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Rotationally Symmetric Stress and Strain in Shells of Revolution

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In generalization of existing results of the linear theory of symmetrical bending and torsion of thin elastic shells of revolution and of the theory of pure bending and twisting of ring sector shells, this paper considers the differential equations of the theory of shells of revolution under the supposition of rotationally symmetric stress and strain, allowing for transverse shear deformation and for moments turning about the normals to the middle surface. In this way, making use of the simplified static-geometric analogy which is associated with the introduction of the two mentioned effects, known results are rederived in a new way and some new results are obtained, in particular for the dislocation-theory-type problems of pure bending and twisting.

1. Introduction

The present paper is concerned with a somewhat more general approach to rotationally symmetric problems of the linear theory of shells of revolution than seems heretofore to have been considered. The starting point of our approach is the vector-analytical formulation of general linear shell theory, including the effect of moment stress stress couples as presented by Günther [3] and one of the present authors [14]. The most significant aspect of this approach is thought to be the extremely simple and direct form of the static-geometric analogy, or duality, which in its original form was proposed by Goldenveiser [2], and which may be read from the vectorial formulation of equilibrium and compatibility equations. The present treatment of the rotationally symmetric problem of the shell of revolution is designed to take full advantage of this form of the static-geometric duality.

The nature of our results may be described briefly as follows. We find that scalar equilibrium and compatibility differential equations each appear in the form of two uncoupled systems of three differential equations. The uncoupled nature of the two systems persists for quite general classes of stress strain relations. One set of three equilibrium and three compatibility equations allows some generalization of the results of H. Reissner [16] and Meissner [7] for symmetric bending, and of the results of Tueda [18] and of one of the present authors [10] for pure bending of curved tubes. The complementary set of six equations is such as to contain as special cases the equations leading to simple results for the problem of torsion, as mentioned in Love's treatise [6], and also to the results for pure twisting of flat plates [12] and of curved membranes [11] which were obtained by one of the present authors.

An important part of the results of H. Reissner and Meissner was the reduction of the original sixth order problem to a fourth order problem by way of first integrals for equilibrium and compatibility equations. The first integral of the equilibrium equations is readily anticipated from considerations of statics. The first integral of the compatibility equations at that time was an unexpected consequence of judicious transformations of suitable strain displacement relations (which much later were considerably simplified by one of the present authors through use of radial and axial instead of meridional and normal displacement components [9]). In our present considerations the first integral of the compatibility equations is recognized simply as the static geometric dual of the first integral of the equilibrium equations.

As far as the authors know, the problem of torsion and pure twisting has not previously been stated explicitly in the generality of a sixth order system of equilibrium and compatibility equations, and the results given here are essentially new. For this system, too, a first integral of the equilibrium equations exists and has a simple statical significance. Again a first integral of the compatibility equations follows as its static-geometric dual. Additionally, for the present problem the fourth-order system degenerates into a zeroth order system as soon as it is assumed that the shell cannot support moment stress stress couples and does not allow transverse shear deformations.

The present paper includes as specific applications (1) an explicit solution for the point load problem of the uniform spherical shell, including the effect of transverse shear deformation and moment stress stiffness, and (2) a treatment of the zeroth order system for the problem of pure twisting, including explicit expressions for stress resultants and couples and an explicit force deflection relation, containing as special cases previously known formulas for the flat plate [12] and the closed-cross-section membrane shell [11].

2. Vectorial form of shell equations for orthogonal middle surface coordinates

With middle surface equation $\mathbf{r} = \mathbf{r}(\xi_1, \xi_2)$ and linear element coefficients α_j^2 we have as equations of equilibrium for stress resultant and stress couple vectors $\mathbf{N}_j$ and $\mathbf{M}_j$ and load intensity vectors $\mathbf{p}$ and $\mathbf{q}$,

$$(\alpha_2 \mathbf{N}_1)_{,1} + (\alpha_1 \mathbf{N}_2)_{,2} + \alpha_1 \alpha_2 \mathbf{p} = 0 \tag{1}$$

$$(\alpha_2 \mathbf{M}_1)_{,1} + (\alpha_1 \mathbf{M}_2)_{,2} + \mathbf{r}_{,1} \times (\alpha_2 \mathbf{N}_1) + \mathbf{r}_{,2} \times (\alpha_1 \mathbf{N}_2) + \alpha_1 \alpha_2 \mathbf{q} = 0 \tag{2}$$

Virtual work considerations lead to the introduction of translational and rotational displacement vectors $\mathbf{u}$ and $\boldsymbol{\phi}$ and to the introduction of strain resultant and strain couple vectors $\boldsymbol{\varepsilon}_j$ and $\boldsymbol{\kappa}_j$, which are related to each other in the form

$$\alpha_j \boldsymbol{\kappa}_j = \boldsymbol{\phi}_{,j}, \qquad \alpha_j \boldsymbol{\varepsilon}_j = \mathbf{u}_{,j} + \mathbf{r}_{,j} \times \boldsymbol{\phi} \tag{3}$$

From (3) follow by inspection two vector compatibility equations

$$(\alpha_2 \boldsymbol{\kappa}_2)_{,1} - (\alpha_1 \boldsymbol{\kappa}_1)_{,2} = 0 \tag{4}$$

$$(\alpha\boldsymbol{\varepsilon}_2)_{,1} - (\alpha_1\boldsymbol{\varepsilon}_1)_{,2} + \mathbf{r}_{,1} \times (\alpha_2\boldsymbol{\kappa}_2) - \mathbf{r}_{,2} \times (\alpha_1\boldsymbol{\kappa}_1) = 0 \quad (5)$$

The homogeneous equations (1) and (2) together with (4) and (5) imply a static-geometric duality expressed schematically as follows

$\mathbf{N}_1$	$\mathbf{N}_2$	$\mathbf{M}_1$	$\mathbf{M}_2$
$\boldsymbol{\kappa}_2$	$-\boldsymbol{\kappa}_1$	$\boldsymbol{\varepsilon}_2$	$-\boldsymbol{\varepsilon}_1$

The corresponding scalar duality system is obtained upon writing, with $\alpha_j\mathbf{t}_j = \mathbf{r}_{,j}$, and $\mathbf{n} = \mathbf{t}_1 \times \mathbf{t}_2$,

$$\mathbf{N}_j = \sum N_{jk}\mathbf{t}_k + Q_j\mathbf{n}, \qquad \mathbf{M}_j = \mathbf{n} \times \sum M_{jk}\mathbf{t}_k + P_j\mathbf{n} \quad (6)$$

$$\boldsymbol{\varepsilon}_j = \sum \varepsilon_{jk}\mathbf{t}_k + \gamma_j\mathbf{n}, \qquad \boldsymbol{\kappa}_j = \mathbf{n} \times \sum \kappa_{jk}\mathbf{t}_k + \lambda_j\mathbf{n} \quad (7)$$

in the form

N_{11}	N_{12}	Q_1	N_{21}	N_{22}	Q_2	M_{11}	M_{12}	P_1	M_{21}	M_{22}	P_2
$-\kappa_{22}$	κ_{21}	λ_2	κ_{12}	$-\kappa_{11}$	$-\lambda_1$	ε_{22}	$-\varepsilon_{21}$	γ_2	$-\varepsilon_{12}$	ε_{11}	$-\gamma_1$

