Exact Reductions of the Equations of Linear Theory of Shells of Revolution

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1. Introduction

Ever since the work of H. Reissner [12] and E. Meissner [7] on the rotationally symmetric problem of shells of revolution, efforts have been made to reduce more general systems of shell equations to two simultaneous equations for a stress function and a displacement function. Results of this nature have been obtained for shallow shells [6], for shells of revolution under wind type loading [2, 15] and for spherical [1, 3, 5, 14] and circular cylindrical [4, 13] shells under arbitrary loads.

The present work is concerned with an exact reduction for general shells of revolution under arbitrary loads. This reduction takes advantage of the possibility of a harmonic analysis with respect to the polar angle in the base plane of the shell, and yields as final results two simultaneous fourth order ordinary differential equations for a stress function and a displacement function.

In Section 4 of this paper, we describe the exact reduction procedure for general shells of revolution. In view of the known reductions for symmetric bending and torsion [7, 11, 12] and for wind-type loads [2, 15], we need only to consider shells with stresses varying as $e^{in\theta}$ with $n \geq 2$.

In Section 5, the detailed reduction will be carried out for a circular cylindrical shell. First, the simple geometry of the shell enables us to demonstrate the method of reduction clearly and concisely. Secondly, the explicit exact results allow us to delineate the relations between Simmonds' equations [13] and those of an exact reduction. Finally, while the method of reduction can be carried out exactly for all shells of revolution, the complexity and tedium of the necessary routine calculations make it desirable to be able to neglect terms which are small compared to unity (e.g., terms of the order of the square of the thickness-to-radius of curvature ratio) in the intermediate stages of the reduction. It is therefore of some interest to know that there is a complete agreement (to within the accuracy of shell theory) of the results for the cylindrical shell obtained by the exact and the simplified procedure.

Of parallel interest is the possibility of reducing the shell equations exactly to one eighth order differential equation for the normal displacement component. We will show in Section 9 that this is also possible for all shells of revolution by a procedure which is somewhat different from that leading to the two simultaneous equations formulation.

2. Differential equations and static-geometric duality

In cylindrical coordinates (r, θ, z), the middle surface of a shell of revolution is described parametrically by $r = r(\xi)$ and $z = z(\xi)$. The deformation of the shell is described by three translational displacement components u, v and w in the meridional, circumferential and normal direction, respectively, and three rotational displacement components ψ, ϕ, and ω, turning about these same directions respectively. With primes and dots indicating differentiation with respect to ξ and θ respectively, and with

$$\alpha = \sqrt{r'^2 + z'^2}, \qquad \frac{1}{R_\theta} = -\frac{z'}{r\alpha}, \qquad \frac{1}{R_\xi} = -\frac{z''r' - r''z'}{\alpha^3} \tag{1}$$

midsurface strain and curvature change measures are given in terms of these displacement components as follows [8, 11]:

$$\varepsilon_\xi = \frac{u'}{\alpha} + \frac{W}{R_\xi}, \qquad \varepsilon_\theta = \frac{v^{\cdot}}{r} + \frac{r'u}{r\alpha} + \frac{W}{R_\theta}, \tag{2}$$

$$\varepsilon_{\xi\theta} = \frac{v'}{\alpha} - \omega, \qquad \varepsilon_{\theta\xi} = \frac{u^{\cdot}}{r} - \frac{r'v}{r\alpha} + \omega, \tag{3}$$

$$\kappa_\xi = \frac{\phi'}{\alpha}, \qquad \kappa_{\xi\theta} = \frac{\psi'}{\alpha} - \frac{\omega}{R_\xi}, \qquad \kappa_{\theta\xi} = \frac{\phi^{\cdot}}{r} - \frac{r'\psi}{r\alpha} + \frac{\omega}{R_\theta}, \qquad \kappa_\theta = \frac{\psi^{\cdot}}{r} + \frac{r'\phi}{r\alpha}, \tag{4}$$

$$\lambda_\xi = \frac{\omega'}{\alpha} + \frac{\psi}{R_\xi}, \qquad \lambda_\theta = \frac{\omega^{\cdot}}{r} - \frac{\phi}{R_\theta}. \tag{5}$$

The conditions of vanishing transverse shearing strains, which will be assumed to hold in what follows, give ϕ and ψ in terms of the translational displacement components,

$$\phi = -\frac{w'}{\alpha} + \frac{u}{R_\xi}, \qquad \psi = -\frac{w^{\cdot}}{r} + \frac{v}{R_\theta}. \tag{6}$$

Within the framework of a theory which includes the effect of stress couples turning about the midsurface normal, the two quantities λ_ξ and λ_θ have the meaning of the normal components of the relevant curvature change vectors [11].

The strain and curvature change measures are assumed to be related to stress resultants and stress couples in the form

$$\varepsilon_\xi = A(N_\xi - \nu_s N_\theta), \qquad \varepsilon_\theta = A(N_\theta - \nu_s N_\xi), \qquad \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = A_s(N_{\xi\theta} + N_{\theta\xi}), \tag{7}$$

$$M_\theta = D(\kappa_\theta + \nu_b\kappa_\xi), \qquad M_\xi = D(\kappa_\xi + \nu_b\kappa_\theta), \qquad M_{\theta\xi} = M_{\xi\theta} = D_S(\kappa_{\theta\xi} + \kappa_{\xi\theta}). \tag{8}$$

The stress resultants and couples N and M together with two transverse shear resultants Q_ξ and Q_θ are subject to six differential equations of equilibrium [8]. In the absence of surface loads*, these may be satisfied by writing resultants and couples in terms of stress functions in the form [10]:

$$M_\theta = \frac{G'}{\alpha} + \frac{F}{R_\xi}, \qquad M_\xi = \frac{H^{\cdot}}{r} + \frac{r'G}{r\alpha} + \frac{F}{R_\theta}, \tag{9}$$

* For the objectives stated in the introduction, we need not consider surface loads.

$$M_{\theta\xi} = -\frac{H'}{\alpha} + J, \quad M_{\xi\theta} = -\frac{G^{\cdot}}{r} + \frac{r'H}{r\alpha} - J, \tag{10}$$

$$N_\theta = -\frac{I'}{\alpha}, \quad N_{\theta\xi} = \frac{K'}{\alpha} - \frac{J}{R_\xi}, \quad N_{\xi\theta} = \frac{I^{\cdot}}{r} - \frac{r'K}{r\alpha} + \frac{J}{R_\theta}, \quad N_\xi = -\frac{K^{\cdot}}{r} - \frac{r'I}{r\alpha}, \tag{11}$$

$$Q_\theta = -\frac{J'}{\alpha} - \frac{K}{R_\xi}, \quad Q_\xi = \frac{J^{\cdot}}{r} - \frac{I}{R_\theta}, \tag{12}$$

where, in the absence of moments turning about the midsurface normal,

$$I = -\frac{F'}{\alpha} + \frac{G}{R_\xi}, \quad K = -\frac{F^{\cdot}}{r} + \frac{H}{R_\theta}. \tag{13}$$

We will make use of the fact that there is a static geometric duality associating the strain displacement relations (2) through (6) on the one hand and the stress–stress functions relations (9) through (13) on the other hand. This means that if we replace all strain measures and displacement components in the strain displacement relations by their dual stress measures and stress functions in accordance with the following table, we will get the stress function representation (9) through (13) and vice versa.

