as usual; so can the singularities at the corner of the clamped edge. More than 30 of the eigen-pairs have been obtained in (Gu 1994) with M_{11} and V_1 at the corners of clamped end having the same singular behavior as in the constant thickness case. These results allow us to obtain the weight factors v_1^Y and m_k^Y in (6) as shown in Table 1 for typical Poisson's ratios by the modified method of projection. They in turn enable us to determine the constants $\{c_i\}$ from (3) and (5) and therewith the shear center location from (4). Numerical results for y_s are given in Table 2; they should be compared with the corresponding results by other methods reported in (Gu and Wan 1993) where more extensive results and discussion can be found.

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Table 1: Weight factors m_k^Y, v_2^Y, for plates with $h(x_2) = h_0 e^{x_2}$

	$\nu = 1/4$	$\nu = 1/3$	$\nu = 1/2$
m_1^T	-0.2458017809	-0.2603925702	-0.2972721605
m_1^B	0.6915848476	0.6961998431	0.6915848476
m_1^F	0.1840535014	0.1960301092	0.2373493011
m_2^F	-0.1177361357	-0.1249234597	-0.1432208160
m_2^B	0.2715267170	0.2680569321	0.2568658827
m_2^F	0.0772631501	0.0794662854	0.090675624
v_2^T	0.6222571812	0.6229754226	0.6259374771
v_2^B	0.1174217780	0.1557787466	0.2355569053
v_2^F	-0.0832635757	-0.0566803311	-0.0367669001

Table 2: Location of y_s for homogeneous isotropic plates with $h = h_0 e^{x_2}, (-1 \leq x_2 \leq 1)$

a/L ($a = 1$)	$\nu = 1/4$	$\nu = 1/3$	$\nu = 1/2$
1.000	0.6834	0.6852	0.6853
0.100	0.6907	0.6950	0.7007
0.010	0.6913	0.6959	0.7020
0.001	0.6914	0.6960	0.7021