In reducing (1) to (5) to scalar form use is made of differentiation formulas

$$\mathbf{t}_{1,1} = -\frac{\alpha_{1,2}}{\alpha_2}\mathbf{t}_2 - \frac{\alpha_1}{R_{11}}\mathbf{n}, \qquad \mathbf{t}_{1,2} = \frac{\alpha_{2,1}}{\alpha_1}\mathbf{t}_2 - \frac{\alpha_2}{R_{12}}\mathbf{n}, \qquad \mathbf{n}_{,1} = \frac{\alpha_1}{R_{11}}\mathbf{t}_1 + \frac{\alpha_1}{R_{12}}\mathbf{t}_2, \text{etc.} \quad (8)$$

and of the additional representations

$$\mathbf{p} = \sum p_k\mathbf{t}_k + p_n\mathbf{n}, \qquad \mathbf{q} = \mathbf{n} \times \sum q_k\mathbf{t}_k + q_n\mathbf{n}$$

The system (1) to (5) is supplemented by stress strain relations which are here taken to be special cases of a system

$$N_{jk} = \frac{\partial A}{\partial \varepsilon_{jk}}, \qquad Q_j = \frac{\partial A}{\partial \gamma_j}, \qquad M_{jk} = \frac{\partial A}{\partial \kappa_{jk}}, \qquad P_j = \frac{\partial A}{\partial \lambda_j} \quad (10)$$

where A is a given function of the twelve arguments ε_{jk}, γ_j, κ_{jk} and λ_j.

3. Scalar form of shell equations for rotationally symmetric stress and strain in shells of revolution

We write $\xi_1 = \theta$, $\xi_2 = \xi$, $\mathbf{r} = r(\xi)\,\mathbf{i}_r + z(\xi)\,\mathbf{k}$ and $r' = \alpha\cos\psi$, $z' = \alpha\sin\psi$, where primes denote differentiation with respect to ξ and ψ is the tangent angle of the meridian curves. We have then further

$$\alpha_1 = r, \qquad \alpha_2 \equiv \alpha = \sqrt{(r')^2 + (z')^2} \quad (1)$$

and

$$\frac{1}{R_{11}} \equiv \frac{1}{R_\theta} = \frac{\sin\psi}{r}, \qquad \frac{1}{R_{22}} \equiv \frac{1}{R_\xi} = \frac{\psi'}{\alpha}, \qquad \frac{1}{R_{12}} = \frac{1}{R_{21}} = 0 \quad (2)$$

Introduction of (1) and (2) into the general scalar equilibrium and compatability equations, together with the assumption that resultants and couples are independent of θ leads to the following two uncoupled systems of equations

$$\frac{(rN_{\xi\xi})' - r'N_{\theta\theta}}{r\alpha} + \frac{Q_\xi}{R_\xi} + p_\xi = 0, \qquad \frac{(r\kappa_{\theta\theta})' - r'\kappa_{\xi\xi}}{r\alpha} + \frac{\lambda_\theta}{R_\xi} = 0 \tag{3}$$

$$\frac{(rQ_\xi)'}{r\alpha} - \frac{N_{\xi\xi}}{R_\xi} - \frac{N_{\theta\theta}}{R_\theta} + p_n = 0, \qquad \frac{(r\lambda_\theta)'}{r\alpha} - \frac{\kappa_{\theta\theta}}{R_\xi} - \frac{\kappa_{\xi\xi}}{R_\theta} = 0 \tag{4}$$

$$\frac{(rM_{\xi\xi})' - r'M_{\theta\theta}}{r\alpha} - Q_\xi - \frac{P_\theta}{R_\theta} + q_\xi = 0, \qquad \frac{(r\varepsilon_{\theta\theta})' - r'\varepsilon_{\xi\xi}}{r\alpha} + \lambda_\theta - \frac{\gamma_\xi}{R_\theta} = 0 \tag{5}$$

and

$$\frac{(rN_{\xi\theta})' + r'N_{\theta\xi}}{r\alpha} + \frac{Q_\theta}{R_\theta} + p_\theta = 0, \qquad \frac{(r\kappa_{\theta\xi})' + r'\kappa_{\xi\theta}}{r\alpha} + \frac{\lambda_\xi}{R_\theta} = 0 \tag{6}$$

$$\frac{(rM_{\xi\theta})' + r'M_{\theta\xi}}{r\alpha} - Q_\theta + \frac{P_\xi}{R_\xi} + q_\theta = 0, \qquad \frac{(r\varepsilon_{\theta\xi})' + r'\varepsilon_{\xi\theta}}{r\alpha} + \lambda_\xi + \frac{\gamma_\theta}{R_\xi} = 0 \tag{7}$$

$$\frac{(rP_\xi)'}{r\alpha} + N_{\theta\xi} - N_{\xi\theta} + \frac{M_{\theta\xi}}{R_\theta} - \frac{M_{\xi\theta}}{R_\xi} + q_n = 0, \qquad \frac{(r\gamma_\theta)'}{r\alpha} + \kappa_{\theta\xi} - \kappa_{\xi\theta} + \frac{\varepsilon_{\xi\theta}}{R_\theta} - \frac{\varepsilon_{\theta\xi}}{R_\xi} = 0 \tag{8}$$

Evidently, the system (3) to (5) is that which governs the theory of symmetric bending. At the same time the system (6) to (8) is seen to contain those resultants and couples which are expected to occur in problems of torsion and twisting.

We consider as a system of stress strain relations which preserves absence-of-coupling properties and which implies that the shell is isotropic with reference to directions in the tangent planes to the middle surface, the following

$$\varepsilon_{\xi\xi} = A(N_{\xi\xi} - \nu_N N_{\theta\theta}), \qquad \varepsilon_{\theta\theta} = A(N_{\theta\theta} - \nu_N N_{\xi\xi}), \qquad \gamma_\xi = A_Q Q_\xi \tag{9}$$

$$M_{\theta\theta} = D(\kappa_{\theta\theta} + \nu_M \kappa_{\xi\xi}), \qquad M_{\xi\xi} = D(\kappa_{\xi\xi} + \nu_M \kappa_{\theta\theta}), \qquad P_\theta = D_P \lambda_\theta \tag{10}$$

and

$$\varepsilon_{\xi\theta} = A_s[N_{\xi\theta} - \mu_N(N_{\xi\theta} - N_{\theta\xi})], \varepsilon_{\theta\xi} = A_s[N_{\theta\xi} - \mu_N(N_{\theta\xi} - N_{\xi\theta})], \gamma_\theta = A_Q Q_\theta \tag{11}$$

$$M_{\theta\xi} = D_s[\kappa_{\theta\xi} + \mu_M(\kappa_{\theta\xi} - \kappa_{\xi\theta})], M_{\xi\theta} = D_s[\kappa_{\xi\theta} + \mu_M(\kappa_{\xi\theta} - \kappa_{\theta\xi})], P_\xi = D_P \lambda_\xi \tag{12}$$

In this $A_s = (1 + \nu_N)A$ and $D_s = (1 - \nu_M)D$ and μ_N and μ_M are additional parameters, which for uniform homogeneous materials are usually taken to have the values $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Alternately, for shells for which $A_Q = O(A)$ and $D_P = O(D)$ the admissibility of the assumption $\mu_N = \mu_M = 0$ [13, 15], on the basis of the form of equations (8), is a plausible one.

