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MEMBRANE AND BENDING STRESSES IN SHALLOW SPHERICAL SHELLS*

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Abstract—The exact solutions of the stress and displacement boundary value problem for a shallow spherical thin elastic shell of constant thickness subject to self-equilibrating edge loads are obtained. The asymptotic behavior of these exact solutions delineates the dependence of the interior and edge zone stress state on the applied loads. Relations between these results and previous results obtained by a direct asymptotic analysis of these boundary value problems are established.

1. INTRODUCTION

THE nature of the interior stress state of a thin elastic shell under external loads has received considerable attention in recent years [2–5]. The interior of the shell may be in a membrane state, an inextensional bending state or a mixed state of stress depending on the applied loads. The present work offers a different perspective to this dependence. We consider a complete shallow spherical cap without surface loads. The edge of the shell is subjected to self-equilibrating loads which vary sinusoidally in the circumferential direction. We solve the boundary value problem exactly and study the asymptotic behavior of the exact solution in order to delineate the dependence of the interior and edge zone stress state on the applied edge loads. In this way we establish relations between results obtained in the earlier works by a direct asymptotic analysis of the boundary value problem and the asymptotic behavior of the exact solution to the same problem which is obtained herein.

2. FORMULATION OF PROBLEM

The system of differential equations governing the small deformations of an isotropic, shallow spherical shell with constant wall thickness and without surface loads [3] may be written in the form

$$D\nabla^2\nabla^2 w - R^{-1}\nabla^2 F = 0, \qquad A\nabla^2\nabla^2 F + R^{-1}\nabla^2 w = 0 \tag{2.1}$$

where $\nabla^2(\ \ ) = (\ \ )_{,rr} + r^{-1}(\ \ )_{,r} + r^{-2}(\ \ )_{,\theta\theta}$ and where

w = the normal component of the middle surface displacement,
F = a stress function representing the direct stress resultants,
R = the radius of the spherical middle surface,

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r, θ = polar coordinates in the plane tangent to the apex of the shell,
$D = E_b h^3/12(1-\nu^2), \qquad A = 1/E_s h,$
ν = Poisson's ratio.

Some degree of nonhomogeneity is included in the above formulation by allowing an independent choice of bending and stretching moduli of elasticity. For a completely homogeneous shell, $E_b = E_s = E$, where E is Young's modulus.

The relevant stress resultants and couples are given in terms of F and w by

$$\begin{aligned} N_r &= r^{-1}F_{,r} + r^{-2}F_{,\theta\theta}, \qquad N_\theta = F_{,rr}, \qquad N_{r\theta} = -(r^{-1}F_{,\theta})_{,r}, \\ Q_r &= -D(\nabla^2 w)_{,r}, \qquad Q_\theta = -Dr^{-1}(\nabla^2 w)_{,\theta}, \\ M_r &= -D[w_{,rr} + \nu(r^{-1}w_{,r} + r^{-2}w_{,\theta\theta})], \\ M_\theta &= -D[\nu w_{,rr} + r^{-1}w_{,r} + r^{-2}w_{,\theta\theta}], \\ M_{r\theta} &= -D(1-\nu)(r^{-1}w_{,\theta})_{,r}, \qquad R_r = Q_r + r^{-1}M_{r\theta,\theta}. \end{aligned} \tag{2.2}$$

The general solution to (2.1) has been obtained in [3]. For our purpose, we consider the solution contribution which leads to finite stresses and displacements at the apex, in the form

$$\begin{aligned} w &= \left\{ \frac{a^2 B_n}{D(1-\nu)}\rho^n + \frac{1}{\sqrt{D}}[C_n ber_n(\lambda\rho) + D_n bei_n(\lambda\rho)] \right\} \cos n\theta \\ F &= \left\{ a^2 A_n \rho^n - \frac{1}{\sqrt{A}}[C_n bei_n(\lambda\rho) - D_n ber_n(\lambda\rho)] \right\} \cos n\theta \end{aligned} \tag{2.3}$$

where

$$\rho = \frac{r}{a}, \qquad \lambda = \frac{a}{\sqrt[4]{(DAR^2)}} \tag{2.4}$$

and where A_n, B_n, C_n, D_n are constants of integration to be determined by the boundary conditions at $r = a$. Since the shell is complete in the circumferential direction, we take n to be a non-negative integer.

From (2.3) and (2.2) follows

$$\begin{aligned} N_r &= -\left\{ n(n-1)A_n\rho^{n-2} + \frac{\lambda^2}{a^2(\sqrt{A})}[C_n f_{rc}(\lambda\rho) + D_n f_{rd}(\lambda\rho)] \right\} \cos n\theta \\ N_\theta &= \left\{ n(n-1)A_n\rho^{n-2} - \frac{\lambda^2}{a^2(\sqrt{A})}[C_n f_{\theta c}(\lambda\rho) + D_n f_{\theta d}(\lambda\rho)] \right\} \cos n\theta \\ N_{r\theta} &= \left\{ n(n-1)A_n\rho^{n-2} + \frac{\lambda^2}{a^2(\sqrt{A})}[C_n f_{sc}(\lambda\rho) + D_n f_{sd}(\lambda\rho)] \right\} \sin n\theta \\ Q_r &= \frac{\lambda}{aR(\sqrt{A})}\{C_n bei_n'(\lambda\rho) - D_n ber_n'(\lambda\rho)\} \cos n\theta \\ Q_\theta &= \frac{\lambda}{aR(\sqrt{A})}\left(\frac{n}{\lambda\rho}\right)\{C_n bei_n(\lambda\rho) - D_n ber_n(\lambda\rho)\} \sin n\theta \\ M_r &= \left\{ -n(n-1)B_n\rho^{n-2} + \frac{1}{R(\sqrt{A})}[C_n g_{rc}(\lambda\rho) + D_n g_{rd}(\lambda\rho)] \right\} \cos n\theta \end{aligned} \tag{2.5}$$

