F. Y. M. WAN
Assistant Professor,
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, Mass. Assoc. Mem. ASME

The Side-Force Problem for Shallow Helicoidal Shells[1]

The side-force problem for a shallow helicoidal shell is shown to be the complete static-geometric analog of the pure bending problem for the same shell. The solution of the former in terms of elementary functions is obtained, without another set of independent calculations, simply by applying the rules of the static-geometric duality to that of the latter. The analogy also enables us to use the same computer program developed for the pure bending problem (without any modification) to generate numerical results for the side-force problem.

Introduction

In cylindrical coordinates (r, θ, z), the middle surface of a helicoidal shell is given by the equation $z = a\theta$ where the constant $2\pi a$ is the pitch of the helicoid. The present work is concerned with a shell bounded by $r = r_i$, $r = r_0$ (with $r_0 > r_i > 0$), and $\theta = \pm\theta_0$. The shell is clamped to a fixed rigid cylinder at the outer radial edge, r_0, and to a movable rigid cylinder at the inner radial edge, r_i. The inner cylinder is displaced laterally, an amount δ_x in the x-direction ($\theta = 0$). To be determined is the side force per unit winding P_x which produces the lateral displacement of the inner cylinder as well as the resulting stress distributions in the shell.

While several methods of solution are possible for this problem, our approach takes advantage of the idea of a static-geometric duality for shells as discussed in [1][2] and elsewhere. We first observe that the side-force problem is the static-geometric dual of the problem of pure bending studied in [6]. Solutions in terms of elementary functions for the pure bending problem have been obtained for shells with a large [2] and small [7] pitch-to-width ratio for which the Marguerre shallow shell equations are applicable. For shells with a small pitch, the solution is still sufficiently complicated that a computer program was developed to calculate the influence coefficient and the stress distributions. In what follows, we confine ourselves to this range of pitch-to-width ratios and obtain without another set of independent calculations, the solution of the side-force problem simply by translating the solution of the pure bending problem given in [7] according to the rules of static-geometric duality. Moreover, this same duality enables us to use (without any modification!) the computer program developed for the pure bending problem to generate the desired numerical results for the side-force problem. To the knowledge of this author, no such computational use of the static-geometric duality has appeared in the literature.

The limiting case of a ring plate (a shell with zero pitch) of our problem was discussed in [5]. Our results show that when the pitch of the shell is large compared to its thickness, the behavior of the structure differs significantly from that of a flat plate, and except for edge effects, is asymptotic to that of a membrane theory.

Formulation

The governing differential equations of a linear theory of shallow helicoidal shells may be found in [3]. In this theory the curvature change measures κ_θ, κ_r, and $\kappa_{r\theta} = \kappa_{\theta r}$ are given in terms of the axial middle surface displacement component w by

$$\kappa_\theta = -(r^{-1}w' + r^{-2}w^{\cdot\cdot}), \qquad \kappa_r = -w'', \qquad \kappa_{\theta r} = \kappa_{r\theta} = -(r^{-1}w^{\cdot})' \tag{1}$$

where primes and dots indicate differentiation with respect to r and θ, respectively. The stress couples M_θ, M_r, and $M_{r\theta} = M_{\theta r}$ are related to w through the stress-strain relations

$$\begin{aligned} M_\theta &= D(\kappa_\theta + \nu_b\kappa_r) = -D[r^{-1}w' + r^{-2}w^{\cdot\cdot} + \nu_b w''] \\ M_r &= D(\kappa_r + \nu_b\kappa_\theta) = -D[w'' + \nu_b(r^{-1}w' + r^{-2}w^{\cdot\cdot})] \\ M_{\theta r} &= M_{r\theta} = D(1-\nu_b)\kappa_{\theta r} = -D(1-\nu_b)(r^{-1}w^{\cdot})' \end{aligned} \tag{2}$$

where D is the bending stiffness of the shell and ν_b is the corresponding effective Poisson's ratio. For moment equilibrium, we have the following expressions for the transverse shear resultants,

$$Q_r = -D(\nabla^2 w)', \qquad Q_\theta = -Dr^{-1}(\nabla^2 w)^{\cdot} \tag{3}$$

where $\nabla^2(\ ) = (\ )'' + r^{-1}(\ )' + r^{-2}(\ )^{\cdot\cdot}$.