We note that the stress strain relations (9) and (11) are static geometric duals of the relations (10) and (12) if we take A, A_Q, ν_N and μ_N to be the dual quantities for

$-D$, $-D_P - \nu_M$ and $-\mu_M$. This additional duality is helpful in arriving at a final formulation of the two classes of problems in a highly symmetric form.

4. The system of first integrals

The consideration of overall axial force equilibrium implies the following first integral of the bending equilibrium differential equations (3.3) to (3.5),

$$r(N_{\xi\xi}\sin\psi - Q_\xi\cos\psi) + \int(p_\xi\sin\psi - p_n\cos\psi)r\alpha\,d\xi = 0 \qquad (1)$$

It is then a consequence of the static-geometric duality that the compatibility differential equations (3.3) to (3.5) possess a first integral of the form

$$r(\kappa_{\theta\theta}\sin\psi - \lambda_\theta\cos\psi) + c_B = 0 \qquad (2)$$

where c_B is a constant of integration.

The condition of overall equilibrium for moments turning about the axis of revolution of the shell implies the following first integral for the twisting equilibrium differential equations (3.6) to (3.8)

$$r^2N_{\xi\theta} + r(M_{\xi\theta}\sin\psi - P_\xi\cos\psi) + \int[r^2p_\theta + r(q_\theta\sin\psi - q_n\cos\psi)]\alpha\,d\xi = 0 \qquad (3)$$

Again because of the static-geometric duality, equation (3) implies as a first integral of the compatibility differential equations (3.6) to (3.8) the relation

$$r^2\kappa_{\theta\xi} - r(\varepsilon_{\theta\xi}\sin\psi - \gamma_\theta\cos\psi) + c_T = 0 \qquad (4)$$

where c_T is a constant of integration.

Equations (1) to (4) allow the reduction of equations (3.3) to (3.8) together with (3.9) to (3.12) to two separate systems, each consisting of two simultaneous second order differential equations, as will next be shown.

5. Reduction of bending differential equations

The fundamental early reductions of the problem [16], [7] were in terms of the transverse shear stress resultant Q_ξ and of an angular displacement variable ϕ. It was later shown [9] that from the point of view of a continuous transition from shell to plate problems it would be of advantage to use a radial stress resultant H in place of Q_ξ, together with an axial stress resultant V, and that the final results assumed a particularly symmetric form upon writing them in terms of a quantity $\chi = rH$.

In what follows, we will again use the quantity χ. From our present point of view it is particularly significant that the appropriate angular displacement variable ϕ then appears by itself as the static-geometric dual of the variable $-\chi$!

From $H = N_{\xi\xi}\cos\psi + Q_\xi\sin\psi$ and $V = N_{\xi\xi}\sin\psi - Q_\xi\cos\psi$ and from equation (4.1) follows

$$rV = -\int (p_\xi \sin\psi - p_n \cos\psi) r\alpha \, d\xi \equiv -\int p_V r\alpha \, d\xi \tag{1}$$

and

$$rN_{\xi\xi} = \chi \cos\psi + rV \sin\psi, \qquad rQ_\xi = \chi \sin\psi - rV \cos\psi \tag{2}$$

Introduction of (1) and (2) into the force equilibrium equations (3.3) and (3.4) leads to an expression for the remaining stress resultant $N_{\theta\theta}$ (which can be derived somewhat more simply directly, as in [9]) of the form

$$\alpha N_{\theta\theta} = \chi' + r\alpha p_H, \qquad p_H = p_\xi \cos\psi + p_n \sin\psi \tag{3}$$

We now introduce the static geometric dual ϕ of the function $-\chi$ by writing, in analogy to (1) and (2) above

$$r\kappa_{\theta\theta} = \phi \cos\psi - c_B \sin\psi, \qquad r\lambda_\theta = \phi \sin\psi + c_B \cos\psi \tag{4}$$

and note that this, as it must be, is compatible with (4.2).

Having (4) we write further in analogy to (3)

$$\alpha\kappa_{\xi\xi} = \phi' \tag{5}$$

Equation (4) and (5) together may also be considered as direct consequences of the compatibility equations (3.3) and (3.4).

It remains to establish differential equations for the two dependent variables χ and ϕ, and this is accomplished by means of the remaining moment equilibrium and compatibility equations (3.5).

Introducing (4) and (5) into the stress strain relations (3.10) and taking Q_ξ from (2) there follows from equation (3.5) as the first of two simultaneous differential equations for ϕ and χ

$$\begin{aligned} &\frac{1}{\alpha}\left[rD\left(\frac{\phi'}{\alpha} + \nu_M \frac{\phi\cos\psi}{r} - \nu_M \frac{c_B \sin\psi}{r} \right)\right]' \\ &\quad - \cos\psi \left[D\left(\frac{\phi\cos\psi}{r} - \frac{c_B\sin\psi}{r} + \nu_M \frac{\phi'}{\alpha} \right)\right] \\ &\quad - \sin\psi \left[D_P \left(\frac{\phi\sin\psi}{r} + \frac{c_B\cos\psi}{r} \right)\right] - \chi\sin\psi + (rV)\cos\psi + rq_\xi = 0 \end{aligned} \tag{6}$$

The associated second differential equation follows upon substituting for $\varepsilon_{\xi\xi}$, $\varepsilon_{\theta\theta}$ and γ_θ in (3.5) the expressions which result by combining (2) and (3) with the stress strain relations (3.9) and by taking λ_θ from (4)

$$\begin{aligned} &\frac{1}{\alpha}\left[rA\left(\frac{\chi'}{\alpha} + rp_H - \nu_N \frac{\chi\cos\psi}{r} - \nu_N \frac{rV\sin\psi}{r} \right)\right]' \\ &\quad - \cos\psi \left[A\left(\frac{\chi\cos\psi}{r} + \frac{rV\sin\psi}{r} - \nu_N \frac{\chi'}{\alpha} - \nu_N + p_H \right)\right] \\ &\quad - \sin\psi \left[A_Q \left(\frac{\chi\sin\psi}{r} - \frac{rV\cos\psi}{r} \right)\right] + \phi\sin\psi + c_B\cos\psi = 0 \end{aligned} \tag{7}$$