$$M_\theta = \left\{n(n-1)B_n\rho^{n-2} + \frac{1}{R(\sqrt{A})}[C_n g_{\theta c}(\lambda\rho) + D_n g_{\theta d}(\lambda\rho)]\right\}\cos n\theta$$

$$M_{r\theta} = \left\{n(n-1)B_n\rho^{n-2} + \frac{1}{R(\sqrt{A})}[C_n g_{sc}(\lambda\rho) + D_n g_{sd}(\lambda\rho)]\right\}\sin n\theta \qquad (2.5 \text{ cont'd.})$$

$$R_r = \frac{1}{a}\left\{n^2(n-1)B_n\rho^{n-3} + \frac{\lambda}{R(\sqrt{A})}[C_n g_{nc}(\lambda\rho) + D_n g_{nd}(\lambda\rho)]\right\}\cos n\theta$$

where

$$f_{rc}(x) = x^{-1}[bei_n'(x) - n^2x^{-1}bei_n(x)], \qquad f_{rd}(x) = -x^{-1}[ber_n'(x) - n^2x^{-1}ber_n(x)]$$

$$f_{\theta c}(x) = ber_n(x) - x^{-1}[bei_n'(x) - n^2x^{-1}bei_n(x)]$$

$$f_{\theta d}(x) = bei_n(x) + x^{-1}[ber_n'(x) - n^2x^{-1}ber_n(x)]$$

$$f_{sc}(x) = -nx^{-1}[bei_n'(x) - x^{-1}bei_n(x)], \qquad f_{sd}(x) = nx^{-1}[ber_n'(x) - x^{-1}ber_n(x)]$$

$$g_{rc}(x) = bei_n(x) + (1-\nu)x^{-1}[ber_n'(x) - n^2x^{-1}ber_n(x)]$$

$$g_{rd}(x) = -ber_n(x) + (1-\nu)x^{-1}[bei_n'(x) - n^2x^{-1}bei_n(x)]$$

$$g_{\theta c}(x) = \nu bei_n(x) - (1-\nu)x^{-1}[ber_n'(x) - n^2x^{-1}ber_n(x)]$$

$$g_{\theta d}(x) = -\nu ber_n(x) - (1-\nu)x^{-1}[bei_n'(x) - n^2x^{-1}bei_n(x)]$$

$$g_{sc}(x) = (1-\nu)nx^{-1}[ber_n'(x) - x^{-1}ber_n(x)]$$

$$g_{sd}(x) = (1-\nu)nx^{-1}[bei_n'(x) - x^{-1}bei_n(x)]$$

$$g_{nc}(x) = bei_n'(x) + (1-\nu)n^2x^{-2}[ber_n'(x) - x^{-1}ber_n(x)]$$

$$g_{nd}(x) = -ber_n'(x) + (1-\nu)n^2x^{-2}[bei_n'(x) - x^{-1}bei_n(x)].$$

Primes here indicate differentiation with respect to the argument of the function.

For shallow shells with sufficiently small bending–stretching stiffness ratio, the dimensionless parameter λ is large compared to unity. For an isotropic homogeneous shell of constant thickness

$$\lambda = \frac{a}{\sqrt{(Rh)}}\sqrt[4]{[12(1-\nu^2)]} \qquad (2.6)$$

so that $\lambda \to \infty$ as $h \to 0$. In the present work, we are concerned particularly with shells for which $\lambda \gg n > 1$. It is known [3] that for this range of λ, the effect of terms involving the Kelvin functions ber_n and bei_n and their derivatives is confined to a narrow region adjacent to the edge of the shell; it is therefore referred to as the edge effect. Terms associated with the constants A_n and B_n are referred to as the membrane and inextensional bending solution contributions respectively. The effect of these terms becomes dominant away from the edge of the shell.

3. THE STRESS BOUNDARY VALUE PROBLEM

Consider a shell acted upon by edge loads and moments at $\rho = 1$ in such a way that

$$(N_r, R_r, M_r) = (N_n, R_n, M_n)\cos n\theta, \qquad N_{r\theta} = S_n \sin n\theta \qquad (3.1)$$

where N_n, S_n, R_n and M_n are prescribed constants and $n \geq 2$. With equation (2.5), the boundary conditions (3.1) assume the form

$$
\begin{aligned}
n(n-1)A_n + \frac{\lambda^2}{a^2(\sqrt{A})}[C_n f_{rc}(\lambda) + D_n f_{rd}(\lambda)] &= -N_n \\
n(n-1)A_n + \frac{\lambda^2}{a^2(\sqrt{A})}[C_n f_{sc}(\lambda) + D_n f_{sd}(\lambda)] &= S_n \\
n^2(n-1)B_n + \frac{\lambda}{R(\sqrt{A})}[C_n g_{nc}(\lambda) + D_n g_{nd}(\lambda)] &= aR_n \\
-n(n-1)B_n + \frac{1}{R(\sqrt{A})}[C_n g_{rc}(\lambda) + D_n g_{rd}(\lambda)] &= M_n.
\end{aligned}
\tag{3.2}
$$

The solution of this system of four equations for the four unknowns A_n, B_n, C_n and D_n may be written in the form

$$
\begin{aligned}
A_n &= \frac{1}{2n(n-1)}\left[S_n X_1 - N_n X_2 + \frac{n-1}{\alpha a}(aR_n + nM_n)X_3\right] \\
B_n &= \frac{\alpha a}{2n^2(n^2-1)}\left[(S_n + N_n)X_4 + \frac{n+1}{\alpha a}(aR_n X_1 - nM_n X_2)\right]
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
C_n &= -\frac{a^2(\sqrt{A})}{q_n\Delta_1}\left[(S_n + N_n)[g_{nd}(\lambda) + \frac{n}{\lambda}g_{rd}(\lambda)] - \frac{\lambda}{\alpha a}(aR_n + nM_n)[f_{rd}(\lambda) - f_{sd}(\lambda)]\right] \\
D_n &= \frac{a^2(\sqrt{A})}{q_n\Delta_1}\left[(S_n + N_n)[g_{nc}(\lambda) + \frac{n}{\lambda}g_{rc}(\lambda)] - \frac{\lambda}{\alpha a}(aR_n + nM_n)[f_{rc}(\lambda) - f_{sc}(\lambda)]\right]
\end{aligned}
\tag{3.4}
$$