N_ξ	N_θ	$N_{\xi\theta}$	$N_{\theta\xi}$	Q_ξ	Q_θ	M_ξ	M_θ	$M_{\xi\theta}$	$M_{\theta\xi}$
$-\kappa_\theta$	$-\kappa_\xi$	$\kappa_{\theta\xi}$	$\kappa_{\xi\theta}$	λ_θ	$-\lambda_\xi$	ε_θ	ε_ξ	$-\varepsilon_{\theta\xi}$	$-\varepsilon_{\xi\theta}$

I	K	J	G	H	F	A	ν_s	A_S
ϕ	ψ	ω	u	v	w	$-D$	$-\nu_b$	$-D_S$

We note that without the appearance of λ_ξ and λ_θ, there would be no static geometric duals for Q_ξ and Q_θ.

The duality carries over to the stress strain relations (7) and (8) if we take A, A_S and ν_s to be the duals of $-D$, $-D_S$ and $-\nu_b$.

The relation $\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}$ allows us to express ω in terms of the translational displacement components:

$$\omega = \frac{1}{2}\left(\frac{v'}{\alpha} - \frac{u^{\cdot}}{r} + \frac{r'v}{r\alpha}\right), \tag{14}$$

and at the same time to eliminate ω from the expression for the inplane shearing strains so that

$$\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = \frac{1}{2}\left(\frac{v'}{\alpha} + \frac{u^{\cdot}}{r} - \frac{r'v}{r\alpha}\right). \tag{15}$$

The static geometric duality implies as corresponding relations

$$J = \frac{1}{2}\left(\frac{H'}{\alpha} - \frac{G^{\cdot}}{r} + \frac{r'H}{r\alpha}\right), \tag{16}$$

and

$$M_{\theta\xi} = M_{\xi\theta} = \frac{1}{2}\left(\frac{H'}{\alpha} + \frac{G^{\cdot}}{r} - \frac{r'H}{r\alpha}\right). \tag{17}$$

In view of (6), (14), (13) and (16), the strain displacement relations (2), (4), (15), the dual stress–stress function relations (9), (11), (17), and the stress strain relations (7) and (8) may be thought of as twenty equations for the ten unknowns, ε_ξ, ε_θ, $\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}$, κ_ξ, κ_θ, $\kappa_{\xi\theta}$, $\kappa_{\theta\xi}$, u, v, and w and for their ten static geometric duals.

3. Circumferentially sinusoidal distributions of displacements and stress functions

We consider stress and displacement functions of the form

$$(u, w, \phi, G, F, I) = (U, W, \Phi, g, f, i) \cos n\theta$$
$$(v, \psi, \omega, H, K, J) = (V, \Psi, \Omega, h, k, j) \sin n\theta \tag{1}$$

where $n \geq 2$ and where the quantities inside the parentheses on the right are functions of ξ only. Correspondingly, we have

$$(\varepsilon_\xi, \varepsilon_\theta, \kappa_\xi, \kappa_\theta, \lambda_\theta, N_\xi, N_\theta, Q_\xi, M_\xi, M_\theta)$$
$$= (e_\xi, e_\theta, \chi_\xi, \chi_\theta, l_\theta, n_\xi, n_\theta, q_\xi, m_\xi, m_\theta) \cos n\theta$$
$$(\varepsilon_{\xi\theta} = \varepsilon_{\theta\xi}, \kappa_{\xi\theta}, \kappa_{\theta\xi}, \lambda_\xi, N_{\xi\theta}, N_{\theta\xi}, Q_\theta, M_{\xi\theta} = M_{\theta\xi})$$
$$= (e, \chi_{\xi\theta}, \chi_{\theta\xi}, l_\xi, n_{\xi\theta}, n_{\theta\xi}, q_\theta, m) \sin n\theta \tag{2}$$

The strain-displacement relations (2.2), (2.15) and (2.4) and the stress–stress function relations (2.9), (2.17) and (2.11) become, with (1) and (2),

$$e_\xi = \frac{U'}{\alpha} + \frac{W}{R_\xi}, \qquad e_\theta = \frac{nV}{r} + \frac{r'U}{r\alpha} + \frac{W}{R_\theta}, \qquad e = \frac{1}{2}\left(\frac{V'}{\alpha} - \frac{r'V}{r\alpha} - \frac{nU}{r}\right), \tag{3, 4, 5}$$

$$\chi_\xi = -\frac{W''}{\alpha^2} + \frac{\alpha' W'}{\alpha^3} + \frac{U'}{\alpha R_\xi} + \left(\frac{1}{R_\xi}\right)' \frac{U}{\alpha}, \qquad \chi_\theta = -\frac{r'W'}{r\alpha^2} + \frac{r'U}{r\alpha R_\xi} + \frac{n^2 W}{r^2} + \frac{nV}{rR_\theta}, \tag{6, 7}$$

$$\chi_{\xi\theta} = \frac{nW'}{r\alpha} - \frac{nr'W}{r^2\alpha} + \frac{1}{2\alpha}\left(\frac{2}{R_\theta} - \frac{1}{R_\xi}\right)\left(V' - \frac{r'}{r}V\right) - \frac{nU}{2rR_\xi}, \tag{8}$$

$$\chi_{\theta\xi} = \frac{nW'}{r\alpha} - \frac{nr'W}{r^2\alpha} + \frac{1}{2\alpha R_\theta}\left(V' - \frac{r'}{r}V\right) + \left(\frac{1}{R_\theta} - \frac{2}{R_\xi}\right)\frac{nU}{2r}, \tag{9}$$