Equations (6) and (7) may be rewritten somewhat more compactly in the form

$$\phi'' + \frac{(Dr/\alpha)'}{(Dr/\alpha)}\phi' - \left[\left(\frac{r'}{r}\right)^2 - \frac{(\nu_M Dr'/\alpha)'}{(Dr/\alpha)} + \frac{D_P}{D}\left(\frac{z'}{r}\right)^2\right]\phi - \frac{z'}{(Dr/\alpha)}\chi$$
$$= -\left[\frac{r'z'}{r^2}\left(1 - \frac{D_P}{D}\right)c_B - \frac{(\nu_M D z' c_B/\alpha)'}{(Dr/\alpha)}\right] - \frac{r'(rV) + \alpha r q_\xi}{(Dr/\alpha)} \tag{8}$$

and

$$\chi'' + \frac{(Ar/\alpha)'}{(Ar/\alpha)}\chi' - \left[\left(\frac{r'}{r}\right)^2 + \frac{(\nu_N Ar'/\alpha)'}{(Ar/\alpha)} + \frac{A_Q}{A}\left(\frac{z'}{r}\right)^2\right]\chi + \frac{z'}{(Ar/\alpha)}\phi$$
$$= \left[\frac{r'z'}{r^2}\left(1 - \frac{A_Q}{A}\right)(rV) + \frac{(\nu_N A z' r V/\alpha)'}{(Ar/\alpha)}\right] - \frac{r' c_B}{(Ar/\alpha)} - \left[\nu_N r' \alpha p_H + \frac{(Ar^2 p_H)'}{(Ar/\alpha)}\right] \tag{9}$$

Equations (8) and (9), upon setting $q_\xi = 0$ and $p_H = 0$, clearly show that they are the static geometric duals of each other. We note in particular the symmetrical and simple appearance of the terms involving A_Q and D_P (the effect of $A_Q \neq 0$ has previously been considered, somewhat less simply, by Naghdi [8]), and that, upon setting $D_P = A_Q = c_B = 0$ and $\nu_N = \nu_M = \nu =$ constant, equations (8) and (9) coincide with equations stated in [9].

6. Reduction of twisting differential equations

Inspection of the form of the first integrals (4.3) and (4.4) together with the equilibrium and compatibility equations (3.6) suggests to write these equations with the help of the stress strain relations (3.11) and (3.12) in terms of quantities

$$\chi_T = rN_{\xi\theta}, \qquad \phi_T = r\kappa_{\theta\xi}, \qquad T = \int (r^2 p_\theta + r q_\theta \sin\psi - r q_n \cos\psi)\alpha \, d\xi \tag{1}$$

in the form

$$N_{\theta\xi}\cos\psi + Q_\theta \sin\psi = -\frac{\chi_T'}{\alpha} - rp_\theta \tag{2}$$

$$(1 - \mu_N)A_S N_{\theta\xi}\sin\psi - A_Q Q_\theta \cos\psi = \phi_T + \frac{c_T}{r} - \mu_N A_S \sin\psi\,\frac{\chi_T}{r} \tag{3}$$

$$\kappa_{\xi\theta}\cos\psi + \lambda_\xi \sin\psi = -\frac{\phi_T'}{\alpha} \tag{4}$$

$$(1 + \mu_M)D_S\kappa_{\xi\theta}\sin\psi - D_P\lambda_\xi\cos\xi = -\chi_T - \frac{T}{r} + \mu_M D_S \sin\psi\,\frac{\phi_T}{r} \tag{5}$$

From this

$$A^* N_{\theta\xi} = \left(\phi_T + \frac{c_T}{r}\right)\sin\psi - \left[A_Q\cos\psi\left(\frac{\chi_T'}{\alpha} + rp_\theta\right) + \mu_N A_S \sin\psi\,\frac{\chi_T}{R_\theta}\right] \tag{6}$$

$$A^* Q_\theta = -\left(\phi_T + \frac{c_T}{r}\right)\cos\psi - A_S\left[(1 - \mu_N)\sin\psi\left(\frac{\chi_T'}{\alpha} + rp_\theta\right) - \mu_N \cos\psi\,\frac{\chi_T}{R_\theta}\right] \tag{7}$$

and

$$D^*\kappa_{\xi\theta} = -\left(\chi_T + \frac{T}{r}\right)\sin\psi - \left[D_P\cos\psi\,\frac{\phi_T'}{\alpha} - \mu_M D_S \sin\psi\,\frac{\phi_T}{R_\theta}\right] \tag{8}$$

$$D^*\lambda_\xi = \left(\chi_T + \frac{T}{r}\right)\cos\psi - D_s\left[(1+\mu_M)\sin\psi\,\frac{\phi_T'}{\alpha} + \mu_M\cos\psi\,\frac{\phi_T}{R_\theta}\right] \tag{9}$$

where

$$A^* = (1-\mu_N)A_S\sin^2\psi + A_Q\cos^2\psi, \qquad D^* = (1+\mu_M)D_S\sin^2\psi + D_P\cos^2\psi \tag{10}$$

Equations (1) together with (6) to (9) are taken in conjunction with the remaining equilibrium and compatibility equations (3.8). Writing the latter in the form

$$\frac{(rD_P\lambda_\xi)'}{r\alpha} + D_S\left[\left(\frac{1+\mu_M}{R_\theta} + \frac{\mu_M}{R_\xi}\right)\kappa_{\theta\xi} - \left(\frac{1+\mu_M}{R_\xi} + \frac{\mu_M}{R_\theta}\right)\kappa_{\xi\theta}\right] + N_{\theta\xi} - N_{\xi\theta} + q_n = 0 \tag{11}$$

$$\frac{(rA_QQ_\theta)'}{r\alpha} + A_S\left[\left(\frac{1-\mu_N}{R_\theta} - \frac{\mu_N}{R_\xi}\right)N_{\xi\theta} - \left(\frac{1-\mu_N}{R_\xi} - \frac{\mu_N}{R_\theta}\right)N_{\theta\xi}\right] - \kappa_{\xi\theta} + \kappa_{\theta\xi} = 0 \tag{12}$$

it is apparent that (11) and (12) are in effect two simultaneous second order differential equations for χ_T and ϕ_T. We note that χ_T and ϕ_T are static geometric duals and that the final differential equations, except for the terms with p_θ, q_θ and q_n, are also static geometric duals.