In the absence of surface loads, the in-plane stress resultants N_r, N_θ, and $N_{r\theta} = N_{\theta r}$ are given in terms of a stress function F by

$$N_r = r^{-1}F' + r^{-2}F^{\cdot\cdot}, \quad N_\theta = F'', \quad N_{r\theta} = N_{\theta r} = -(r^{-1}F^{\cdot})' \tag{4}$$

The in-plane strain measures are related to F by

$$\begin{aligned} \epsilon_r &= A(N_r - \nu_s N_\theta) = A[r^{-1}F' + r^{-2}F^{\cdot\cdot} - \nu_s F''] \\ \epsilon_\theta &= A(N_\theta - \nu_s N_r) = A[F'' - \nu_s(r^{-1}F' + r^{-2}F^{\cdot\cdot})] \\ \epsilon_{r\theta} &= \epsilon_{\theta r} = A(1+\nu_s)N_{r\theta} = -A(1+\nu_s)(r^{-1}F^{\cdot})' \end{aligned} \tag{5}$$

where $1/A$ is the stretching stiffness and ν_s is the corresponding effective Poisson's ratio.

In addition, we introduce two new quantities λ_r and λ_θ through

$$\begin{aligned} \lambda_\theta &= r^{-1}[(r\epsilon_\theta)' - \epsilon_{r\theta}^{\cdot} - \epsilon_r] = A(\nabla^2 F)', \\ \lambda_r &= r^{-1}[(r\epsilon_{\theta r})' - \epsilon_r^{\cdot} + \epsilon_{r\theta}] = -Ar^{-1}(\nabla^2 F)^{\cdot}. \end{aligned} \tag{6}$$

In the generalized theory of shallow shells [4], they have the meaning of the normal components of the curvature change vectors.

---

[1] Preparation of this paper was supported by the Office of Naval Research of the United States Navy.

[2] Numbers in brackets designate References at end of paper.

Contributed by the Applied Mechanics Division and presented at the Applied Mechanics and Fluids Engineering Conference, Evanston, Ill., June 16–18, 1969, of The American Society of Mechanical Engineers.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until July 25, 1969. Discussion received after the closing date will be returned. Manuscript received by ASME Applied Mechanics Division, July 31, 1968. Paper No. 69-APM-11.

The two unknown functions w and F are determined by two coupled partial differential equations

$$A\nabla^2\nabla^2 F = -2ar^{-2}(r^{-1}w^{\cdot})', \qquad D\nabla^2\nabla^2 w = 2ar^{-2}(r^{-1}F^{\cdot})' \quad (7)$$

which are consequences of the axial equilibrium and compatibility conditions.

Finally, the radial and circumferential displacements u_r and u_θ, are related to the strain measures by

$$\epsilon_r = u_r', \qquad \epsilon_\theta = r^{-1}(u_r + u_\theta^{\cdot}) + ar^{-2}w^{\cdot}$$
$$\epsilon_{r\theta} = \epsilon_{\theta r} = \tfrac{1}{2}[r^{-1}u_r^{\cdot} + r(r^{-1}u_\theta)' + ar^{-1}w'] \quad (8)$$

It is not difficult to verify by direct substitution that the strain and curvature change measures as defined by (1), (6), and (8) satisfy the following compatibility conditions

$$-(r\kappa_\theta)' + \kappa_{r\theta}^{\cdot} + \kappa_r = 0, \qquad (r\kappa_{\theta r})' - \kappa_r^{\cdot} + \kappa_{r\theta} = 0$$
$$(r\lambda_\theta)' - \lambda_r^{\cdot} - 2ar^{-1}\kappa_{\theta r} = 0 \quad (9)$$
$$(r\epsilon_\theta)' - \epsilon_{r\theta}^{\cdot} - \epsilon_r - r\lambda_\theta = 0, \qquad -(r\epsilon_{\theta r})' + \epsilon_r^{\cdot} - \epsilon_{r\theta} + r\lambda_r = 0$$

The foregoing differential equations are supplemented by an appropriate set of boundary conditions. For our problem, the shell is clamped to a fixed rigid cylinder at the outer radial edge; we have therefore

$$u_r(r_0) = u_\theta(r_0) = w(r_0) = w'(r_0) = 0 \quad (10)$$

At the inner radial edge, the shell is clamped to a rigid cylinder which is displaced laterally in the x-direction an amount δ_x; we have therefore

$$u_r(r_i) = \delta_x \cos\theta, \qquad u_\theta(r_i) = -\delta_x \sin\theta,$$
$$w(r_i) = w'(r_i) = 0 \quad (11)$$

We will assume the shell is such that we can appeal to Saint Venant's principle and ignore the boundary conditions at $\theta = \pm\theta_0$.