where $\alpha = a/R$ and

$$
\begin{aligned}
X_1 &= \frac{1}{\Delta_1}\left\{1 - \frac{1-\nu}{\lambda}\left[\alpha_2 - \frac{n(n+1)}{\lambda}\alpha_3 + \frac{n^3}{\lambda^2}\alpha_4\right]\right\}, \\
X_2 &= \frac{1}{\Delta_1}\left\{1 - \frac{n(1-\nu)}{\lambda}\left[\alpha_2 - \frac{n+1}{\lambda}\alpha_3 + \frac{n}{\lambda^2}\alpha_4\right]\right\}, \qquad X_3 = \frac{1}{\Delta_1}, \\
X_4 &= \frac{1}{\Delta_1}\left\{1 - \frac{1-\nu}{\lambda}\left[\alpha_2 - \frac{2n^2}{\lambda}\alpha_3 + \frac{n^2}{\lambda^2}\alpha_4 + \frac{n^2(n^2-1)(1-\nu)}{\lambda^3}\right]\right\}
\end{aligned}
\tag{3.5}
$$

with

$$
\Delta_1 = 1 - \frac{(n+1)(1-\nu)}{2\lambda}\left(\alpha_2 - \frac{2n}{\lambda}\alpha_3 + \frac{n}{\lambda^2}\alpha_4\right) \tag{3.6}
$$

$$
\alpha_2 = s_n/q_n, \qquad \alpha_3 = r_n/q_n, \qquad \alpha_4 = p_n/q_n \tag{3.7}
$$

and, in the notation of Ref. [6],

$$\begin{aligned} p_n &= ber_n^2(\lambda)+bei_n^2(\lambda), \qquad q_n = ber_n(\lambda)bei_n'(\lambda)-bei_n(\lambda)ber_n'(\lambda) \\ r_n &= ber_n(\lambda)ber_n'(\lambda)+bei_n(\lambda)bei_n'(\lambda), \qquad s_n = [ber_n'(\lambda)]^2+[bei_n'(\lambda)]^2. \end{aligned} \tag{3.8}$$

From the expressions for C_n and D_n, we observe immediately that there is no edge zone contribution to the stresses and displacements of the shell if $S_n+N_n = aR_n+nM_n = 0$. If in addition, $R_n = M_n = 0$, we see from the expression for A_n and B_n that the shell is in a pure membrane state of stress. On the other hand, if we have $S_n = N_n = 0$ instead, then the shell is in a state of pure inextensional bending.

In this work, we are in particular concerned with the range $\lambda \gg n > 1$. For this range of values of λ, we obtain simpler expressions for the solution of the problem, by way of the asymptotic expansion of the relevant Kelvin functions [6],

$$ber_n(\lambda) \sim \frac{e^{\lambda/(\sqrt{2})}}{\sqrt{(2\pi\lambda)}}\left[\cos\left(\frac{\lambda}{\sqrt{2}}-\frac{\pi}{8}+\frac{n\pi}{2}\right)-\frac{(4n^2-1)}{8\lambda}\cos\left(\frac{\lambda}{\sqrt{2}}-\frac{3\pi}{8}+\frac{n\pi}{2}\right)+O\left(\frac{1}{\lambda^2}\right)\right] \tag{3.9}$$

etc. Correspondingly, we have

$$\frac{\alpha_2}{\sqrt{2}} \sim 1-\frac{1}{(\sqrt{2})\lambda}+O\left(\frac{1}{\lambda^2}\right), \qquad \alpha_3 \sim 1-\frac{1}{(\sqrt{2})\lambda}+O\left(\frac{1}{\lambda^2}\right), \qquad \frac{\alpha_4}{\sqrt{2}} \sim 1+O\left(\frac{1}{\lambda^2}\right). \tag{3.10}$$

We shall henceforth confine ourselves to homogeneous shells and introduce at this point for convenience sake a new parameter $\mu = \lambda(\sqrt{2})$ to take the place of λ in all subsequent asymptotic considerations. In view of (2.6),

$$\mu = \frac{a}{\sqrt{(Rh)}}\sqrt[4]{[3(1-\nu^2)]}. \tag{3.11}$$

The following asymptotic expressions for the X_i's are obtained with the help of (3.10) when $\mu \gg n > 1$:

$$\begin{aligned} X_1 &\sim 1+\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right), \qquad X_2 \sim 1-\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right) \\ X_3 &\sim 1+\frac{(n+1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right), \qquad X_4 \sim 1+\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right). \end{aligned} \tag{3.12}$$

With these, there follows from (3.3)

$$\begin{aligned} A_n &= \frac{1}{2n(n-1)}\left\{\left[1+\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]S_n-\left[1-\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]N_n\right. \\ &\quad \left.+\frac{n-1}{\alpha a}\left[1+\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](aR_n+nM_n)\right\} \\ &= \frac{1}{2n(n-1)}\left\{\left[1+O\left(\frac{1}{\mu^2}\right)\right](S_n-N_n)+\frac{(n-1)(1-\nu)}{2\mu}\left[1+O\left(\frac{1}{\mu}\right)\right](S_n+N_n)\right. \\ &\quad \left.+\frac{n-1}{\alpha a}\left[1+O\left(\frac{1}{\mu}\right)\right](aR_n+nM_n)\right\} \end{aligned} \tag{3.13}$$

$$B_n = \frac{h\mu^2}{2n^2(n^2-1)\sqrt{[3(1-\nu^2)]}}\left\{\left[1+\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](S_n+N_n)\right.$$

$$\left.+\frac{n+1}{\alpha a}\left[1+\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](aR_n)-\frac{n+1}{\alpha a}\left[1-\frac{(n-1)(1-\nu)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](nM_n)\right\}$$

$$= \frac{h\mu^2}{2n^2(n^2-1)\sqrt{[3(1-\nu^2)]}}\left\{\left[1+O\left(\frac{1}{\mu}\right)\right](S_n+N_n)+\frac{n+1}{\alpha a}\left[1+O\left(\frac{1}{\mu^2}\right)\right](aR_n-nM_n)\right.$$

$$\left.+\frac{(n^2-1)(1-\nu)}{2\mu\alpha a}\left[1+O\left(\frac{1}{\mu}\right)\right](aR_n+nM_n)\right\}. \tag{3.14}$$