In contrast to what is encountered in dealing with the problem of bending, the assumptions $A_Q = D_P = 0$ lead to a significant change of character of the problem of twisting. When $A_Q = D_P = 0$ we have from (6), (8), (11) and (12) as a system of zeroth order differential equations for χ_T and ϕ_T

$$\left[(1+\mu_M) + (1-\mu_N)(1+2\mu_M)\frac{D_SA_S}{R_\theta^2}\right]\phi_T - \left[\frac{1+\mu_M\mu_N}{R_\theta} - \frac{(1+\mu_M)(1-\mu_N)}{R_\xi}\right]A_S\chi_T$$
$$+ (1+\mu_M)\frac{c_T}{r} + (1-\mu_N)\left(\frac{\mu_M}{R_\theta} + \frac{1+\mu_M}{R_\xi}\right)\frac{A_ST}{r} + q_n = 0 \tag{13}$$

$$\left[(1-\mu_N) + (1+\mu_M)(1-2\mu_N)\frac{D_SA_S}{R_\theta^2}\right]\chi_T + \left[\frac{1+\mu_M\mu_N}{R_\theta} - \frac{(1+\mu_M)(1-\mu_N)}{R_\xi}\right]D_S\phi_T$$
$$+ (1-\mu_N)\frac{T}{r} + (1+\mu_M)\left(\frac{\mu_N}{R_\theta} - \frac{1-\mu_N}{R_\xi}\right)\frac{D_Sc_T}{r} = 0 \tag{14}$$

Further specialization, by assuming $D_S = 0$ and $q_n = 0$, leads to relations of the form

$$\chi_T + \frac{T}{r} = 0, \qquad \phi_T + \frac{c_T}{r} = A_S\sin\psi\,\frac{\chi_T}{r} \tag{15}$$

from which may be deduced the previously known membrane-theoretical solution of the problem of torsion [6] and of the problem of pure twist of closed-cross-section tubes [11].

7. Determination of displacements

Setting in equation (2.3) $\mathbf{u} = u_\xi \mathbf{t}_\xi + u_\theta \mathbf{t}_\theta + w \mathbf{n}$ and $\boldsymbol{\phi} = \mathbf{n} \times (\phi_\xi \mathbf{t}_\xi + \phi_\theta \mathbf{t}_\theta) + \omega \mathbf{n}$ leads to the scalar strain displacement relations

$$\varepsilon_{\theta\theta} = \frac{u_\theta^\bullet}{r} + \frac{r' u_\xi}{r\alpha} + \frac{w}{R_\theta}, \qquad \varepsilon_{\xi\xi} = \frac{u_\xi'}{\alpha} + \frac{w}{R_\xi}, \qquad \gamma_\xi = \phi_\xi + \frac{w'}{\alpha} - \frac{u_\xi}{R_\xi} \tag{1}$$

$$\kappa_{\theta\theta} = \frac{\phi_\theta^\bullet}{r} + \frac{r'\phi_\xi}{r\alpha}, \qquad \kappa_{\xi\xi} = \frac{\phi_\xi'}{\alpha}, \qquad \lambda_\theta = \frac{\omega^\bullet}{r} + \frac{\phi_\xi}{R_\theta} \tag{2}$$

and

$$\varepsilon_{\theta\xi} = \frac{u_\xi^\bullet}{r} - \frac{r' u_\theta}{r\alpha} - \omega, \qquad \varepsilon_{\xi\theta} = \frac{u_\theta'}{\alpha} + \omega, \qquad \gamma_\theta = \phi_\theta + \frac{w^\bullet}{r} - \frac{u_\theta}{R_\theta} \tag{3}$$

$$\kappa_{\theta\xi} = \frac{\phi_\xi^\bullet}{r} - \frac{r'\phi_\theta}{r\alpha} - \frac{\omega}{R_\theta}, \qquad \kappa_{\xi\theta} = \frac{\phi_\theta'}{\alpha} + \frac{\omega}{R_\xi}, \qquad \lambda_\xi = \frac{\omega'}{\alpha} - \frac{\phi_\theta}{R_\xi} \tag{4}$$

in which dots indicate differentiation with respect to θ.

It may be seen from (1) to (4) that rotationally symmetric strains for problems of bending are associated with displacement states

$$(u_\xi, w, \phi_\xi) = (u_\xi, w, \phi_\xi)(\xi), \qquad (u_\theta, \phi_\theta, \omega) = (f_u, f_\phi, f_\omega)(\xi)\,\theta \tag{5}$$

while rotationally symmetric strains for twisting are associated with displacement states

$$(u_\xi, w, \phi_\xi) = (g_u, g_w, g_\phi)(\xi)\,\theta, \qquad (u_\theta, \phi_\theta, \omega) = (u_\theta, \phi_\theta, \omega)(\xi) \tag{6}$$

Introduction of (5) and (6) into (1) to (4) and observation of the integrated compatibility equations (4.2) and (4.4) leads to the conclusion that the most general non-rotationally symmetric displacement contributions in (5) and (6) are, respectively

$$u_\theta = -c_B \theta\, r, \qquad \phi_\theta = -c_B \theta \sin\psi, \qquad \omega = c_B \theta \cos\psi \tag{7}$$

and

$$u_\xi = c_T \theta \sin\psi, \qquad w = -c_T \theta \cos\psi, \qquad \phi_\xi = 0 \tag{8}$$

Introduction of (5) and (7) into (1) to (4) gives as expressions for strain resultants and couples of the problem of bending

$$\varepsilon_{\theta\theta} = \frac{u_\xi \cos\psi + w \sin\psi}{r} - c_B, \qquad \varepsilon_{\xi\xi} = \frac{u_\xi' + \psi' w}{\alpha} \tag{9}$$

$$\gamma_\xi = \frac{w' - \psi' u_\xi}{\alpha} + \phi_\xi \qquad \lambda_\theta = \frac{\phi_\xi \sin\psi + c_B \cos\psi}{r} \tag{10}$$

$$\kappa_{\theta\theta} = \frac{\phi_\xi \cos\psi - c_B \sin\psi}{r} \qquad \kappa_{\xi\xi} = \frac{\phi_\xi'}{\alpha} \tag{11}$$

A comparison of (10) and (11) with (5.4) and (5.5) indicates that the quantity ϕ which was introduced in (5.5) and (5.6) as geometric dual to the stress function $-\chi$ is in fact the angular displacement component ϕ_ξ.*

It is seen that for the problem of bending the angular displacement components ϕ_ξ, ϕ_θ and ω, as well as the translational displacement component u_θ, are directly determined through the solution of the two simultaneous differential equations (5.8) and (5.9). Determination of the translational components u_ξ and w is effected by expressing the components of strain $\varepsilon_{\theta\theta}$, $\varepsilon_{\xi\xi}$ and γ_ξ in terms of the stress function χ by means of the stress strain relations (3.9). In this connection it is convenient to replace u_ξ and w by radial and axial displacement components u_H and u_V [9]. With

$$u_H = u_\xi \cos\psi + w\sin\psi, \qquad u_V = u_\xi \sin\psi - w\cos\psi \tag{12}$$

there follow from (9) and (10) as expressions for u_H and u_V

$$\frac{u_H}{r} = \varepsilon_{\theta\theta} + c_B, \qquad \frac{u_V'}{\alpha} = \varepsilon_{\xi\xi}\sin\psi + (\phi_\xi - \gamma_\xi)\cos\psi \tag{13}$$

Strain resultants and couples for the problem of twisting follow from (8), (6), (4) and (3) in the form

$$\varepsilon_{\theta\xi} = \frac{c_T \sin\psi - u_\theta\cos\psi}{r} - \omega, \qquad \varepsilon_{\xi\theta} = \frac{u_\theta'}{\alpha} + \omega \tag{14}$$

$$\gamma_\theta = \phi_\theta - \frac{c_T\cos\psi + u_\theta\sin\psi}{r}, \qquad \lambda_\xi = \frac{\omega'}{\alpha} \tag{15}$$

$$\kappa_{\theta\xi} = -\frac{\phi_\theta\cos\psi + \omega\sin\psi}{r}, \qquad \kappa_{\xi\theta} = \frac{\phi_\theta' + \psi'\omega}{\alpha} \tag{16}$$

Introduction of u_ξ and w from (8) into equations (12) gives

$$u_H = 0, \qquad u_V = c_T\theta \tag{17}$$

which makes clear the geometrical significance of the constant c_T.

