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Rotationally Symmetric Shearing and Bending of Helicoidal Shells

By Frederic Y. M. Wan

1. Introduction

Past work on the subject of rotationally symmetric stress and strain in elastic helicoidal shells has been concerned mainly with two distinct classes of problems. The first of these involves in-plane normal stress resultants and twisting couples (and non-rotationally symmetric displacement components), and includes the problem of twisting and stretching of helicoidal strips [3, 6, 8, 12]. The second involves in-plane shear stress resultants and bending couples and includes the problem of a shell with one fixed helical edge while the other concentric helical edge is subjected to rigid body axial and circumferential edge displacements [1, 4, 5, 7, 9]. The displacement state associated with the second class of problem is also rotationally symmetric.

The purpose of the present work is to exploit the observation that the governing differential equations for the two classes of problems mentioned above are the static geometric duals of each other. Motivated by this duality, we undertake a reduction of the problem of the second type in terms of a dependent variable which is the static geometric dual of another variable with the help of which there has been recent progress on problems of the first type [12]. In this way, simplifications and generalizations of the theoretical aspects of the work done in [1], [9] and [4] are attained.

Going beyond the well known duality among the governing differential equations [2, 10], we establish that there is also a static geometric duality for the two sets of appropriate boundary conditions for the two specific physical problems mentioned earlier, namely, the problem of twisting and stretching by axial end forces and torques and the problem of relative axial and circumferential rigid body edge displacements. This is accomplished by reformulating the displacement boundary conditions of the second problem in terms of strain and curvature change measures, as a special case of a result obtained in [13]. The complete static geometric duality between the two boundary value problems then permits us to obtain analytical results for the problem of relative axial and circumferential edge displacements from those for the problem of axial torsion and extension [12] without another set of independent calculations. Moreover, the computer program developed in [12] for the latter problem can also be used, without any modification, to generate the finite difference solution for the former problem. The numerical results obtained here verify and extend the results obtained in [4] which (as was already pointed out in [4]) were limited by the convergence properties of the power series solution used to a certain range of pitch to width ratio of the shell. It is believed that this is the first example of the use of the static geometric duality in connection with the computational aspect of non-shallow shell problems. A not so elaborate computational use of the static geometric duality has recently been considered for shallow shells in [14].

2. Differential equations and static geometric duality

In cylindrical coordinates (r, θ, z), the middle surface of a helicoidal shell is given by the equation $z = a\theta$ where $2\pi a$ is the pitch of the helicoid. It was found in [12] that the appropriate displacement field for problems of elastic helicoidal shells involving rotationally symmetric stress distributions is of the form

$$
\begin{aligned}
u_r(r,\theta) &= u(r) + ca^2(1 - \cos\theta - \theta\sin\theta)\\
u_\theta(r,\theta) &= v(r) + \frac{ka + \psi r^2}{\alpha} a\theta - \frac{a^2 r}{\alpha}(\theta\cos\theta - 2\sin\theta)\\
u_n(r,\theta) &= w(r) + \frac{kr - \psi ar}{\alpha} a\theta + \frac{ca}{\alpha}[a^2\theta\cos\theta - (a^2 - r^2)\sin\theta]\\
\phi_\theta(r,\theta) &= \frac{u - kr + \psi ar}{R} + ca\left(\frac{a}{R} - \cos\theta\right)\\
\phi_r(r,\theta) &= -w' + \frac{v}{R} + \frac{\psi a}{\alpha} a\theta - \frac{car}{\alpha}\sin\theta\\
\omega(r,\theta) &= \frac{(\alpha v)'}{2\alpha} + \frac{\psi r}{\alpha} a\theta + \frac{ca^2}{\alpha}\sin\theta
\end{aligned}
\tag{1}
$$

where

$$
\alpha = \sqrt{a^2 + r^2}, \qquad \frac{1}{R} = \frac{a}{\alpha^2}, \tag{2}
$$

and where u_r, u_θ and u_n are the displacement components in the radial, tangential and normal direction, respectively, ϕ_θ, ϕ_r and ω are the rotational displacement components turning about the same directions, respectively, k, ψ and c are constants, and primes indicate differentiation with respect to r.

The midsurface strains and curvature changes corresponding to (1) divide themselves into two separate groups:

$$
\begin{aligned}
\varepsilon_r &= u', \qquad \varepsilon_\theta = \frac{ru}{\alpha^2} + \frac{car + ka + \psi r^2}{R}\\
\kappa_{r\theta} &= \frac{1}{R}\left(u' - \frac{2r}{\alpha^2}u\right) - \frac{2ca^2 r + (k - \psi a)(a^2 - r^2)}{R\alpha^2}\\
\kappa_{\theta r} &= -\frac{1}{R\alpha^2}[ru + ca^2 r - kr^2 - \psi a^3]
\end{aligned}
\tag{3}
$$

and

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2}\left(v' - \frac{rv}{\alpha^2} + \frac{2w}{R}\right)$$
$$\kappa_r = -w'' + \frac{v'}{2R} - \frac{3rv}{2R\alpha^2}, \qquad \kappa_\theta = -\frac{rw'}{\alpha^2} - \frac{v'}{2R} + \frac{rv}{2R\alpha^2}. \tag{4}$$