and

$$m_\theta = \frac{g'}{\alpha} + \frac{f}{R_\xi}, \qquad m_\xi = \frac{nh}{r} + \frac{r'g}{r\alpha} + \frac{f}{R_\theta}, \qquad m = \frac{1}{2}\left(\frac{h'}{\alpha} - \frac{r'h}{r\alpha} - \frac{ng}{r}\right), \tag{10, 11, 12}$$

$$n_\theta = \frac{f''}{\alpha^2} - \frac{\alpha' f'}{\alpha^3} - \frac{g'}{\alpha R_\xi} - \left(\frac{1}{R_\xi}\right)' \frac{g}{\alpha}, \qquad n_\xi = \frac{r'f'}{r\alpha^2} - \frac{r'g}{r\alpha R_\xi} - \frac{n^2 f}{r^2} - \frac{nh}{rR_\theta}, \tag{13, 14}$$

$$n_{\theta\xi} = \frac{nf'}{r\alpha} - \frac{nr'f}{r^2\alpha} + \frac{1}{2\alpha}\left(\frac{2}{R_\theta} - \frac{1}{R_\xi}\right)\left(h' - \frac{r'}{r}h\right) - \frac{ng}{2rR_\xi}, \tag{15}$$

$$n_{\xi\theta} = \frac{nf'}{r\alpha} - \frac{nr'f}{r^2\alpha} + \frac{1}{2\alpha R_\theta}\left(h' - \frac{r'}{r}h\right) + \left(\frac{1}{R_\theta} - \frac{2}{R_\xi}\right)\frac{ng}{2r}, \tag{16}$$

where the relations

$$\Phi = -\frac{W'}{\alpha} + \frac{U}{R_\xi}, \qquad \Psi = \frac{nW}{r} + \frac{V}{R_\theta}, \qquad \Omega = \frac{1}{2}\left(\frac{V'}{\alpha} + \frac{nU}{r} + \frac{r'V}{r\alpha}\right), \tag{17a,b,c}$$

and

$$i = -\frac{f'}{\alpha} + \frac{h}{R_\xi}, \qquad k = \frac{nf}{r} + \frac{h}{R_\theta}, \qquad j = \frac{1}{2}\left(\frac{h'}{\alpha} + \frac{ng}{r} + \frac{r'h}{r\alpha}\right), \tag{18a,b,c}$$

which follow from (2.6), (2.14), (and their dual relations) have been used to eliminate Φ, Ψ and Ω from (2.4), and i, j and k from (2.11).

The stress strain relations (2.7) and (2.8) are written for our purposes in the following more convenient form

$$n_\xi = \frac{e_\xi + \nu_s e_\theta}{A(1-\nu_s^2)}, \qquad n_\theta = \frac{e_\theta + \nu_s e_\xi}{A(1-\nu_s^2)}, \qquad e = A_S(n_{\xi\theta} + n_{\theta\xi}), \tag{19, 20, 21}$$

and

$$\chi_\theta = \frac{m_\theta - \nu_b m_\xi}{D(1-\nu_b^2)}, \qquad \chi_\xi = \frac{m_\xi - \nu_b m_\theta}{D(1-\nu_b^2)}, \qquad m = D_S(\chi_{\theta\xi} + \chi_{\xi\theta}). \tag{22, 23, 24}$$

The expressions (2.5) and (2.12) for the remaining four strain and stress measures imply the relations

$$l_\theta = \frac{1}{\alpha}\left(\frac{W'}{R_\theta} + \frac{nV'}{2r} + \frac{nr'V}{2r^2}\right) + \left(\frac{n^2}{2r^2} + \frac{1}{R_\xi R_\theta}\right)U, \tag{25}$$

$$l_\xi = \frac{1}{2\alpha}\left(\frac{V'}{\alpha} + \frac{r'V}{r\alpha} + \frac{nU}{r}\right)' - \frac{1}{R_\xi}\left(\frac{r'W'}{r\alpha^2} - \frac{n^2 W}{r^2} - \frac{r'U}{r\alpha R_\xi} - \frac{nV}{rR_\theta}\right), \tag{26}$$

$$q_\xi = \frac{1}{\alpha}\left(\frac{f'}{R_\theta} + \frac{nh'}{2r} + \frac{nr'h}{2r^2}\right) + \left(\frac{n^2}{2r^2} - \frac{1}{R_\xi R_\theta}\right)g, \tag{27}$$

$$q_\theta = -\frac{1}{2\alpha}\left(\frac{h'}{\alpha} + \frac{r'h}{r\alpha} + \frac{ng}{r}\right)' + \frac{1}{R_\xi}\left(\frac{r'f'}{r\alpha^2} - \frac{n^2 f}{r^2} - \frac{r'g}{r\alpha R_\xi} - \frac{nh}{rR_\theta}\right). \tag{28}$$

4. Exact reduction to two simultaneous equations for W and f

The key to an exact reduction is the observation that with equations (3.3) through (3.16), the six stress strain relations (3.19) through (3.24) are six equations for the six unknowns W, U, V, f, g and h. Upon substituting (3.3), (3.4), (3.14) into the stress strain relations (3.19) and substituting (3.10), (3.11) and (3.7) into (3.22), we get after some rearrangements,

$$\begin{aligned} U' &= -\nu_s\left(\frac{r'}{r}U + \frac{n\alpha}{r}V\right) - \left(\frac{\alpha}{R_\xi} + \frac{\nu_s\alpha}{R_\theta}\right)W - A(1-\nu_s^2)\left(\frac{r'}{rR_\xi}g + \frac{n\alpha}{rR_\theta}h + \frac{n^2\alpha}{r^2}f - \frac{r'}{r\alpha}f'\right), \\ &= L_U(U, V, g, h, W, f, f'), \end{aligned} \tag{1}$$

$$\begin{aligned} g' &= \nu_b\left(\frac{r'}{r}g + \frac{n\alpha}{r}h\right) - \left(\frac{\alpha}{R_\xi} - \frac{\nu_b\alpha}{R_\theta}\right)f + D(1-\nu_b^2)\left(\frac{r'}{rR_\xi}U + \frac{n\alpha}{rR_\theta}V + \frac{n^2\alpha}{r^2}W - \frac{r'}{r\alpha}W'\right), \\ &= L_g(g, h, U, V, f, W, W'). \end{aligned} \tag{2}$$