Determination of the warping displacement u_θ in terms of the solutions χ_T and ϕ_T of the differential equations of the twisting problem is effected by integrating the relation

$$\frac{u_\theta'}{\alpha} - \cos\psi\frac{u_\theta}{r} = \frac{r}{\alpha}\left(\frac{u_\theta}{r}\right)' = \varepsilon_{\xi\theta} + \varepsilon_{\theta\xi} - c_T\frac{\sin\psi}{r} \tag{18}$$

where $\varepsilon_{\xi\theta}$ and $\varepsilon_{\theta\xi}$ are expressed in terms of χ_T and ϕ_T through equations (3.11), (6.7) and (6.8). Thereafter ω and ϕ_θ may be obtained in the form $\omega = \varepsilon_{\xi\theta} - u_\theta'/\alpha$ and $\phi_\theta = \gamma_\theta + (c_T\cos\psi + u_\theta\sin\psi)/r$.

Equation (18) implies for *closed-cross-section shells*, for which u_θ must be a

* Equations (9) to (11) supplement earlier results of one of the authors [10] insofar as in these earlier results the term with c_B in $\kappa_{\theta\theta}$ was absent. This term is insignificant for the bending of tubes with which the earlier work was primarily concerned but must be considered for such problems as pure bending of radially slit conical shells.

univalued function of ξ, the important relation

$$\oint \left(\frac{\varepsilon_{\xi\theta} + \varepsilon_{\theta\xi}}{r} - c_T \frac{\sin\psi}{r^2} \right) \alpha \, d\xi = 0 \tag{19}$$

8. Uniform spherical shell under point loads at its poles

Explicit solutions of this problem, under the assumption $A_Q = D_P = 0$, have recently been given by Koiter [4] and Simmonds [17]. Both these derivations start out with a formulation of the problem of unsymmetrical deformations of spherical shells. In what follows, an explicit solution is obtained somewhat more simply, and with the added generality of nonvanishing A_Q and D_P, through the use of the symmetric bending differential equations (5.8) and (5.9).

We define the spherical shell through the relations $r = a \sin \xi$ and $z = a \cos \xi$ making $\alpha = a$. We assume as loads equal and opposite concentrated forces F acting normal to the shell surface at the poles $\xi = 0$ and $\xi = \pi$. Setting

$$2\pi r V = F, \qquad p_H = c_B = q_\xi = 0 \tag{1}$$

and further assuming constant D, D_P, A, A_Q, ν_M and ν_N, the differential equations (5.8) and (5.9) reduce to

$$\phi'' + \cot\xi\,\phi' - \left[\csc^2\xi - \left(1 - \nu_M - \frac{D_P}{D}\right)\right]\phi + \frac{a}{D}\chi = -\frac{Fa}{2\pi D}\cot\xi \tag{2}$$

$$\chi'' + \cot\xi\,\chi' - \left[\csc^2\xi - \left(1 + \nu_N - \frac{A_Q}{A}\right)\right]\chi - \frac{a}{A}\phi = -\frac{F}{2\pi}\left(1 + \nu_N - \frac{A_Q}{A}\right)\cot\xi \tag{3}$$

Expressions for stress resultants and stress couples corresponding to the above follow from equations (5.2) to (5.5) together with (3.10) as

$$N_{\theta\theta} = \frac{\chi'}{a}, \qquad N_{\xi\xi} = \cot\xi\,\frac{\chi}{a} - \frac{F}{2\pi a}, \qquad Q_\xi = \frac{\chi}{a} + \cot\xi\,\frac{F}{2\pi a} \tag{4}$$

$$M_{\xi\xi} = -\frac{D}{a}(\phi' + \nu_M \cot\xi\,\phi), \qquad M_{\theta\theta} = -\frac{D}{a}(\nu_M\phi' + \cot\xi\,\phi), \qquad P_\theta = \frac{D_P}{a}\phi \tag{5}$$

We take as boundary conditions for the solution of the system (2) and (3), which is to be continuous in the interval $0 < \xi < \pi$, the symmetry conditions of vanishing meridional displacements *throughout the shell thickness* at $\xi = 0$ and $\xi = \pi$. On the basis of the considerations in Section 7 this means that u_ξ and ϕ_ξ are assumed to vanish for $\xi = 0$ and $\xi = \pi$. According to equations (7.9) and (7.11) these conditions are equivalent to the conditions of vanishing $r\varepsilon_{\theta\theta}$ and $r\kappa_{\theta\theta}$ for $\xi = 0$ and $\xi = \pi$. Expressed in terms of the variables ϕ and χ we have then as boundary conditions

$$\xi = 0, \pi: \qquad \phi = 0, \qquad \chi' \sin\xi - \nu_N\,\chi = 0 \tag{6}$$

Alternately we may, because of symmetry, limit ourselves to a consideration of the interval $0 \le \xi \le \frac{1}{2}\pi$, with the conditions at $\xi = 0$ as above and with the conditions of vanishing ϕ_ξ and Q_ξ for $\xi = \pi/2$, i.e.

$$\xi = \frac{\pi}{2}: \qquad \phi = 0, \qquad \chi = 0 \tag{7}$$

For the solution of the system (2) and (3), it is convenient to recognize the existence of the closed-form particular solution

$$\phi_p = 0, \qquad \chi_p = -\frac{F}{2\pi}\cot\xi \tag{8}$$

(corresponding to the membrane theory solution of the point load problem), and to rewrite the homogeneous system as a complex differential equation

$$\Psi_h'' + \cot\xi\,\Psi_h' + [n(n+1) - \csc^2\xi]\,\Psi_h = 0 \tag{9}$$

where

$$\Psi_h = \phi_h + \rho\,\chi_h \tag{10}$$

and where

$$\rho = i\sqrt{\frac{A}{D}}\sqrt{1 - \frac{DA}{4a^2}\left(\nu_M + \nu_N + \frac{D_P}{D} - \frac{A_Q}{A}\right)^2} - \frac{A}{2a}\left(\nu_M + \nu_N + \frac{D_P}{D} - \frac{A_Q}{A}\right) \tag{11}$$

and

$$n(n+1) = -i\frac{a}{\sqrt{DA}}\sqrt{1 - \frac{DA}{4a^2}\left(\nu_M + \nu_N + \frac{D_P}{D} - \frac{A_Q}{A}\right)^2} + \left(1 + \frac{\nu_N - \nu_M}{2} - \frac{D_P}{2D} - \frac{A_Q}{2A}\right) \tag{12}$$

The two linearly independent solutions of (9) are known to be the associated Legendre functions $P_n^1(\cos\xi)$ and $Q_n^1(\cos\xi)$. For simplicity's sake we shall limit ourselves to what may be called the usual situation that is to $Im(\rho) \neq 0$.