If only the leading terms of the expansions for the X_i's are retained in (3.13) and (3.14), the expressions for A_n and B_n become

$$A_n = \frac{1}{2n(n-1)}\left\{(S_n-N_n)+\frac{n-1}{\alpha a}(aR_n+nM_n)\right\} \tag{3.15}$$

$$B_n = \frac{h\mu^2}{2n^2(n^2-1)\sqrt{[3(1-\nu^2)]}}\left\{(S_n+N_n)+\frac{n+1}{\alpha a}(aR_n-nM_n)\right\}. \tag{3.16}$$

Equations (3.15) and (3.16) coincide with results obtained by Reissner [2] by a direct asymptotic analysis. Having (3.13) and (3.14), we now see that (3.15) and (3.16) are valid first approximation to A_n and B_n, respectively, provided

$$\frac{1}{\mu}\left|(S_n+N_n)+\frac{n+1}{\alpha a}(aR_n+nM_n)\right| \ll \left|(S_n-N_n)+\frac{n-1}{\alpha a}(aR_n+nM_n)\right| \tag{3.17}$$

and

$$\frac{1}{\mu}\left|(S_n+N_n)+\frac{n+1}{\alpha a}(aR_n+nM_n)\right| \ll \left|(S_n+N_n)+\frac{n+1}{\alpha a}(aR_n-nM_n)\right|. \tag{3.18}$$

Equations (3.17) and (3.18) are, in particular, always satisfied if all but one of the prescribed stress quantities vanish.

4. DIRECT AND BENDING STRESSES

To examine the direct and bending stresses in the shell, we consider two representative quantities $\sigma_{\theta D}$ and σ_{rB} given by

$$\sigma_{\theta D} = \left.\frac{N_\theta}{h}\right|_{\rho=1} = [\sigma^i_{Dn}+\sigma^e_{Dn}]\cos n\theta$$

$$\sigma_{rB} = \left.\frac{6M_r}{h^2}\right|_{\rho=1} = [\sigma^i_{Bn}+\sigma^e_{Bn}]\cos n\theta \tag{4.1}$$

where

$$\sigma^i_{Dn} = \frac{n(n-1)A_n}{h} = \frac{1}{2h}\left[S_n X_1 - N_n X_2 + \frac{n-1}{\alpha a}(aR_n + nM_n)X_3\right]$$
$$\sigma^i_{Bn} = -\frac{6n(n-1)B_n}{h^2} = -\frac{3\alpha a}{n(n+1)h^2}\left[(S_n+N_n)X_4 + \frac{n+1}{\alpha a}(aR_n X_1 - nM_n X_2)\right] \tag{4.2}$$

$$\sigma^e_{Dn} = -\frac{\lambda^2}{ha^2(\sqrt{A})}[C_n f_{\theta c}(\lambda) + D_n f_{\theta d}(\lambda)]$$
$$= \frac{\lambda^2}{h}\left[(S_n+N_n)Z_1 + \frac{n+1}{\alpha a}(aR_n + nM_n)Z_2\right]$$
$$\sigma^e_{Bn} = \frac{6}{h^2 R(\sqrt{A})}[C_n g_{rc}(\lambda) + D_n g_{rd}(\lambda)] \tag{4.3}$$
$$= \frac{6\alpha a}{h^2}\left[(S_n+N_n)Z_3 + \frac{n+1}{\alpha a}(aR_n + nM_n)Z_4\right]$$

with

$$Z_1 = \frac{1}{\Delta_1}\left\{\alpha_3 + \frac{n}{\lambda}\alpha_4 - \frac{n(n+1)}{\lambda^2} - \frac{n^2(n+1)^2(1-\nu)}{\lambda^4}\alpha_3 + \frac{n^4(n+1)(1-\nu)}{\lambda^5}\alpha_4\right\}$$
$$Z_2 = \frac{1}{\Delta_1}\left\{\alpha_3 - \frac{n}{\lambda}\alpha_4 - \frac{n(n-1)}{\lambda^2}\right\}$$
$$Z_3 = \frac{1}{\Delta_1}\left\{1 - \frac{1-\nu}{\lambda}\alpha_2 - \frac{n(1-\nu)}{\lambda^2}[(n+1)-\nu(n-1)]\alpha_3 + \frac{n^2(1-\nu)}{\lambda^3}[(n+1)+\nu n]\alpha_4 + \frac{n^2(n^2-1)(1-\nu)^2}{\lambda^4}\right\} \tag{4.4}$$
$$Z_4 = \frac{1}{\Delta_1}\left\{1 - \frac{1-\nu}{\lambda}\alpha_2 + \frac{n(n+1)(1-\nu)}{\lambda^2}\alpha_3 - \frac{n^3(1-\nu)}{\lambda^3}\alpha_4\right\}.$$

For $\mu \gg n > 1$, the following asymptotic expansion for the Z_i's are obtained with the help of (3.10)

$$Z_1 \sim 1 + \frac{(2n+1)-2\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right), \qquad Z_2 \sim 1 - \frac{(2n-1)+2\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right)$$
$$Z_3 \sim Z_4 \sim 1 + \frac{(n+1)+2\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right). \tag{4.5}$$

It is not difficult to see from (2.5) that other direct and bending stress quantities are at most of the same order of magnitude as $\sigma_{\theta D}$ and σ_{rB}, respectively. In what follows, we will first study the relative importance of the interior membrane and inextensional bending stress through the asymptotic expansion of the X_i's for $\mu \gg n > 1$ and indicate certain limitations on the range of validity of the classical results obtained in [2]. We then turn to a study of the edge zone stresses to show that beyond the range of validity of the classical results for the interior zone state, the dominant stresses in the shell are associated with the edge zone state.