Static-Geometric Duality

It is well known that there is a duality between the quantities F, N_r, $N_{r\theta} = N_{\theta r}$, N_θ, Q_r, Q_θ, M_r, $M_{r\theta} = M_{\theta r}$, M_θ, A, and ν_s on the one hand, and the quantities w, $-\kappa_\theta$, $\kappa_{\theta r} = \kappa_{r\theta}$, $-\kappa_r$, λ_θ, $-\lambda_r$, ϵ_θ, $-\epsilon_{\theta r} = -\epsilon_{r\theta}$, ϵ_r, $-D$, and $-\nu_b$ on the other hand in the sense that if we replace all quantities in equations (1)–(3) and the first equation of (7), by their dual quantities, we get equations (4)–(6) and the second equation of (7). In the same sense, the compatibility equations (9) are the static-geometric duals of the equilibrium equations as listed in equations (6) of [7].

We will extend this duality to the relevant boundary conditions in order to establish a complete static-geometric duality between the side-force problem and the pure-bending problem. This last duality, which is the key to our final results, is not at all obvious from the displacement boundary conditions (10) and (11) and the Kirchhoff stress boundary conditions of the pure-bending problem. To establish the desired duality, we first observe that the boundary conditions (10) and (11) for the side-force problem imply the following conditions on the strain and curvature change measures at the two radial edges:

$$r = r_i, r_0: \qquad -\kappa_\theta = \kappa_{\theta r} = \epsilon_\theta = \lambda_\theta - r^{-1}\epsilon_{\theta r}^{\cdot} = 0 \quad (12)$$

The conditions (12) (which can be verified by direct substitution) are just the static-geometric duals of the homogeneous Kirchhoff stress conditions for the same edges for the pure-bending problem; see [7], equation (7).

We may, if we wish, establish next that the boundary conditions (10) and (11) are in fact equivalent to (12) and six overall conditions which are the static-geometric duals of the overall equilibrium conditions for the pure-bending problem as listed in equations (8) of [7]. Altogether, we have that, upon reformulating them in terms of strain measures, all the boundary conditions for the side-force problem are exactly those of the pure-bending problem except the quantities which appear in the former are the duals of those which appear in the latter. Along with the well-known duality among the differential equations as just summarized, the solution of the side-force problem must therefore be the solution of the pure-bending problem with the roles of all relevant quantities taken over by their duals.

We will not concern ourselves here with all six overall compatability conditions since five of them are identically satisfied by way of (9) (see [7]), and are therefore of no consequence in the final solution of the side-force problem.[3] We merely note that it follows from the first equation of (8)

$$u_r = -\int_r^{r_0} \epsilon_r dr \quad (13a)$$

with limits of integration chosen so that the first condition of (10) is satisfied. The sixth and only nontrivial overall compatibility condition can then be obtained from this and the first equation of (11) in the form

$$-\delta_x \cos\theta = \int_{r_i}^{r_0} \epsilon_r dr \quad (13b)$$

Solution

Motivated by the discussion of the last secti[illegible] we now obtain the solution of the side-force problem by replacing all quantities of the solution of the pure-bending problem by their duals. We get, with $\rho = r/r_0$, the following expressions for w and F:[4]

$$F(r, \theta) = \hat{f}(r)\cos\theta + \tfrac{1}{2}Par\theta\sin\theta$$
$$= \frac{Par_0\rho}{\alpha_R^2}\{\hat{c}_1\rho^{\alpha_R} + \hat{c}_2\rho^{-\alpha_R}$$
$$+ \rho^{-\alpha_R/2}[\hat{c}_3\cos(\alpha_I\ln\rho) + \hat{c}_4\sin(\alpha_I\ln\rho)]$$
$$+ \rho^{\alpha_R/2}[\hat{c}_5\cos(\alpha_I\ln\rho) + \hat{c}_6\sin(\alpha_I\ln\rho)]\}$$
$$+ \tfrac{1}{2}Par_0\rho\theta\sin\theta \quad (14a)$$