The general solution of the system (2) and (3) may then be written in the form

$$\Psi = \phi + \rho\chi = \frac{\rho F}{2\pi}[C_1 P_n^1(\cos\xi) + C_2 Q_n^1(\cos\xi) - \cot\xi] \tag{13}$$

where C_1 and C_2 are complex constants of integration.

For the evaluation of these constants of integration, we make use of the following relations for associated Legendre functions [1]

$$P_n^1(0) = -\frac{2\sin\frac{1}{2}n\pi}{\sqrt{\pi}}\frac{\Gamma(1+\frac{1}{2}n)}{\Gamma(\frac{1}{2}+\frac{1}{2}n)}, \qquad Q_n^1(0) = -\sqrt{\pi}\cos\tfrac{1}{2}n\pi\frac{\Gamma(1+\frac{1}{2}n)}{\Gamma(\frac{1}{2}+\frac{1}{2}n)} \tag{14}$$

and [5]

$$P_n^1(\cos\xi) \sim -\tfrac{1}{2}n(n+1)\tan\tfrac{1}{2}\xi\,[1 + O(\sin^2\xi)]$$

$$Q_n^1(\cos\xi) \sim -\frac{1}{\sin\xi}[1 + n(n+1)\sin^2\xi\log(\tan\tfrac{1}{2}\xi)][1 + O(\sin^2\xi)] \tag{15}$$

for $\xi \ll \pi/2$.

Introduction of the boundary conditions (7) into (13) and observation of (14) gives

$$C_1 = -\tfrac{1}{2}\pi C_2 \cot \tfrac{1}{2}n\pi \tag{16}$$

Inspection of (15) indicates that the boundary conditions (6) for $\xi = 0$ are effectively the same as the one complex condition

$$\xi = 0: \qquad \Psi = 0 \tag{17}$$

Consequently, equations (13), (16) and (17) imply the simple relation

$$C_2 = 1 \tag{18}$$

and therewith altogether

$$\Psi = \frac{\rho F}{2\pi}\left[\frac{\pi}{2}\cot\frac{n\pi}{2}P_n^1(\cos\xi) - Q_n^1(\cos\xi) - \cot\xi\right] \tag{19}$$

For sufficiently small values of ξ, we have

$$\Psi \sim \frac{\rho n(n+1)F}{2\pi}\sin\xi\log(\tan\tfrac{1}{2}\xi) \tag{20}$$

and correspondingly, with $\chi = \mathrm{Im}(\Psi)/\mathrm{Im}(\rho)$ and $\phi = \mathrm{Re}(\Psi) - \mathrm{Re}(\rho)\chi$,

$$\phi \sim \frac{Fa}{2\pi D}\log(\tan\tfrac{1}{2}\xi) \qquad \chi \sim \frac{F}{2\pi}\left(1 + \nu_N - \frac{A_Q}{A}\right)\log(\tan\tfrac{1}{2}\xi) \tag{21}$$

From these, we obtain the following behavior of the stress resultants and stress couples near the pole $\xi = 0$

$$N_{\theta\theta} \sim N_{\xi\xi} \sim \frac{F}{2\pi a}\left(1 + \nu_N - \frac{A_Q}{A}\right)\log(\tan\tfrac{1}{2}\xi), \qquad Q_\xi \sim \frac{F}{2\pi a}\cot\xi$$
$$M_{\xi\xi} \sim M_{\theta\theta} \sim -\frac{F}{2\pi}(1 + \nu_M)\log(\tan\tfrac{1}{2}\xi), \qquad P_\theta \sim \frac{FD_P}{2\pi D}\sin\xi\log(\tan\tfrac{1}{2}\xi) \tag{22}$$

In these, we observe that neither A_Q nor D_P occurs in $M_{\xi\xi}$ and $M_{\theta\theta}$ and that $N_{\xi\xi}$ and $N_{\theta\theta}$ are free of D_P.

We further note that (22) implies, in agreement with Koiter [4], that near the points of load application the stresses $\sigma_D = N/h$ are small, of relative order h/a, compared to the stresses $\sigma_B = 6M/h^2$.

9. Pure twisting of meriodionally nonhomogeneous closed cross section tubes without transverse shear deformation and moment stress stress couples

We take account of the fact that meridional non-homogeneity may include, as a limiting case, absence of stiffness over portions of the meridional curves and thereby treat open-cross-section tubes as special cases of meridionally non-homogeneous closed-cross-section tubes. As for open cross sections the effect of the couples $M_{\xi\theta}$ and $M_{\theta\xi}$ will dominate, while for closed cross sections the effect of the resultants $N_{\xi\theta}$ and $N_{\theta\xi}$ will ordinarily dominate, it is indicated that for the present treatment both couples and resultants should be retained.

For the purpose of what follows, it is convenient to rewrite the zeroth order twisting differential equation (6.13) and (6.14), with $p_\theta = q_\theta = q_n = 0$, in terms of $N_{\xi\theta}$ and $\kappa_{\theta\xi}$.

We have from (4.3) and (4.4), when $P_\xi = \gamma_\theta = 0$,

$$M_{\xi\theta} = -R_\theta\left(N_{\xi\theta} + \frac{T}{r^2}\right), \qquad \varepsilon_{\theta\xi} = R_\theta\left(\kappa_{\theta\xi} + \frac{c_T}{r^2}\right) \tag{1}$$

With this, the stress strain relations (3.11) and (3.12) give us

$$(1 - \mu_N)N_{\theta\xi} = \frac{R_\theta}{A_S}\kappa_{\theta\xi} - \mu_N N_{\xi\theta} + \frac{c_T R_\theta}{A_S r^2},$$

$$(1 + \mu_M)\kappa_{\xi\theta} = -\frac{R_\theta}{D_S}N_{\xi\theta} + \mu_M\kappa_{\theta\xi} - \frac{TR_\theta}{D_S r^2} \tag{2}$$

$$(1 + \mu_M)M_{\theta\xi} = (1 + 2\mu_M)D_S\kappa_{\theta\xi} + \mu_M R_\theta N_{\xi\theta} + \mu_M\frac{TR_\theta}{r^2},$$

$$(1 - \mu_N)\varepsilon_{\xi\theta} = (1 - 2\mu_N)A_S N_{\xi\theta} + \mu_N R_\theta\kappa_{\theta\xi} + \mu_N\frac{c_T R_\theta}{r^2} \tag{3}$$