For $\mu \gg n > 1$ so that A_n and B_n are given by (3.13) and (3.14), we have from (4.2)

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = -\frac{n(n+1)\sqrt{[3(1-\nu^2)]}}{6\mu^2} \times \left\{ \frac{\left[1+O\left(\frac{1}{\mu^2}\right)\right](S_n-N_n)+\frac{(n-1)(1-\nu)}{2\mu}\left[1+O\left(\frac{1}{\mu}\right)\right](S_n+N_n)+\frac{n-1}{\alpha a}\left[1+O\left(\frac{1}{\mu}\right)\right](aR_n+nM_n)}{\left[1+O\left(\frac{1}{\mu}\right)\right](S_n+N_n)+\frac{n+1}{\alpha a}\left[1+O\left(\frac{1}{\mu^2}\right)\right](aR_n-nM_n)+\frac{(n^2-1)(1-\nu)}{2\mu\alpha a}\left[1+O\left(\frac{1}{\mu}\right)\right](aR_n+nM_n)} \right\}. \tag{4.6}$$

If all but one of the prescribed quantities N_n, S_n, R_n and M_n vanish, then

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = O\left(\frac{n^2}{\mu^2}\right)$$

so that the interior of the shell is in a state of inextensional bending. However, if more than one of the prescribed quantities do not vanish, the situation needs not be equally simple. It will be instructive to consider separately two special classes of problems, the class for which $R_n = M_n = 0$ and the class for which $S_n = N_n = 0$.

For the first class of problems, (4.6) becomes

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = -\frac{n(n+1)\sqrt{[3(1-\nu^2)]}}{6\mu^2} \left\{ \frac{\left[1+O\left(\frac{1}{\mu^2}\right)\right](S_n-N_n)+\frac{(n-1)(1-\nu)}{2\mu}\left[1+O\left(\frac{1}{\mu}\right)\right](S_n+N_n)}{\left[1+O\left(\frac{1}{\mu}\right)\right](S_n+N_n)} \right\}. \tag{4.7}$$

Equation (4.7) shows that the nature of the interior stress state depends on the relative magnitude of the two quantities (S_n+N_n) and (S_n-N_n).

If $|S_n-N_n| = O[(n/\mu)|S_n+N_n|]$, we have

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = O\left(\frac{n^3}{\mu^3}\right).$$

The interior membrane stresses are therefore small compared with the interior bending stresses.

On the other hand, if $(n/\mu)|S_n+N_n| \ll |S_n-N_n|$, we have

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = O\left(\frac{n^2}{\mu^2}\frac{S_n-N_n}{S_n+N_n}\right). \tag{4.8}$$

Several possibilities arise depending on the magnitude of the ratio $|S_n-N_n|/|S_n+N_n|$.

(1) If $|S_n-N_n|$ and $|S_n+N_n|$ are of the same order of magnitude which is the normal situation, then

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = O\left(\frac{n^2}{\mu^2}\right)$$

and the membrane stresses are again small compared with the bending stresses.

(2) If $|S_n+N_n|$ and $(n/\mu)|S_n-N_n|$ are of the same order of magnitude, then

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = O\left(\frac{n}{\mu}\right).$$

(3) If $|S_n+N_n|$ and $(n/\mu)^2|S_n-N_n|$ are of the same order of magnitude, then $\sigma^i_{Dn}/\sigma^i_{Bn} = O(1)$. That is, membrane and bending stresses are equally important in the interior of the shell.

(4) Finally, if $|S_n+N_n| \ll (n/\mu)^2|S_n-N_n|$, then $\sigma^i_{Dn}/\sigma^i_{Bn} \ll 1$, so that the interior stresses are predominantly membrane stresses. In particular, if $S_n + N_n = 0$ (in addition to $R_n = M_n = 0$), then there is no inextensional bending deformation contribution in the shell interior (see equation (3.3)).

We note that as long as the interior bending stresses dominate the interior membrane stresses, we will be interested in the leading term approximation for B_n and not in that for A_n. Equation (3.18) indicates that when $R_n = M_n = 0$, the leading term approximation (3.16) is always adequate.

From the above analysis, we see that interior membrane stresses will be significant when $|S_n+N_n| = O[(n/\mu)^2|S_n-N_n|]$. For this case, we are interested in the adequacy of the leading term approximation (3.15) as well as (or rather than) (3.16). Equation (3.17) indicates that when $R_n = M_n = 0$ (3.15) is always adequate for the range of values of S_n and N_n to be considered.

Accordingly, when $R_n = M_n = 0$, equations (3.15) and (3.16) together furnish the correct first order approximation for the interior state for all possible combinations of S_n and N_n.

Turning now to the second class of problems, for which $S_n = N_n = 0$, we have from (4.5)

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = -\frac{n(n-1)\sqrt{[3(1-\nu^2)]}}{6\mu^2} \times \left\{ \frac{\left[1+O\left(\frac{1}{\mu}\right)\right](aR_n+nM_n)}{\left[1+O\left(\frac{1}{\mu^2}\right)\right](aR_n-nM_n)+\frac{(n-1)(1-\nu)}{2\mu}\left[1+O\left(\frac{1}{\mu}\right)\right](aR_n+nM_n)} \right\}. \tag{4.9}$$

(1) If $(n/\mu)|aR_n+nM_n| \ll |aR_n-nM_n|$ which is the normal situation, the second term in the denominator of (4.9) can be neglected and we have

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = O\left(\frac{n^2}{\mu^2}\right).$$

Therefore, we are interested in an accurate first approximation of B_n. We see that (3.18) does in fact provide this accurate first approximation.