$$w(r, \theta) = \hat{W}(r)\sin\theta$$
$$= \frac{Par_0\rho}{\alpha_R^2}\sqrt{\frac{A}{D}}\{\hat{c}_1\rho^{\alpha_R} - \hat{c}_2\rho^{-\alpha_R}$$
$$+ \rho^{-\alpha_R/2}[\hat{c}_3\cos(\alpha_I\ln\rho) + \hat{c}_4\sin(\alpha_I\ln\rho)]$$
$$-\rho^{\alpha_R/2}[\hat{c}_5\cos(\alpha_I\ln\rho) + \hat{c}_6\sin(\alpha_I\ln\rho)]\}\sin\theta \quad (14b)$$

and correspondingly,

$$(\kappa_\theta, \kappa_{\theta r}) = (\hat{\kappa}_\theta\sin\theta, \hat{\kappa}_{\theta r}\cos\theta)$$
$$= -\frac{Pa}{\alpha_R r_0\rho}\sqrt{\frac{A}{D}}\{\hat{c}_1\rho^{\alpha_R} + \hat{c}_2\rho^{-\alpha_R} + \rho^{-\alpha_R/2}[\hat{c}_3 G_{r3}(\rho)$$
$$+ \hat{c}_4 G_{r4}(\rho)] - \rho^{\alpha_R/2}[\hat{c}_5 G_{r5}(\rho) + \hat{c}_6 G_{r6}(\rho)]\}(\sin\theta, \cos\theta)$$

$$\kappa_r = \hat{\kappa}_r\sin\theta$$
$$= -\frac{Pa}{r_0\rho}\sqrt{\frac{A}{D}}\{\hat{c}_1(1 + \alpha_R^{-1})\rho^{\alpha_R} - \hat{c}_2(1 - \alpha_R^{-1})\rho^{-\alpha_R}$$
$$+ \rho^{-\alpha_R/2}[\hat{c}_3 G_{\theta 3} + \hat{c}_4 G_{\theta 4}] - \rho^{\alpha_R/2}[\hat{c}_5 G_{\theta 5} + \hat{c}_6 G_{\theta 6}]\}\sin\theta \quad (15)$$

---

[3] To indicate the nature of the analysis involved, we have, for example, from (1), (10), and (11)

$$\int_{r_i}^{r_0} (\kappa_{r\theta}\cos\theta + \kappa_r\sin\theta)dr = -[(r^{-1}w^{\cdot})\cos\theta + w\sin\theta]_{r_i}^{r_0} = 0$$

which is the dual of the first condition of (8) in [7].

[4] The additional terms associated with the constants c_7 and c_8 in [7] have been omitted as they are inconsequential for the present problem.

$$\epsilon_\theta = \hat{\epsilon}_\theta \cos\theta = A[\hat{f}'' - \nu_s(r^{-1}\hat{f})'] \cos\theta \qquad (15)\ (Cont.)$$

$$= \frac{PaA}{r_0\rho}\{\hat{c}_1[1 + (1-\nu_s)\alpha_R^{-1}]\rho^{\alpha_R} + \hat{c}_2[1 - (1-\nu_s)\alpha_R^{-1}]\rho^{-\alpha_R}$$
$$+ \rho^{-\alpha_R/2}[\hat{c}_3(G_{\theta 3} - \nu_s\alpha_R^{-1}G_{r3}) + \hat{c}_4(G_{\theta 4} - \nu_s\alpha_R^{-1}G_{r4})]$$
$$+ \rho^{\alpha_R/2}[\hat{c}_5(G_{\theta 5} - \nu_s\alpha_R^{-1}G_{r5}) + \hat{c}_6(G_{\theta 6} - \nu_s\alpha_R^{-1}G_{r6})] - \nu_s\} \cos\theta$$

$$\epsilon_r = \hat{\epsilon}_r \cos\theta = A[(r^{-1}\hat{f})' - \nu_s\hat{f}''] \cos\theta$$
$$= -\frac{PaA}{r_0\rho}\{\hat{c}_1[\nu_s - (1-\nu_s)\alpha_R^{-1}]\rho^{\alpha_R} + \hat{c}_2[\nu_s + (1-\nu_s)\alpha_R^{-1}]\rho^{-\alpha_R}$$
$$+ \rho^{-\alpha_R/2}[\hat{c}_3(\nu_s G_{\theta 3} - \alpha_R^{-1}G_{r3}) + \hat{c}_4(\nu_s G_{\theta 4} - \alpha_R^{-1}G_{r4})]$$
$$+ \rho^{\alpha_R/2}[\hat{c}_5(\nu_s G_{\theta 5} - \alpha_R^{-1}G_{r5}) + \hat{c}_6(\nu_s G_{\theta 6} - \alpha_R^{-1}G_{r6})] - 1\} \cos\theta$$