Having (1), (2) and (3), we now write the equilibrium and compatibility equations (3.8) (with $\gamma_\theta = P_\xi = q_n = 0$) as two equations for $N_{\xi\theta}$ and $\kappa_{\theta\xi}$:

$$\left[(1 + \mu_M) + (1 - \mu_N)(1 + 2\mu_M)\frac{D_S A_S}{R_\theta^2}\right]\kappa_{\theta\xi} - A_S\left[\frac{1 + \mu_M\mu_N}{R_\theta} - \frac{(1 + \mu_M)(1 - \mu_N)}{R_\xi}\right]N_{\xi\theta} = -(1 + \mu_M)\frac{c_T}{r^2} - (1 - \mu_N)\frac{A_S T}{r^2}\left(\frac{\mu_M}{R_\theta} + \frac{1 + \mu_M}{R_\xi}\right) \tag{4}$$

$$\left[(1 - \mu_N) + (1 + \mu_M)(1 - 2\mu_N)\frac{D_S A_S}{R_\theta^2}\right]N_{\xi\theta} + D_S\left[\frac{1 + \mu_M\mu_N}{R_\theta} - \frac{(1 + \mu_M)(1 - \mu_N)}{R_\xi}\right]\kappa_{\theta\xi} = -(1 + \mu_M)\frac{D_S c_T}{r^2}\left(\frac{\mu_N}{R_\theta} - \frac{1 - \mu_N}{R_\xi}\right) - (1 - \mu_N)\frac{T}{r^2} \tag{5}$$

We omit listing the exact solution of (4) and (5). Instead, we state the effective solution which takes account of the fact that $A_S D_S R_\theta^2 \ll 1$ and which retains dominant terms coming from both c_T and T,

$$N_{\xi\theta} = \frac{D_S c_T}{r^2 R_\theta} - \frac{T}{r^2}, \qquad N_{\theta\xi} = \frac{D_S c_T}{r^2}\left(\frac{2}{R_\theta} - \frac{1}{R_\xi}\right) - \frac{T}{r^2} \tag{6}$$

$$M_{\xi\theta} = -\frac{D_S c_T}{r^2} - \frac{D_S A_S T}{r^2}\left(\frac{2+\mu_M}{R_\theta} - \frac{1+\mu_M}{R_\xi}\right)$$

$$M_{\theta\xi} = -\frac{D_S c_T}{r^2} - \frac{D_S A_S T}{r^2}\left(\frac{1-\mu_M}{R_\theta} + \frac{\mu_M}{R_\xi}\right) \tag{7}$$

The corresponding expression for Q_θ is obtained from the moment equilibrium equation (3.7) in the form

$$Q_\theta = -\frac{c_T}{r\alpha}\left\{\left(\frac{D_S}{r}\right)' + \frac{r' D_S}{r^2}\right\}$$
$$-\frac{T}{r\alpha}\left\{\left[\frac{D_S A_S}{r}\left(\frac{2+\mu_M}{R_\theta} - \frac{1+\mu_M}{R_\xi}\right)\right]' + \frac{D_S A_S r'}{r^2}\left[\frac{1-\mu_M}{R_\theta} + \frac{\mu_M}{R_\xi}\right]\right\} \tag{8}$$

It remains to express the constants c_T and T in terms of the applied axial load F which is given by the integral

$$F = \oint (N_{\theta\xi}\sin\psi - Q_\theta\cos\psi)\alpha\, d\xi \tag{9}$$

For this purpose we have in addition to equation (9) the univaluedness condition (7.19), expressed in terms of stress resultants,

$$\oint \left(A_S \frac{N_{\xi\theta} + N_{\theta\xi}}{r} - c_T\frac{\sin\psi}{r^2}\right)\alpha\, d\xi = 0 \tag{10}$$

Considering the form of the second term in the integral in (10), we find that insofar as (10) is concerned, we may replace $N_{\xi\theta}$ and $N_{\theta\xi}$ given by (6) by the still simpler relations $N_{\xi\theta} = N_{\theta\xi} = -T/r^2$. With this we have from (10) as a relation between T and c_T

$$T\oint \frac{2A_S}{r^3}\alpha\, d\xi + c_T\oint\frac{\sin\psi}{r^2}\alpha\, d\xi = 0 \tag{11}$$

Taking $N_{\theta\xi}$ and Q_θ from the equilibrium equations (3.8) and (3.7) one obtains, after rearrangements, an integration by parts, and observation of the first-integral relation $r^2 N_{\xi\theta} + rM_{\xi\theta}\sin\psi + T = 0$, that F may be written in the alternate form

$$F = -\oint\left\{\frac{M_{\xi\theta} + M_{\theta\xi}}{r} + T\frac{\sin\psi}{r^2}\right\}\alpha\, d\xi \tag{12}$$

In evaluating (12) one finds that now $M_{\xi\theta}$ and $M_{\theta\xi}$ in (7) may be replaced by the simpler statement $M_{\xi\theta} = M_{\theta\xi} = -D_S c_T/r^2$. Therewith equation (12) becomes

$$F = c_T\oint\frac{2D_S}{r^3}\alpha\, d\xi - T\oint\frac{\sin\psi}{r^2}\alpha\, d\xi \tag{13}$$

Eliminating T by means of (11) we finally have as relation between the axial force F and the axial displacement component $c_T\theta$,

$$\frac{F}{c_T} = \oint\frac{2D_S}{r^3}\alpha\, d\xi + \left(\oint\frac{\sin\psi}{r^2}\alpha\, d\xi\right)^2 \bigg/ \oint\frac{2A_S}{r^3}\alpha\, d\xi \tag{14}$$

Upon setting $D_s = 0$ equation (14) reduces to the previously known result of membrane theory [11].

Alternatively, the appropriate result for *open* cross sections is obtained as follows as a special case of (14). An open cross section may be taken to be equivalent to a closed cross section for which $A_S = \infty$ over a portion of the path of integration in (11) to (14). Consequently, for an open cross-section equations (11) and (12) reduce to

$$T = 0, \qquad F = c_T \int \frac{2D_S}{r^3} \alpha \, d\xi \tag{15}$$

the latter relation including a previously known result of plate theory [9].

We finally note that the terms with μ_M in (7) are actually negligible. They are dominated in the load deflection relation (12) by the explicitly appearing T-term, while insofar as the distribution of stress is concerned the effect of the terms with T in $M_{\xi\theta}$ and $M_{\theta\xi}$ is small compared to the effect of the terms with T in $N_{\xi\theta}$ and $N_{\theta\xi}$. Altogether, the calculations are in agreement with the general conclusion stated earlier that for sufficiently small values of A_Q and D_P the results, effectively, do not depend on the assumed values of μ_M and μ_N.

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