Upon substituting (3.5), (3.15) and (3.16) into (3.21), and (3.12), (3.8) and (3.9) into (3.24), and upon solving the resulting equations for V' and h', we get V' and h' as *linear combinations* of U, V, g, h, W, W', f and f':

$$\begin{aligned} V' &= \frac{n\alpha(1+\varepsilon_0^2)}{r(1+\varepsilon_1^2)}U + \frac{r'}{r}V - \frac{4nD_SA_S}{r(1+\varepsilon_1^2)}\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left(W' - \frac{r'}{r}W\right) \\ &\quad + \frac{4n\alpha\rho A_S}{r(1+\varepsilon_1^2)}h + \frac{4nA_S}{r(1+\varepsilon_1^2)}\left(f' - \frac{r'}{r}f\right), \\ &= L_V(U, V, h, W, W', f, f'), \end{aligned} \tag{3}$$

$$\begin{aligned} h' &= \frac{n\alpha(1-\varepsilon_0^2)}{r(1+\varepsilon_1^2)}g + \frac{r'}{r}h - \frac{4nD_SA_S}{r(1+\varepsilon_1^2)}\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left(f' - \frac{r'}{r}f\right) \\ &\quad - \frac{4n\alpha\rho D_S}{r(1+\varepsilon_1^2)}V - \frac{4nD_S}{r(1+\varepsilon_1^2)}\left(W' - \frac{r'}{r}W\right), \\ &= L_h(g, h, V, f, f', W, W'). \end{aligned} \tag{4}$$

where

$$\rho = \frac{1}{R_\theta} - \frac{1}{R_\xi}, \qquad \varepsilon_0^2 = D_SA_S\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left(\frac{1}{R_\theta} - \frac{3}{R_\xi}\right), \qquad \varepsilon_1^2 = D_SA_S\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)^2. \tag{5}$$

Finally, the substitution of (3.3), (3.4) and (3.13) into (3.20) and the substitution of (3.10), (3.11) and (3.6) into (3.23) give us after some rearrangements

$$\begin{aligned} W'' &= \frac{\alpha}{R_\xi}U' + \alpha\left(\frac{1}{R_\xi}\right)'U + \frac{\alpha'}{\alpha}W' - \frac{\alpha^2}{D(1-\nu_b^2)}\left\{\frac{n}{r}h - \frac{\nu_b}{\alpha}g' - \frac{r'}{r\alpha}g + \left(\frac{1}{R_\theta} - \frac{\nu_b}{R_\xi}\right)f\right\}, \\ &= L_W(U, U', g, g', h, W', f), \end{aligned} \tag{6}$$

$$\begin{aligned} f'' &= \frac{\alpha}{R_\xi}g' + \alpha\left(\frac{1}{R_\xi}\right)'g + \frac{\alpha'}{\alpha}f' + \frac{\alpha^2}{A(1-\nu_s^2)}\left\{\frac{n}{r}V + \frac{\nu_s}{\alpha}U' - \frac{r'}{r\alpha}U + \left(\frac{1}{R_\theta} + \frac{\nu_s}{R_\xi}\right)W\right\}, \\ &= L_f(g, g', U, U', V, f', W). \end{aligned} \tag{7}$$

The fact that the only second derivatives which appear in (1), (2), (3), (4), (6) and (7) are W'' and f'' suggests that the reduction be to simultaneous equations for W and f. The first step in such a reduction is to eliminate U', V', g' and h' from (6) and (7) by means of equations (1) to (4). In this way, we obtain

$$W'' = L_{W2}(g, h, U, V, f, f', W, W'), \tag{8}$$

$$f'' = L_{f2}(U, V, g, h, W, W', f, f'). \tag{9}$$

We next differentiate (8) and (9) and eliminate U', V', g' and h' from the right hand sides of the resulting equations by means of (1), (2), (3) and (4), yielding

$$W''' = L_{W3}(g, h, U, V, f, f', f'', W, W', W''), \tag{10}$$

$$f''' = L_{f3}(U, V, g, h, W, W', W'', f, f', f''). \tag{11}$$

We differentiate both sides of (10) and (11) once more and again eliminate U', V', g' and h'. In this way, we obtain

$$W^{\mathrm{IV}} = L_{W4}(g, h, U, V, f, f', f'', f''', W, W', W'', W'''), \tag{12}$$

$$f^{\mathrm{IV}} = L_{f4}(U, V, g, h, W, W', W'', W''', f, f', f'', f'''). \tag{13}$$

Equations (8), (9), (10) and (11) are now considered as four linear algebraic equations for U, V, g and h. Finally, the solution of this system is used to eliminate these four quantities from (12) and (13). In this way, we obtain the system

$$W^{\mathrm{IV}} = \hat{L}_{W4}(W, W', W'', W''', f, f', f'', f'''), \tag{14}$$

$$f^{\mathrm{IV}} = \hat{L}_{f4}(f, f', f'', f''', W, W', W'', W'''), \tag{15}$$

as a direct consequence of the original system of shell equations.

In Section 5 and 6, this reduction to two simultaneous equations for W and f, as well as the derivation of the auxiliary equations expressing the stress resultants and couples in terms of W and f will be carried out in detail for the case of a circular cylindrical shell.

5. The exact reduction for circular cylindrical shells

We set $r(\xi) = a$ and $z(\xi) = a\xi$ where a is a constant, therewith

$$\alpha = a, \qquad \frac{1}{R_\theta} = -\frac{1}{a}, \qquad \frac{1}{R_\xi} = 0. \tag{1}$$

Equations (4.1) to (4.4) become

$$U' = -\frac{A(1-\nu_s^2)}{a}(n^2 f - nh) + \nu_s(W - nV), \tag{2}$$

$$g' = \frac{D(1-\nu_b^2)}{a}(n^2 W - nV) - \nu_b(f - nh), \tag{3}$$

$$V' = -\frac{4nA_S}{a(1+\varepsilon_1^2)}(g - f') + \frac{4n\varepsilon_1^2}{3(1+\varepsilon_1^2)}W' + \frac{n(3-\varepsilon_1^2)}{3(1+\varepsilon_1^2)}U, \tag{4}$$

$$h' = \frac{4nD_S}{a(1+\varepsilon_1^2)}(U - W') + \frac{4n\varepsilon_1^2}{3(1+\varepsilon_1^2)}f' + \frac{n(3-\varepsilon_1^2)}{3(1+\varepsilon_1^2)}g, \tag{5}$$

where now

$$\varepsilon_1^2 = \frac{9D_S A_S}{a^2}. \tag{6}$$

Equations (4.8) and (4.9) become

$$f'' = \frac{a}{A}(nV - W) - \nu_s(n^2 f - nh), \tag{7}$$

$$W'' = -\frac{a}{D}(nh - f) + \nu_b(n^2 W - nV). \tag{8}$$

The further reduction will be carried out, for the sake of simplicity, under the assumption of constant A, D, $\nu_s = \nu_b = \nu$ and $D_S A_S = DA(1-\nu^2)/4$.