$$\epsilon_{r\theta} = \hat{\epsilon}_{r\theta} \sin\theta = A(1+\nu_s)(r^{-1}\hat{f})' \sin\theta$$
$$= \frac{PaA(1+\nu_s)}{\alpha_R r_0\rho}\{\hat{c}_1\rho^{\alpha_R} - \hat{c}_2\rho^{-\alpha_R} + \rho^{-\alpha_R/2}[\hat{c}_3G_{r3} + \hat{c}_4G_{r4}] + \rho^{\alpha_R/2}[\hat{c}_5G_{r5} + \hat{c}_6G_{r6}]\} \sin\theta$$

$$\lambda_r = \hat{\lambda}_r \sin\theta$$
$$= \frac{PaA}{(r_0\rho)^2}\{\hat{c}_1(1 + 2\alpha_R^{-1})\rho^{\alpha_R} + \hat{c}_2(1 - 2\alpha_R^{-1})\rho^{-\alpha_R}$$
$$+ \rho^{-\alpha_R/2}[\hat{c}_3(G_{\theta 3} + \alpha_R^{-1}G_{r3}) + \hat{c}_4(G_{\theta 4} + \alpha_R^{-1}G_{r4})]$$
$$+ \rho^{\alpha_R/2}[\hat{c}_5(G_{\theta 5} + \alpha_R^{-1}G_{r5}) + \hat{c}_6(G_{\theta 6} + \alpha_R^{-1}G_{r6})] + 1\} \sin\theta$$

$$\lambda_\theta = \hat{\lambda}_\theta \cos\theta$$
$$= \frac{\alpha_R PaA}{(r_0\rho)^2}\{\hat{c}_1(1 + \alpha_R^{-1} - 2\alpha_R^{-2})\rho^{\alpha_R}$$
$$- \hat{c}_2(1 - \alpha_R^{-1} - 2\alpha_R^{-2})\rho^{-\alpha_R}$$
$$+ \rho^{-\alpha_R/2}[\hat{c}_3G_{q3} + \hat{c}_4G_{q4}]$$
$$+ \rho^{\alpha_R/2}[\hat{c}_5G_{q5} + \hat{c}_6G_{q6}] - \alpha_R^{-1}\} \cos\theta$$

and the corresponding stress resultants and couples can be obtained through the stress-strain relations (2) and (5). In these, the constants α_R and α_I and the G-functions are exactly those defined in [7] and will not be listed here again as they are not essential to the present development. We note however that α_R and α_I depend only on a dimensionless parameter $\delta = (a^2/DA)^{1/6}$. For an isotropic homogeneous medium, we have $\delta = \sqrt[3]{a/h}/\sqrt[6]{12(1-\nu^2)}$ where h is the shell thickness. For $\delta \ll 1$, the shell is nearly a flat plate so we are mainly interested in the range $\delta \gg 1$.

In this range, we have

$$\alpha_R \sim \sqrt[3]{2}\delta, \qquad \alpha_I \sim \sqrt{3}\delta/\sqrt[3]{4} \qquad (16)$$

so that α_R and α_I are $0\left(\sqrt[3]{a/h}\right)$.

The arbitrary constant P associated with the nonperiodic portion of F is the static-geometric dual of the constant k associated with the nonperiodic portion of the displacement function w for the pure-bending problem. The relation between the constant P and the side force per unit winding P_x applied through the movable rigid cylinder can be seen from

$$P_x = \int_0^{2\pi} (N_r \cos\theta - N_{r\theta}\sin\theta) r d\theta$$
$$= \int_0^{2\pi} [(r^{-1}\hat{f})'(\cos^2\theta - \sin^2\theta) + Par^{-1}\cos^2\theta] r d\theta = Pa\pi \qquad (17)$$

The six constants of integration $\hat{c}_k$ are to be chosen so that the boundary conditions at the radial edges (12) are satisfied. In view of (15), these conditions become, with $\rho_i = r_i/r_0$,

$$\rho = \rho_i,\ 1: \qquad -\hat{\kappa}_\theta = \hat{\kappa}_{\theta r} = \hat{\epsilon}_\theta = \hat{\lambda}_\theta - r^{-1}\hat{\epsilon}_{\theta r} = 0 \qquad (18)$$

Noting that $\hat{\kappa}_\theta = \hat{\kappa}_{\theta r} = -r^{-1}(\hat{W}' - r^{-1}\hat{W})$ we have, analogous to the pure-bending problem, that the boundary conditions (18) consist of only three independent conditions. Clearly, the $\hat{c}_k$'s depend only on ρ_i, δ, and ν_s and are simply the c_k's of [7] with ν_b replaced by ν_s since the c_k's themselves allow the satisfaction of the duals of (18).