(2) If, on the other hand, the exceptional case $|aR_n-nM_n| \ll (n/\mu)|aR_n+nM_n|$ is given, then the first term in the denominator of (4.6) can be neglected and we have

$$\frac{\sigma^i_{Dn}}{\sigma^i_{Bn}} = O\left(\frac{n}{\mu}\right).$$

While we are again interested in an accurate first approximation to B_n, equation (3.16) no longer provides this approximation.

(3) For the rather special case of $|aR_n - nM_n|$ and $(n/\mu)|aR_n + nM_n|$ being of the same order of magnitude, both terms in the denominator of (4.9) are equally important. If their sum is again of the same order of magnitude as the individual terms, the ratio $\sigma^i_{Dn}/\sigma^i_{Bn}$ is again of the order n/μ. For this case, (3.16) does not lead to an accurate first approximation; it nevertheless gives the correct order of magnitude of the interior stresses. If the same sum is such that the denominator is $O[(n/\mu)^2|aR_n + nM_n|]$, neither (3.16) nor (3.14) is sufficient for our purpose and we will have to investigate the higher order terms in the asymptotic expansion of B_n to determine whether the interior of the shell is in a mixed stress state of a membrane stress state.

Altogether, the above analysis shows that for the class of problems in which the contribution of S_n and N_n to A_n and B_n are negligible, the interior of the shell is almost always in a state of inextensional bending. The only exception occurs when

$$\left| aR_n - nM_n + \frac{(n-1)(1-\nu)}{2\mu}(aR_n + nM_n) \right| = O\left(\frac{n}{\mu} |aR_n + nM_n| \right). \tag{4.10}$$

For this exception, the relevant third order terms of the asymptotic expression for B_n must be considered in order to determine whether the interior is in a mixed state or a membrane state of stress. Comparing this with the results for the first class of problems, we see that, while self-equilibrating tangential edge loads alone may lead to any one of the three possible interior stress states, it takes a rather special combination of transverse load and moment (without tangential edge loads) to induce an interior state other than an inextensional bending state.

The general case where none of the prescribed edge loads are zero can be investigated by a similar analysis. However, we shall confine ourselves to some remarks in connection with the validity of (3.15) and (3.16) as first approximations for A_n and B_n respectively. It is not difficult to see that they in fact lead to an accurate first approximation of the interior stresses as long as

$$\left| (S_n + N_n) + \frac{n+1}{\alpha a}\left[(aR_n - nM_n) + \frac{(n-1)(1-\nu)}{2\mu}(aR_n + nM_n) \right] \right| \gg \frac{n}{\mu}|aR_n + nM_n|. \tag{4.11}$$

Therefore, insofar as the determination of the dominant interior stresses is concerned, the restrictions (3.17) and (3.18) on the applicability of (3.15) and (3.16) as accurate first approximations can be relaxed to that given by (4.11). When (4.11) is violated, it is not difficult to see that the interior bending stresses will be much weaker than the stresses of the more representative cases with only one of the prescribed loads alone acting on the shell. We may therefore take the sign of exceptionally low (but still dominant) interior bending stresses as a warning of possible breakdown of (3.16) as an accurate first approximation to B_n.

Turning now to the edge stresses, we first consider the exceptional case of $S_n = N_n = 0$ and (4.10) being satisfied, so that the classical results for the interior stresses do not apply. For this case, we have from (4.2), (4.3), (4.5) and (4.10)

$$\frac{\sigma^i_{Bn}}{\sigma^e_{Bn}} = -\frac{1}{2n(n+1)}\left[\frac{aR_n X_1 - nM_n X_2}{(aR_n + nM_n)Z_4} \right]$$

$$= O\left[\frac{(aR_n - nM_n) + \dfrac{(n-1)(1-\nu)}{2\mu}(aR_n + nM_n)}{(aR_n + nM_n)\left\{1 + O\left(\dfrac{1}{\mu}\right)\right\}} \right] = O\left(\frac{n}{\mu}\right).$$

Therefore, the edge zone bending stresses are at least an order of magnitude larger than the interior bending stresses. Moreover,

$$\frac{\sigma^i_{Dn}}{\sigma^e_{Bn}} = O\left(\frac{n^2}{\mu^2}\frac{X_3}{Z_4}\right) = O\left(\frac{n^2}{\mu^2}\right)$$

so that the edge bending stresses also dominate the interior direct stresses. Finally

$$\frac{\sigma^e_{Dn}}{\sigma^e_{Bn}} = O\left(\frac{Z_2}{Z_4}\right) = O(1)$$

so that σ^e_{Bn} is in fact the representative stress level developed in the shell.

It is not difficult to verify that the same conclusions also hold for the more general case in which none of the prescribed loads is zero and together they satisfy the restriction (4.11). We conclude therefore that for the exceptional cases for which (3.15) and (3.16) do not represent a correct first approximation to the interior state, it is the edge zone state rather than the interior state which is associated with the dominant stress of the problem.

On the other hand, it can be shown by similar considerations that in all cases for which the classical results are a valid first approximation of the interior state, the edge zone contribution is at most of the same order of magnitude as the interior stresses.

5. THE DISPLACEMENT BOUNDARY VALUE PROBLEM

We consider now a shell subject to edge deformations at $\rho = 1$ so that

$$(u, w, \beta_r) = (u_n, w_n, \beta_n)\cos n\theta, \qquad v = v_n \sin n\theta \tag{5.1}$$

where u and v are meridional and circumferential displacement components, $\beta_r = -w_{,r}$ is the meridional rotation, u_n, v_n, w_n and β_n are prescribed constants and $n \geq 2$. To solve this displacement boundary value problem, we need explicit expressions for u, v and β_r.