We differentiate (7) and (8) once with respect to ξ to get

$$f''' = \frac{a}{A}(nV' - W') - \nu_s(n^2 f' - nh'), \tag{9}$$

$$W''' = -\frac{a}{D}(nh' - f') + \nu_b(n^2 W' - nV'). \tag{10}$$

Upon using (4) and (5) to eliminate h' and V' from (9) and (10), we have further

$$f''' = n^2 A_1(f' - g) - \frac{a}{A}(A_2 W' - n^2 A_3 U), \tag{11}$$

$$W''' = n^2 A_1^*(W' - U) + \frac{a}{D}(A_2^* f' - n^2 A_3^* g), \tag{12}$$

where

$$A_1 = 2 + \nu_s - \frac{2\varepsilon_1^2(3 + \nu_s)}{3(1 + \varepsilon_1^2)}, \qquad A_2 = 1 - \frac{4n^2\varepsilon_1^2(3 + \nu_s)}{9(1 + \nu_s)(1 + \varepsilon_1^2)},$$

$$A_3 = 1 - \frac{4\varepsilon_1^2(3 + \nu_s)}{9(1 + \nu_s)(1 + \varepsilon_1^2)}, \tag{13}$$

and where the A_i^* are obtained from the A_i by replacing all the elasticity constants in A_i by their duals.

We now differentiate (11) and (12) and use (2) and (3) to eliminate U' and g'. Therewith, we have

$$f^{\mathrm{IV}} = n^2(B_1 f + B_2 f'') + \frac{a}{A}(n^2 B_3 W + B_4 W'') + n^2 B_5 h - \frac{n^3 a}{A} B_6 V, \tag{14}$$

$$W^{\mathrm{IV}} = n^2(B_1^* W + B_2^* W'') - \frac{a}{D}(n^2 B_3^* f + B_4^* f'') + n^2 B_5^* V + \frac{n^3 a}{D} B_6^* h, \tag{15}$$

where

$$B_1 = \nu_b A_1 - n^2(1 - \nu^2)A_3, \qquad B_2 = A_1, \qquad B_3 = \nu_s A_3 + \frac{4n^2\varepsilon_1^2}{9} A_1,$$

$$B_4 = -A_2, \qquad B_5 = (1 - \nu^2)A_3 - \nu_b A_1, \qquad B_6 = \nu_s A_3 + \frac{4\varepsilon_1^2}{9} A_1. \tag{16}$$

and where the B_i^* are the static geometric duals of the B_i in the same manner that the A_i^* are the duals of A_i.

Equations (14) and (15) will become two simultaneous differential equations for W and f if we can express V and h in terms of W and f and their derivatives. The latter is accomplished by solving equations (7) and (8) for V and h, yielding*

$$(1 - \nu^2\varepsilon_2^2)nV = (1 - n^2\nu^2\varepsilon_2^2)W + \nu_s\varepsilon_2^2 W'' + \frac{A}{a}[\nu_s(n^2 - 1)f + f''], \tag{17}$$

$$(1 - \nu^2\varepsilon_2^2)nh = (1 - n^2\nu^2\varepsilon_2^2)f - \nu_b\varepsilon_2^2 f'' + \frac{D}{a}[\nu_b(n^2 - 1)W - W''], \tag{18}$$

where

$$\varepsilon_2^2 = \frac{DA}{a^2}. \tag{19}$$

Upon substituting (17) and (18) into (14) and (15), we end up with two simultaneous

* In contrast to the general case (see (8), (9), (10), and (11) of Section (4)), the simple geometry of the cylindrical shell uncouples (7) and (8) from (11) and (12).

equations for f and W of the form

$$\boxed{\begin{aligned} f^{\mathrm{IV}} - C_2 f'' + C_1 f &= -\frac{a}{A}(C_4 W'' + C_3 W), \\ W^{\mathrm{IV}} - C_2^* W'' + C_1^* W &= \frac{a}{D}(C_4^* f'' + C_3^* f). \end{aligned}} \qquad \begin{aligned} (20) \\ (21) \end{aligned}$$

In these,

$$C_1 = \nu_s(n^2 - 1)B_6 - B_1 - B_5 + \frac{(n^2 - 1)\nu^2\varepsilon_2^2}{1 - \nu^2\varepsilon_2^2}(B_5 + \nu_s B_6),$$

$$C_2 = A_1 - B_6 - \frac{\varepsilon_2^2}{1 - \nu^2\varepsilon_2^2}[(1 - \nu^2)B_6 + \nu_b B_5],$$

$$C_3 = B_6 - B_3 - \frac{(n^2 - 1)\varepsilon_2^2}{1 - \nu^2\varepsilon_2^2}[(1 - \nu^2)B_6 + \nu_b B_5],$$

$$C_4 = A_2 + \frac{n^2\varepsilon_2^2}{1 - \nu^2\varepsilon_2^2}(B_5 + \nu_s B_6). \qquad (22)$$

and the C_i^* are the duals of the C_i.

6. Auxiliary equations for cylindrical shells

To complete our analysis, we now obtain equations which express the stress resultants and couples and the two tangential displacement components in terms of W, f and their derivatives.

From equation (3.13), we have immediately

$$n_\theta = \frac{f''}{a^2}. \qquad (1)$$

From (3.14) and (5.18), we have

$$\begin{aligned} n_\xi &= -\frac{n}{a^2}(nf - h), \\ &= -\frac{1}{a^2(1 - \nu^2\varepsilon_2^2)}\left\{(n^2 - 1)f + \nu_b\varepsilon_2^2 f'' + \frac{D}{a}[W'' - \nu_b(n^2 - 1)W]\right\}. \end{aligned} \qquad (2)$$

By the static geometric duality, we have correspondingly

$$\chi_\xi = -\frac{W''}{a^2}, \qquad \chi_\theta = \frac{1}{a^2(1 - \nu^2\varepsilon_2^2)}\left\{(n^2 - 1)W - \nu_s\varepsilon_2^2 W'' - \frac{A}{a}[f'' + \nu_s(n^2 - 1)f]\right\}.$$