We now relate the constant P to the lateral displacement δ_x by way of (13b) which, upon using the solution for ϵ_r in (15), becomes

$$\delta_x = -\int_{r_i}^{r_0} \hat{\epsilon}_r dr = -\int_{r_i}^{r_0} A[(r^{-1}\hat{f})' - \nu_s\hat{f}'' + Par^{-1}]dr \qquad (19a)$$

or

$$-\frac{\delta_x}{A} = [-\nu_s\hat{f}' + r^{-1}\hat{f} + Pa\ln r]_{r_i}^{r_0} \qquad (19b)$$

These are just the duals of equations (17) and (18) of [7], respectively. The results of [7] (see [7], equations (26)) enables us to write down immediately the following flexibility relation:

$$\delta_x \equiv \hat{B}P = -PaA\left[-\ln\rho_i + \alpha_R^{-1}\sum_{k=1}^{6}\hat{c}_k\hat{\Delta}_k\right] \qquad (20)$$

or, since it seems more physically meaningful to write (20) in terms of the side force per unit winding P_x,

$$\delta_x \equiv B^*P_x = -\frac{P_xA}{\pi}\left[-\ln\rho_i + \alpha_R^{-1}\sum_{k=1}^{6}\hat{c}_k\hat{\Delta}_k\right] \qquad (21)$$

where $\hat{\Delta}_k$ are obtained from the Δ_k's of [7] by replacing ν_b in the latter by $-\nu_s$:

Numerical Results

Because of the complexity of the calculations involved, a computer program was developed to calculate the influence coefficient and the stress distributions for the pure-bending problem. In view of the static-geometric duality between the pure-bending problem and the present side-force problem, we may obtain numerical results for the latter from the same computer program (without any modification) if we use for the input quantities D, A, ν_b, and ν_s the values of $-A$, $-D$, $-\nu_s$, and $-\nu_b$, respectively, and interpret all the output quantities as outputs for the dual quantities. The results of some typical calculations are presented herein.

The variation of the flexibility coefficient B^* as a function of δ is shown in Fig. 1 for representative values of ρ_i and for $\nu_s = 0.3$. This graph shows that for $\delta \gg 1$, we have

$$B^* \sim \frac{A}{\pi}\ln\rho_i \equiv B_M^* \qquad (22)$$

Thus only the nonperiodic portion of the solution in (14) contributes significantly to B^* for $\delta \gg 1$. As such, the shell behaves essentially as a membrane for this range of δ insofar as the overall load deformation relation (21) is concerned. As a tends to zero so that δ tends to zero, Fig. 1 shows that B^* tends to the corresponding result for a flat plate [5]:[5]

$$B_p^* = \frac{A(1+\nu_s)}{4\pi(3-\nu_s)}\left[(3-\nu_s)^2\ln\rho_i + (1+\nu_s)^2\frac{1-\rho_i^2}{1+\rho_i^2}\right] \qquad (23)$$

We see from the graph that a shell is more flexible than the corresponding plate. For shells with same A, ν_s, r_i, and r_0, the flexibility can change by as much as 50 percent as one

[5] The solution for the limiting case of a flat plate may be obtained also by applying the rules of static-geometric duality to the results given in the Appendix of [7].