Differentiating equation (2.3), we get

$$\beta_r = -\left\{\frac{naB_n}{D(1-\nu)}\rho^{n-1} + \frac{\lambda}{a(\sqrt{D})}[C_n ber'_n(\lambda\rho) + D_n bei'_n(\lambda\rho)]\right\}\cos n\theta \tag{5.2}$$

In order to obtain u and v, we use the strain displacement relations

$$u_{,r} - \frac{w}{R} = A(N_r - \nu N_\theta), \qquad \frac{v_{,\theta} + u}{r} - \frac{w}{R} = A(N_\theta - \nu N_r)$$
$$\frac{u_{,\theta}}{r} + r\left(\frac{v}{r}\right)_{,r} = 2(1+\nu)AN_{r\theta} \tag{5.3}$$

It follows from (5.3), (2.5) and (2.3) in a manner described in [3] that

$$u = \left\{-na(1+\nu)AA_n\rho^{n-1} + \frac{a^3B_n}{RD(n+1)(1-\nu)}\rho^{n+1} + \frac{\lambda(\sqrt{A})(1+\nu)}{a}[C_n bei'_n(\lambda\rho) - D_n ber'_n(\lambda\rho)]\right\}\cos n\theta \tag{5.4}$$

$$v = \left\{ na(1+\nu)AA_n\rho^{n-1} + \frac{a^3 B_n}{RD(n+1)(1-\nu)}\rho^{n+1} - \frac{\lambda(\sqrt{A})(1+\nu)}{a}\left(\frac{n}{\lambda\rho}\right)[C_n bei_n(\lambda\rho) - D_n ber_n(\lambda\rho)]\right\} \sin n\theta. \quad (5.4 \text{ cont'd.})$$

With (2.3), (5.2) and (5.4), the boundary conditions (5.1) assume the form

$$\begin{aligned}
&\frac{a^2 B_n}{D(1-\nu)} + \frac{1}{\sqrt{D}}[C_n ber_n(\lambda) + D_n bei_n(\lambda)] = w_n \\
&\frac{na^2 B_n}{D(1-\nu)} + \frac{\lambda}{\sqrt{D}}[C_n ber_n'(\lambda) + D_n bei_n'(\lambda)] = -a\beta_n \\
&na(1+\nu)AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} - \frac{n(1+\nu)(\sqrt{A})}{a}[C_n bei_n(\lambda) - D_n ber_n(\lambda)] = v_n \\
&-na(1+\nu)AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} + \frac{(1+\nu)\lambda(\sqrt{A})}{a}[C_n bei_n'(\lambda) - D_n ber_n'(\lambda)] = u_n.
\end{aligned} \quad (5.5)$$

The solution for the interior state coefficients A_n and B_n may be written in the form

$$\begin{aligned}
A_n &= -\frac{1}{2naA(1+\nu)}\{u_n Y_1 - v_n Y_2 - 2\alpha(1+\nu)[w_n Y_3 + a\beta_n Y_4]\} \\
B_n &= \frac{D(n+1)(1-\nu)}{2\alpha a^2}\{(u_n + v_n)Y_5 - 2\alpha(1+\nu)[w_n Y_6 + a\beta_n Y_7]\}
\end{aligned} \quad (5.6)$$

where

$$\begin{aligned}
&Y_1 = \frac{1}{\Delta_2}\left\{1 + \frac{n(n+1)(1+\nu)}{\lambda^2}\left(\alpha_3 - \frac{n}{\lambda}\alpha_4\right)\right\}, \qquad Y_2 = \frac{1}{\Delta_2}\left\{1 - \frac{(n+1)(1+\nu)}{\lambda}\left(\alpha_2 - \frac{n}{\lambda}\alpha_3\right)\right\} \\
&Y_3 = \frac{1}{\lambda\Delta_2}\left\{\alpha_2 + \frac{n}{\lambda}\alpha_3 - \frac{n^2(n+1)(1+\nu)}{\lambda^3}\right\}, \qquad Y_4 = \frac{1}{\lambda^2\Delta_2}\left\{\alpha_3 + \frac{n}{\lambda}\alpha_4 - \frac{n(n+1)(1+\nu)}{\lambda^2}\right\} \\
&Y_5 = \frac{1}{\Delta_2}, \qquad Y_6 = \frac{1}{2\lambda\Delta_2}\left(\alpha_2 - \frac{n}{\lambda}\alpha_3\right), \qquad Y_7 = \frac{1}{2\lambda^2\Delta_2}\left(\alpha_3 - \frac{n}{\lambda}\alpha_4\right)
\end{aligned} \quad (5.7)$$

with

$$\Delta_2 = 1 - \frac{(n+1)(1+\nu)}{2\lambda}\left(\alpha_2 - \frac{2n}{\lambda}\alpha_3 + \frac{n^2}{\lambda^2}\alpha_4\right) \quad (5.8)$$

and where α and α_i's are as defined previously. The solution for the edge zone state coefficients C_n and D_n may be written in the form

$$\begin{aligned}
C_n = -\frac{(n+1)(\sqrt{D})}{2\alpha q_n \Delta_2}\Big\langle &(u_n + v_n)\left[bei_n'(\lambda) - \frac{n}{\lambda}bei_n(\lambda)\right] \\
&- \frac{2\alpha w_n}{n+1}\left[bei_n'(\lambda) + \frac{n(n+1)(1+\nu)}{2\lambda^2}\left\{ber_n'(\lambda) - \frac{n}{\lambda}ber_n(\lambda)\right\}\right] \\
&- \frac{2\alpha a\beta_n}{\lambda(n+1)}\left[bei_n(\lambda) + \frac{(n+1)(1+\nu)}{2\lambda}\left\{ber_n'(\lambda) - \frac{n}{\lambda}ber_n(\lambda)\right\}\right]\Big\rangle
\end{aligned} \quad (5.9)$$

$$D_n = \frac{(n+1)(\sqrt{D})}{2\alpha q_n \Delta_2}\Big\langle (u_n+v_n)\left[ber_n'(\lambda)-\frac{n}{\lambda}ber_n(\lambda)\right]$$
$$-\frac{2\alpha w_n}{n+1}\left[ber_n'(\lambda)-\frac{n(n+1)(1+v)}{2\lambda^2}\left\{bei_n'(\lambda)-\frac{n}{\lambda}bei_n(\lambda)\right\}\right] \qquad (5.9 \text{ cont'd.})$$
$$-\frac{2\alpha a\beta_n}{\lambda(n+1)}\left[ber_n(\lambda)-\frac{(n+1)(1+v)}{2\lambda}\left\{bei_n'(\lambda)-\frac{n}{\lambda}bei_n(\lambda)\right\}\right]\Big\rangle.$$