The stress strain relations (3.22) and (3.23) then give (3, 4)

$$m_\xi = -\frac{D}{a^2(1 - \nu^2\varepsilon_2^2)}\left\{W'' - \nu_b(n^2 - 1)W - \frac{\nu_b A}{a}[f'' + \nu_s(n^2 - 1)f]\right\}, \qquad (5)$$

$$m_\theta = -\frac{D}{a^2(1 - \nu^2\varepsilon_2^2)}\left\{(1 + 4\varepsilon_1^2)\nu_b W'' + (n^2 - 1)W - \frac{A}{a}[f'' + \nu_s(n^2 - 1)f]\right\}. \qquad (6)$$

From (3.15) and (5.18), we have also

$$n_{\theta\xi} = \frac{nf' - h'}{a^2},$$

$$= \frac{1}{na^2(1 - \nu^2\varepsilon_2^2)}\left\{(n^2 - 1)f' - \nu_b\varepsilon_2^2 f''' - \frac{D}{a}[W''' - \nu_b(n^2 - 1)W']\right\}. \tag{7}$$

To obtain the remaining stress measures, we need to express g in terms of W and f. Solving (5.11) and (5.12) for g and U gives us

$$n^2\Delta g = -\frac{D}{a}[A_3 W''' + A_1^*(A_2 - n^2 A_3)W'] - [\varepsilon_2^2 A_1^* f''' - (A_3 A_2^* + n^2\varepsilon_2^2 A_1 A_1^*)f'] \tag{8}$$

$$n^2\Delta U = \frac{A}{a}[A_3^* f''' + A_1(A_2^* - n^2 A_3^*)f'] - [\varepsilon_2^2 A_1 W''' - (A_3^* A_2 + n^2\varepsilon_2^2 A_1^* A_1)W'], \tag{9}$$

where

$$\Delta = A_3 A_3^* + \varepsilon_2^2 A_1 A_1^*. \tag{10}$$

With (8), we get from (3.16), (3.12), (3.27) and (3.28)

$$na^2 n_{\xi\theta} = \varepsilon_2^2\left(\frac{\nu_b}{1 - \nu^2\varepsilon_2^2} + \frac{A_1^*}{\Delta}\right)f''' + \left[\frac{2n^2 - 1 - n^2\nu^2\varepsilon_2^2}{1 - \nu^2\varepsilon_2^2} - \frac{1}{\Delta}(A_3 A_2^* + n^2\varepsilon_2^2 A_1 A_1^*)\right]f'$$
$$+ \frac{D}{a}\left(\frac{1}{1 - \nu^2\varepsilon_2^2} + \frac{A^3}{\Delta}\right)W''' - \frac{D}{a}\left[\frac{\nu_b(n^2 - 1)}{1 - \nu^2\varepsilon_2^2} - \frac{A_1^*}{\Delta}(A_2 - n^2 A_3)\right]W', \tag{11}$$

$$\frac{2na^2}{D_S}m = \varepsilon_2^2\left(\frac{3\nu_s}{1 - \nu^2\varepsilon_2^2} - \frac{A_1}{\Delta}\right)W''' - \left[\frac{4n^2 - 3 - n^2\nu^2\varepsilon_2^2}{1 - \nu^2\varepsilon_2^2} - \frac{1}{\Delta}(A_2 A_3^* + n^2\varepsilon_2^2 A_1 A_1^*)\right]W'$$
$$+ \frac{A}{a}\left(\frac{3}{1 - \nu^2\varepsilon_2^2} + \frac{A_3^*}{\Delta}\right)f''' + \frac{A}{a}\left[\frac{3\nu_s(n^2 - 1)}{1 - \nu^2\varepsilon_2^2} + \frac{A_1}{\Delta}(A_2^* - n^2 A_3^*)\right]f', \tag{12}$$

$$2a^2 q_\xi = -\frac{D}{a}\left(\frac{1}{1 - \nu^2\varepsilon_2^2} + \frac{A_3}{\Delta}\right)W''' + \frac{D}{a}\left[\frac{\nu_b(n^2 - 1)}{1 - \nu^2\varepsilon_2^2} + \frac{A_1^*}{\Delta}(n^2 A_3 - A_2)\right]W'$$
$$- \varepsilon_1^2\left(\frac{\nu_b}{1 - \nu^2\varepsilon_2^2} + \frac{A_1^*}{\Delta}\right)f''' - \left[\frac{1 + (n^2 - 2)\nu^2\varepsilon_2^2}{1 - \nu^2\varepsilon_2^2} + \frac{1}{\Delta}(A_3 A_2^* + n^2\varepsilon_2^2 A_1 A_1^*)\right]f', \tag{13}$$

$$2na^2 q_\theta = -\frac{D}{a}\left[\frac{\nu_b(C_4 + n^2 - 1) - C_2^*}{1 - \nu^2\varepsilon_2^2} + \frac{C_4 A_1^* - C_2^* A_3 - A_1^*(A_2 - n^2 A_3)}{\Delta}\right]W''$$
$$- \frac{D}{a}\left[\frac{C_1^* - \nu_b\varepsilon_2^2 C_3}{1 - \nu^2\varepsilon_2^2} + \frac{C_1^* A_3 + C_3 A_1^*}{\Delta}\right]W$$
$$- \left[\frac{1 - C_4^* - \varepsilon_2^2(n^2\nu^2 + \nu_b C_2)}{1 - \nu^2\varepsilon_2^2} + \frac{A_3 A_2^* + \varepsilon_2^2 A_1^*(n^2 A_1 - C_2)}{\Delta}\right]f''$$
$$+ \left[\frac{C_3^* - \nu_b\varepsilon_2^2 C_1}{1 - \nu^2\varepsilon_2^2} + \frac{A_3 C_3^* - A_1^* C_1\varepsilon_2^2}{\Delta}\right]f. \tag{14}$$

7. Simplified equations for cylindrical shells

In the derivation of the stress strain relations (2.7) and (2.8), terms of relative order h/a were neglected. If $DA = O(h^2)$, which is the case for a homogeneous isotropic shell, we should consistently omit from the exact results of the last two sections all terms of the order $\varepsilon_2^2 = DA/a^2$ compared to unity. In this way, the two exact simultaneous equations for W and f become simplified to

$$W^{\mathrm{IV}} - 2n^2 W'' + n^2(n^2-1)W = \frac{a}{D}\{f'' + \underline{\varepsilon_2^2(n^2-1)[2n^2(1-\nu^2) - \nu_s^3]f}\}, \tag{1}$$

$$f^{\mathrm{IV}} - 2n^2 f'' + n^2(n^2-1)f = -\frac{a}{A}\{W'' + \underline{\varepsilon_2^2(n^2-1)[2n^2(1-\nu^2) + \nu_b^3]W}\}. \tag{2}$$