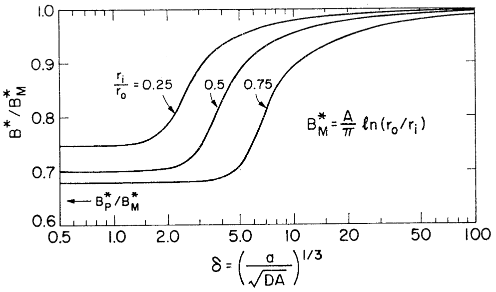

Fig. 1 Dimensionless overall flexibility coefficient versus δ for $\nu_s = 0.3$

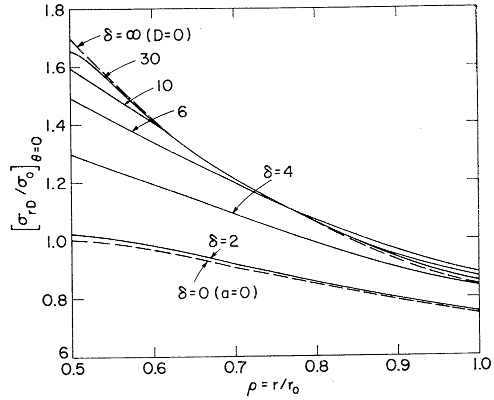

Fig. 2 Variation of dimensionless direct stress σ_{rD}/σ_0 across shell width for $\nu_s = 0.3$, $r_i/r_0 = 0.5$, and $\theta = 0$; see equation (24) for σ_0

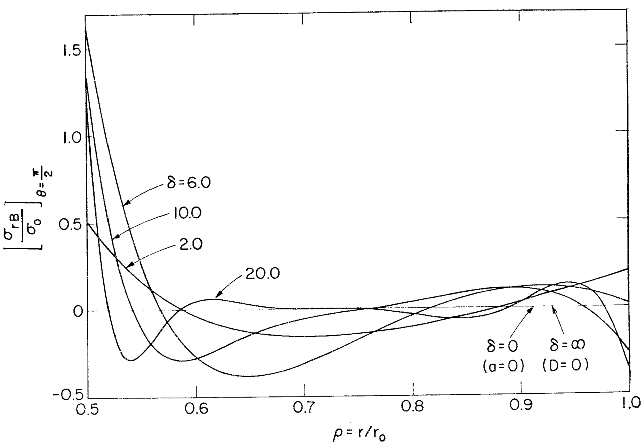

Fig. 3 Variation of dimensionless bending stress σ_{rB}/σ_0 across shell width for $\nu_s = 0.3$, $r_i/r_0 = 0.5$, and $\theta = \pi/2$; see equation (24) for σ_0

varies the pitch-to-thickness ratio δ. Also, for a fixed outer radius r_0, A, ν_s, and δ a relatively wider shell is more flexible.

Turning now to the stress distributions, the expressions for the resultants and couples given by (15) and the stress-strain relations suggest that for $\delta \gg 1$, there are two distinct stress states in the shell. Only the contribution from the term $\frac{1}{2}Par\ \theta \sin\theta$ in (14) to the stress resultants is significant everywhere in the shell and can therefore be considered as the *interior state*. Inasmuch as this same term does not contribute to the curvature changes, therefore not to the stress couples, the interior state of the shell is purely a *membrane state*.

In contrast, the contribution of terms associated with the constants $\hat{c}_k$ becomes insignificant at a distance $0(r_0/\delta)$ from the edges for $\delta \gg 1$. As such, this contribution is a boundary-layer effect and may therefore be considered as an *edge zone state*. The existence of these two distinct states can be seen from the graphs for $(\sigma_{rD}/\sigma_0)_{\theta=0}$ and $(\sigma_{rB}/\sigma_0)_{\theta=\pi/2}$ in Figs. 2 and 3 where $\sigma_{rD} = N_r/h$ and $\sigma_{rB} = 6M_r/h^2$ and where

$$\sigma_0 \equiv [\sigma_{rD}|a=0]_{\max} = \frac{P_x}{2\pi r_i h}\,\frac{(3-\nu_s) + (5+\nu_s)\rho_i^2}{(3-\nu_s)(1+\rho_i^2)} \tag{24}$$

is the maximum σ_{rD} for the limiting case of a flat plate. These figures also show that the stress distributions tend to those for flat plates [5] as a tends to zero.

Inplane Displacements

We now use the strain displacement relations (8) to determine the expression for the displacement components u_r and u_θ. Upon using the expression for ϵ_r in (15), the first equation of (8) (see also (13a)) gives

$$u_r = A[-\nu_s\hat{f} + r^{-1}\hat{f} + Pa\ln r - u_0]\cos\theta \equiv \hat{u}_r(r)\cos\theta \tag{25a}$$

where

$$u_0 = -[-\nu_s\hat{f}'(r_0) + r_0^{-1}\hat{f}(r_0) + Pa\ln r_0] \tag{25b}$$

In view of (19) and (25), u_r satisfies the first condition of both (10) and (11).