For $\mu \gg n > 1$, we have, with the help of (3.10), the following asymptotic expressions for the quantities Y_i's:

$$Y_1 \sim 1+\frac{(n+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right), \qquad Y_2 \sim 1-\frac{(u+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right)$$
$$Y_3 \sim \frac{1}{2\mu}\left\{1+\frac{1}{2\mu}[(n-1)+(n+1)(1+v)]+O\left(\frac{1}{\mu^2}\right)\right\}$$
$$Y_4 \sim \frac{1}{(2\mu)^2}\left\{1+\frac{1}{2\mu}[(2n-1)+(n+1)(1+v)]+O\left(\frac{1}{\mu^2}\right)\right\} \qquad (5.10)$$
$$Y_5 \sim 1+\frac{(n+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right), \qquad Y_6 \sim \frac{1}{2\mu}\left\{1+\frac{v(n+1)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right\}$$
$$Y_7 \sim \frac{1}{(2\mu)^2}\left\{1-\frac{1}{2\mu}[(2n+1)-(n+1)(1+v)]+O\left(\frac{1}{\mu^2}\right)\right\}.$$

Correspondingly, we have as asymptotic expressions for A_n and B_n:

$$A_n = \frac{1}{2naA(1+v)}\left\{\left[1-\frac{(n+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]v_n-\left[1+\frac{(n+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]u_n\right.$$
$$+\frac{2\alpha(1+v)w_n}{2\mu}\left[1+\frac{(n-1)+(n+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right] \qquad (5.11)$$
$$\left.+\frac{2\alpha(1+v)a\beta_n}{(2\mu)^2}\left[1+\frac{(2n-1)+(n+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]\right\}$$

$$B_n = \frac{D(1-v)(n+1)}{2\alpha a^2}\left\{\left[1+\frac{(n+1)(1+v)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right](u_n+v_n)\right.$$
$$-\frac{2\alpha(1+v)w_n}{2\mu}\left[1+\frac{v(n+1)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right] \qquad (5.12)$$
$$\left.-\frac{2\alpha(1+v)a\beta_n}{(2\mu)^2}\left[1+\frac{(n+1)(1+v)-(2n+1)}{2\mu}+O\left(\frac{1}{\mu^2}\right)\right]\right\}.$$

If all but one of prescribed edge displacements vanish, we get from (5.11) and (5.12)

$$\frac{\sigma_{Bn}^i}{\sigma_{Dn}^i} = -\frac{6B_n}{hA_n} = O\left(\frac{n^2}{\mu^2}\right) \qquad (5.13)$$

so that the interior of the shell is in a membrane stress state.

If terms of the order n/μ are discarded in the Y_i's and contribution for the w_n and β_n terms are negligible, (5.11) and (5.12) are further reduced to

$$A_n = \frac{v_n - u_n}{2na(1+\nu)A}, \qquad B_n = \frac{D(1-\nu)(n+1)}{2\alpha a^2}(u_n + v_n) \tag{5.14, 15}$$

respectively. Equations (5.14) and (5.15) are exactly those obtained by a direct asymptotic analysis of the displacement boundary problem assuming that both w_n and $a\beta_n$ are $O(\alpha U)$ where U is the larger (in magnitude) of v_n and u_n [3]. Having (5.11) and (5.12), we now see that this restriction on the magnitude of w_n and β_n would lead to (5.14) and (5.15) provided that $(u_n + v_n)$ and $(u_n - v_n)$ are of the same order of magnitude. However, even if $|u_n + v_n| \ll |u_n - v_n|$, the results of [3] remain valid in that (5.14) continues to give the correct dominant stresses in the shell. On the other hand, if $|u_n - v_n| = O[(n/\mu)(u_n + v_n)]$, in which case the contribution from w_n and β_n is in fact negligible, it is not difficult to see from (5.11) that (5.14) would not give an accurate first approximation to A_n. But at the same time, it is also not difficult to see that the dominant interior (membrane) stresses are at least an order of magnitude weaker than for the typical cases. Direct calculations of the edge zone stresses show again that for this exceptional case for which (5.14) and (5.15) do not represent an accurate first approximation for the interior stress state, it is the edge zone state rather than the interior state which is associated with dominant stresses of the problem.

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Résumé—Les solutions exactes du problème de contrainte et de déplacement de limite pour une enveloppe élastique sphérique peu profonde d'épaisseur constante, sujette à des chargements de bord s'équilibrant tout seuls, sont obtenues. Le comportement asymptotique de ces solutions exactes trace la dépendance de l'état de contrainte de l'intérieur et de la zone de bord sur les charges appliquées. Des relations entre ces résultats et les résultats précedemment obtenus par une analyse directe asymptotique de ces problèmes de valeur de limite sont établies.

Zusammenfassung—Genaue Läsungen werden erhalten für das Verschiebungs und Spannungs Grenzproblem einer flachen elastischen kugelförmigen Schale gleichmässiger Dicke, die ausgewogenen Randbelastungen unterliegt. Das asymptotische Verhalten dieser genauen Lösungen gibt die Abhängigkeit der inneren und Grenzzonen Spannungszustände von der Belastung. Beziehungen zwischen diesen Resultaten und denen die durch direkte asymptotische Analyse der Gernzwertprobleme erhalten wurden werden bestimmt.

Абстракт—Получены точные решения проблемы напряжения и смещения граничного значения для мелкой сферической тонкой эластичной оболочки постоянной толщины при условии нагрузок самоуравновешивающего края. Асимптотическое поведение этих точных решений устанавливает зависимость состояния напряжения внутренней и краевой зоны от применяемых нагрузок. Установлены отношения между этими и предыдущими результатами, полученными прямым асимптотическим анализом этих проблем граничных значений.