Correspondingly, the auxiliary equations become

$$n_\theta = \frac{f''}{a^2}, \qquad n_\xi = -\frac{1}{a^2}\left\{(n^2-1)f + \underline{\nu_s\varepsilon_2^2 f''} + \underline{\underline{\frac{D}{a}[W'' - \nu_b(n^2-1)W]}}\right\},$$

$$n_{\xi\theta} = \frac{1}{na^2}\left\{(n^2-1)f' - \underline{\varepsilon_2^2 f'''} + \frac{D}{a}[W''' - (n^2-1)W']\right\},$$

$$n_{\theta\xi} = \frac{1}{na^2}\left\{(n^2-1)f' + \underline{\nu_s\varepsilon_2^2 f'''} + \frac{D}{a}[W''' - \nu_b(n^2-1)W']\right\},$$

$$m_\xi = -\frac{D}{a^2}\left\{W'' - \nu_b(n^2-1)W - \underline{\underline{\frac{\nu_b A}{a}[f'' + \nu_s(n^2-1)f]}}\right\},$$

$$m_\theta = -\frac{D}{a^2}\left\{\nu_b W'' - (n^2-1)W - \underline{\underline{\frac{A}{a}[f'' + \nu_s(n^2-1)f]}}\right\},$$

$$m = -\frac{2D_S}{na^2}\left\{(n^2-1)W' + \underline{(1-\nu_b)\varepsilon_2^2 W'''} - \frac{A}{a}[f''' - \underline{\underline{\tfrac{1}{2}(n^2-1)(1-\nu_s)f'}}]\right\},$$

$$q_\xi = -\frac{D}{a^3}\left\{W''' - (n^2-1)W' + \underline{\underline{\frac{A}{a}[f''' - \tfrac{1}{2}(n^2-1)(1-2\nu_s-\nu^2)f']}}\right\},$$

$$q_\theta = \frac{nD}{a^3}\left\{W'' - (n^2-1)W + \underline{\underline{O\left(\frac{A}{a}f''\right)}}\right\},$$

$$U_\xi = \frac{1}{n^2}\left\{W' - \underline{\varepsilon_2^2(2+\nu_s)W'''} + \frac{A}{a}[f''' - (2+\nu_s)(n^2-1)f']\right\},$$

$$U_\theta = \frac{1}{n}\left\{W + \underline{\nu_b\varepsilon_2^2 W''} + \frac{A}{a}[f'' + \nu_s(n^2-1)f]\right\}. \tag{3}$$

From the form of the two differential equations (1) and (2), we know that differentiation with respect to ξ changes orders of magnitude by a factor $\lambda = O(\max[n, \sqrt{a/h}])$ at most. Therefore, all singly underlined terms in (1), (2) and (3) will be omitted as they are $O(h/a)$, at most, compared to the dominant terms in the corresponding equations. It turns out that we can also omit all doubly underlined terms. The justification for this is briefly as follows:

Suppose that the doubly underlined terms contributed significantly to the stress couples. It follows then that

$$W = O\left(\frac{A}{a}f,\right) \qquad (m_\xi, m_\theta) = O\left(\frac{\lambda^2 DA}{a^3}f\right). \tag{4, 5}$$

In that case, the doubly underlined terms in n_ξ are $O(h^2/a^2)$ compared to the first term and can therefore be omitted. At the same time, the relative order of magnitude of the bending stresses and the direct stresses is

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{6m_\theta/h^2}{n_\theta/h}\right) = O\left(\frac{\lambda^2 DAf/a^3}{\lambda^2 hf/a^2}\right) = O\left(\frac{h}{a}\right). \tag{6}$$

In view of the error inherent in shell theory, *the accuracy of the stress couples m_ξ and m_θ is not needed (and in fact not meaningful)* and we might as well omit the doubly underlined terms in these quantities also. A similar argument applies to q_ξ and q_θ.

If, on the other hand, the doubly underlined terms in n_ξ are significant, we will have

$$f = O\left(\frac{D\lambda^2}{n^2 a}W\right), \qquad n_\xi = O\left(\frac{\lambda^2 D}{a^3}W\right). \tag{7, 8}$$

The contribution of the doubly underlined terms to the stress couples m_ξ and m_θ and the transverse shear resultants in this case is $O(h/a)$ at most compared to the other terms and can therefore be omitted.

At the same time, the relative order of magnitude of the direct stress associated with n_ξ and pending stress associated with m_ξ is

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{n_\xi/h}{6m_\xi/h^2}\right) = O\left(\frac{\lambda^2 hDW/a^3}{\lambda^2 DW/a^2}\right) = O\left(\frac{h}{a}\right). \tag{9}$$

Therefore, *the accuracy of the membrane stress resultant n_ξ is meaningless* and we might as well omit the doubly underlined terms in it also.

With the omission of all underlined terms, the equations of the theory of circular cylindrical shells for stress distributions of the form (3.1) and (3.2)*, now consist of the two simultaneous differential equations

$$\boxed{\begin{aligned} &W^{\mathrm{IV}} - 2n^2 W'' + n^2(n^2 - 1)W = \frac{a}{D}f'', && (10)\\ &f^{\mathrm{IV}} - 2n^2 f'' + n^2(n^2 - 1)f = -\frac{a}{A}W'', && (11)\end{aligned}}$$

and of the following expressions for the stress measures and the tangential displacement components:

$$n_\theta = \frac{f''}{a^2}, \qquad n_\xi = -\frac{n^2 - 1}{a^2}f,$$

$$n_{\xi\theta} = \frac{1}{na^2}\left\{(n^2 - 1)f' + \frac{D}{a}[W''' - (n^2 - 1)W']\right\},$$

* For the $n = 0$ and $n = 1$ cases, the omission of the doubly underlined terms in the auxiliary equations may lead to incorrect expressions for the resultant forces and moments [9].

$$n_{\theta\xi} = \frac{1}{na^2}\left\{(n^2-1)f' + \frac{D}{a}[W''' - \nu_b(n^2-1)W']\right\},$$

$$m_\xi = -\frac{D}{a^2}\{W'' - \nu_b(n^2-1)W\}, \qquad m_\theta = -\frac{D}{a^2}\{\nu_b W'' - (n^2-1)W\},$$

$$m = -\frac{2D_S}{na^2}\left\{(n^2-1)W' - \frac{A}{a}f'''\right\},$$

$$q_\xi = -\frac{D}{a^3}\{W''' - (n^2-1)W'\}, \qquad q_\theta = \frac{nD}{a^3}\{W'' - (n^2-1)W\},$$

$$U = \frac{1}{n^2}\left\{W' + \frac{A}{a}[f''' - (2+\nu_s)(n^2-1)f']\right\},$$

$$V = \frac{1}{n}\left\{W + \frac{A}{a}[f'' + \nu_s(n^2-1)f]\right\}. \tag{12}$$

Except for the twisting couple m, equations (10), (11) and (12) are a simplified version of the auxiliary equations of [13] for $n \geq 2$. On the other hand, our expression for m differs significantly from that given by Simmonds in that the latter does not contain any term involving f.