From the second equation of (8), we have the following solution for u_θ

$$u_\theta \equiv \hat{u}_\theta(r)\sin\theta = (r\hat{\epsilon}_\theta - \hat{u}_r - ar^{-1}\hat{W})\sin\theta \tag{26}$$

Note that the solutions for u_r, u_θ together with w satisfy the third equation of (8) identically; see [7].

We must now show that u_θ and w given, respectively, by (26) and (14) satisfy the last three displacement boundary conditions of (10) and (11). To see this, we observe that

$$\begin{aligned}\lambda_\theta - r^{-1}\epsilon_{\theta r}^{\cdot} &= r^{-1}[(r\epsilon_\theta)' - \epsilon_{r\theta}^{\cdot} - \epsilon_r - \epsilon_{\theta r}^{\cdot}] \\ &= r^{-3}[ru_\theta^{\cdot} - ru_r^{\cdot\cdot} - aw^{\cdot}]\end{aligned} \tag{27}$$

Upon substituting the expression for u_θ into this, we get

$$\hat{\lambda}_\theta - r^{-1}\hat{\epsilon}_{\theta r} = r^{-1}\hat{\epsilon}_\theta - 2ar^{-3}\hat{W} \tag{28}$$

But we have from (18) $\hat{\lambda}_\theta - r^{-1}\hat{\epsilon}_{\theta r} = 0$ *and* $\hat{\epsilon}_\theta = 0$ *at the two* radial edges; therefore we also have

$$\hat{W}(r_i) = \hat{W}(r_0) = 0 \quad \text{or} \quad w(r_i, \theta) = w(r_0, \theta) = 0 \tag{29}$$

Next, we have from (18)

$$r = r_i, r_0: \quad \hat{\kappa}_\theta = -r^{-1}(\hat{W}' - r^{-1}\hat{W}) = 0 \tag{30}$$

In view of (29), this implies

$$\hat{W}'(r_i) = \hat{W}'(r_0) = 0 \quad \text{or} \quad w'(r_i, \theta) = w'(r_0, \theta) = 0 \tag{31}$$

Since $\hat{\epsilon}_\theta(r_0) = \hat{\epsilon}_\theta(r_i) = \hat{W}(r_0) = \hat{W}(r_i) = \hat{u}_r(r_0) = 0$ and $\hat{u}_r(r_i) = \delta_x$, we have finally

$$u_\theta(r_0, \theta) = [r_0\hat{\epsilon}_\theta(r_0) - \hat{u}_r(r_0) - ar_0^{-1}\hat{W}(r_0)]\sin\theta = 0 \tag{32}$$

and

$$\begin{aligned}u_\theta(r_i, \theta) &= [r_i\hat{\epsilon}_\theta(r_i) - \hat{u}_r(r_i) - ar_i^{-1}\hat{W}(r_i)]\sin\theta \\ &= -\delta_x\sin\theta\end{aligned} \tag{33}$$

References

1 Goldenweizer, A. L., "The Equations of the Theory of Thin Shells," *Prikladnaya Mathematika i Mekhanika*, Vol. 4, 1940, pp. 32–42.

2 Maunder, L., and Reissner, E., "Pure Bending of Pretwisted Rectangular Plates," *Journal of the Mechanics and Physics of Solids*, Vol. 5, 1957, pp. 261–266.

3 Riessner, E., "Small Rotationally Symmetric Deformations of Shallow Helicoidal Shells," Journal of Applied Mechanics, Vol. 22, Trans. ASME, Vol. 77, 1955, pp. 31–34.

4 Reissner, E., and Wan, F. Y. M., "On the Equations of Linear Shallow Shell Theory," *Studies in Applied Mathematics*, Vol. 48, 1969.

5 Reissner, H., and Strauch, Fr., "Ringplatte und Augenstab," *Ingenieur-Archiv*, Vol. 4, 1933, pp. 481–505.

6 Wan, F. Y M., "A Class of Unsymmetric Stress Distributions in Helicoidal Shells," *Quarterly of Applied Mathematics*, Vol. 24, 1967, pp. 374–379.

7 Wan, F. Y. M., "Pure Bending of Shallow Helicoidal Shells," Journal of Applied Mechanics, Vol. 35, Trans. ASME, Vol. 90, Series E, 1968, pp. 387–392.

Reprinted from the June 1969
Journal of Applied Mechanics