8. A simplified reduction for general shells of revolution

The exact reduction as described in section (4) may be seen to involve a considerable amount of routine calculations in most cases. The necessary amount of calculations may be reduced by omitting all terms of the order of $(h/R)^2$ in the intermediate steps (e.g., in (4), (5), (11), (12), (14), (15), (17) and (18) of section (5)), where R is a representative magnitude of the principal radii of curvature. It is not difficult to verify that doing this for the case of the cylindrical shell leads to results which differ from (7.1), (7.2) and (7.3) only by the underlined terms in the governing differential equations and in the expression for q_θ. These terms are of no consequence in a first approximation shell theory.

9. Exact reduction to a single equation for W or f

Parallel to the reduction of Section 4 is the possibility of an exact reduction of the shell equations to a single eighth order differential equation for W or f. Since the latter reduction is in general somewhat different from the reduction to two simultaneous equations, we will briefly describe in this section a suitable procedure.

To accomplish our objective, we first rewrite the various formulas for the first derivatives of the displacement functions as given by (3.17a), (3.23) (but now using (2.4a) for χ_ξ), (4.1) and (4.3) in the form

$$W' = -(\alpha\Phi) + \frac{\alpha}{R_\xi}U, \tag{1}$$

$$(\alpha\Phi)' = -\frac{\nu_b n^2\alpha^2}{r^2}W + \left(\frac{\alpha'}{\alpha} - \frac{\nu_b r'}{r}\right)(\alpha\Phi) + \frac{\nu_b n a^2}{rR_\theta}V + \frac{\alpha^2}{D}\left(\frac{1}{R_\theta}f + \frac{r'}{r\alpha}g + \frac{n}{r}h\right), \tag{2}$$

$$U' = -\left(\frac{\alpha}{R_\xi} + \frac{\nu_s\alpha}{R_\theta}\right)W - \nu_s\left(\frac{r'}{r}U + \frac{\alpha n}{r}V\right) - A(1-\nu^2)\left[\frac{n^2\alpha}{r^2}f + \frac{r'}{r\alpha}(\alpha i) + \frac{n\alpha}{rR_\theta}h\right], \tag{3}$$

$$V' = \frac{4nD_S A_S}{r(1+\varepsilon_1^2)}\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left[\frac{r'}{r}W + (\alpha\Phi)\right] + \frac{n\alpha(1-\varepsilon_3^2)}{r(1+\varepsilon_1^2)}U + \frac{r'}{r}V$$
$$- \frac{4nA_S}{r(1+\varepsilon_1^2)}\left[\frac{r'}{r}f + (\alpha i) - \frac{\alpha}{R_\xi}g\right] + \frac{4n\alpha\rho A_S}{r(1+\varepsilon_1^2)}h, \tag{4}$$

$$\varepsilon_3^2 = D_S A_S\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right)\left(\frac{1}{R_\theta} + \frac{1}{R_\xi}\right). \tag{5}$$

By the static geometric duality, we also have the corresponding formulas for the dual stress functions.

Next, we list once again the expression for W'' given by (4.8) but now in the form

$$W'' = b_{21}W + b_{22}(\alpha\Phi) + b_{23}U + b_{24}V + b_{25}f + b_{26}(\alpha i) + b_{27}g + b_{28}h, \tag{6}$$

where use has been made of (1) and its dual to eliminate W' and f'. We will not list the explicit expressions for the coefficients b_{2j} as they are not needed here.

We now differentiate (6) to get

$$W''' = b_{21}W' + b_{22}(\alpha\Phi)' + b_{23}U' + b_{24}V' + b_{25}f' + b_{26}(\alpha i)' + b_{27}g' + b_{28}h'$$
$$+ b'_{21}W + b'_{22}(\alpha\Phi) + b'_{23}U + b'_{24}V + b'_{25}f + b'_{26}(\alpha i) + b'_{27}g + b'_{28}h. \tag{7}$$

The first derivatives on the right are then eliminated by way of (1) through (4) as well as their duals. Therewith, we get

$$W''' = b_{31}W + b_{32}(\alpha\Phi) + b_{33}U + b_{34}V + b_{35}f + b_{36}(\alpha i) + b_{37}g + b_{38}h. \tag{8}$$

We repeat this process to get for $k = 1, \ldots, 8$

$$W^{(k)} = b_{k1}W + b_{k2}(\alpha\Phi) + b_{k3}U + b_{k4}V + b_{k5}f + b_{k6}(\alpha i) + b_{k7}g + b_{k8}h, \tag{9}$$

with the case $k = 1$ being equation (1).

We can now think of (9) as eight linear algebraic equations for the eight unknowns W, $(\alpha\Phi)$, U, V, f, (αi), g, and h. Upon solving these equations for W, we have the desired eighth order differential equation. The solution for the remaining unknowns will provide us with all necessary information for the determination of the stress and displacement measures.

10. Remarks

Since we are concerned here mainly with an exact reduction of the equations of linear shell theory to two simultaneous equations for a stress function and a displacement function, we mentioned in the introduction only related reductions which have as their point of departure an acceptable set of shell equations and which entail no approximation (at least in principle). Examples of other reductions which do not fit into this category are the well-known work of V. V. Novozhilov [16] and the more recent work of J. L. Sanders [17]. The former omits terms from the original shell equations during intermediate stages of the reduction $\nu \neq 0$, while the latter obtains a sequence of two fourth order complex equations to be solved successively. After reading a preprint of the present article, Professor J. G. Simmonds pointed out that A. L. Goldenveiser has proposed a conceptually similar reduction to two simultaneous sixth order partial differential equations [18]. While the final two equations were not given, it appears that upon specializing to shells of revolution and separating out $e^{in\theta}$, the reduction of [18] would lead to only two sixth order differential equations in ξ.

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