Evidently, the strain measures (3) satisfy the following compatibility equations:

$$(\alpha\kappa_{\theta r})' + \frac{r}{\alpha}\kappa_{r\theta} + \frac{a}{\alpha}\lambda_\theta = 0, \qquad (\alpha\lambda_\theta)' - \frac{a}{\alpha}(\kappa_{\theta r} + \kappa_{r\theta}) = 0 \tag{5, 6}$$

$$(\alpha\varepsilon_\theta)' - \frac{r}{\alpha}\varepsilon_r - \alpha\lambda_\theta = 0, \qquad \kappa_{\theta r} - \kappa_{r\theta} - \frac{\varepsilon_\theta - \varepsilon_r}{R} = 0 \tag{7, 8}$$

while the strain measures (4) satisfy the compatibility equations

$$(\alpha\kappa_\theta)' - \frac{r}{\alpha}\kappa_r + \frac{a}{\alpha}\lambda_r = 0, \qquad (\alpha\varepsilon_{\theta r})' + \frac{r}{\alpha}\varepsilon_{r\theta} - \alpha\lambda_r = 0 \tag{9, 10}$$

where (7) and (10) are to be taken as the defining equations for λ_θ and λ_r, respectively. These quantities, which do not appear in the conventional shell theory, have the meaning of the normal components of the curvature change vectors in a theory which included the effect of stress couples turning about the midsurface normal [10].

The strain measures are related to stress resultants, N, and stress couples, M, of the shell by means of a system of stress–strain relations which is taken in the form

$$M_{r\theta} = M_{\theta r} = \tfrac{1}{2}(1 - \nu_b)D(\kappa_{\theta r} + \kappa_{r\theta}) \tag{11}$$

$$\varepsilon_r = A(N_r - \nu_s N_\theta), \qquad \varepsilon_\theta = A(N_\theta - \nu_s N_r) \tag{12}$$

and

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = \tfrac{1}{2}(1 + \nu_s)A(N_{r\theta} + N_{\theta r}) \tag{13}$$

$$M_\theta = D(\kappa_\theta + \nu_b\kappa_r), \qquad M_r = D(\kappa_r + \nu_b\kappa_\theta). \tag{14}$$

For an isotropic homogeneous shell, we have

$$D = \frac{Eh^3}{12(1 - \nu^2)}, \qquad A = \frac{1}{Eh}, \qquad \nu_s = \nu_b = \nu \tag{15}$$

where E is Young's modulus, ν is Poisson's ratio and h is the shell thickness.

The stress resultants and stress couples together with the transverse shear resultants, Q_r and Q_θ, satisfy six differential equations of equilibrium,

$$(\alpha N_r)' - \frac{r}{\alpha}N_\theta - \frac{a}{\alpha}Q_\theta + \alpha p_r = 0, \qquad (\alpha M_{r\theta})' + \frac{r}{\alpha}M_{\theta r} - \alpha Q_\theta = 0 \tag{16, 17}$$

and

$$(\alpha N_{r\theta})' + \frac{r}{\alpha}N_{\theta r} + \frac{a}{\alpha}Q_r + \alpha p_\theta = 0, \qquad (\alpha Q_r)' - \frac{a}{\alpha}(N_{r\theta} + N_{\theta r}) + \alpha p_n = 0 \tag{18, 19}$$

$$(\alpha M_r)' - \frac{r}{\alpha} M_\theta - \alpha Q_r = 0, \qquad N_{r\theta} - N_{\theta r} - \frac{M_r - M_\theta}{R} = 0 \tag{20, 21}$$

where p_r, p_θ and p_n are the radial, tangential and normal components, respectively, of the distributed surface load vector.

With suitably prescribed boundary conditions, problems of elastic helicoidal shells involving rotationally symmetric stress and strain are evidently divided into two distinct classes, One of these involves the quantities N_r, N_θ, Q_θ, $M_{r\theta} = M_{\theta r}$, $\kappa_{r\theta}$, $\kappa_{\theta r}$, λ_r, ε_r, ε_θ and u (and the constants k, ψ and c) and is governed by equations (3), (5) to (8), (11), (12), (16) and (17). The problem of axial extension and torsion by end forces and torques investigated in [3, 6, 8, 12] belongs to this class of problems. The other class involves the quantities $N_{r\theta}$, $N_{\theta r}$, Q_r, M_r, M_θ, κ_r, κ_θ, λ_θ, $\varepsilon_{r\theta} = \varepsilon_{\theta r}$, v and w and is governed by equations (4), (9), (10), (13), (14) and (18) to (21). Several problems of this class have been investigated in [1, 4, 5, 7, 9]. We are interested here in the latter class of problems, the associated displacement field of which is also rotationally symmetric (see equation (1)).

With the introduction of the additional quantities λ_r and λ_θ, we now see that, except for the inhomogeneous terms, equations (5) to (8), (11), (12), (16) and (17) are the static geometric duals of equations (18) to (21), (13) (14), (9) and (10), respectively. More specifically, we can obtain one of the two sets of equations from the other sets if we replace all quantities in the latter by their dual quantities according to the following table:

N_r	N_θ	Q_θ	$M_{r\theta}$	$M_{\theta r}$	ε_r	ε_θ	$\kappa_{r\theta}$	$\kappa_{\theta r}$	λ_θ	A	ν_s
$-\kappa_\theta$	$-\kappa_r$	$-\lambda_r$	$-\varepsilon_{\theta r}$	$-\varepsilon_{r\theta}$	M_θ	M_r	$N_{\theta r}$	$N_{r\theta}$	Q_r	$-D$	$-\nu_b$

An important simplification for the problem of stretching and twisting of helicoidal shells is the observation in [8] that the expression for the twisting couple

$$M_{r\theta} = M_{\theta r} = (1 - \nu_b) D \left(u' - 3\frac{ru}{\alpha^2} - 3ca^2 \frac{r}{\alpha^2} + k\frac{2r^2 - a^2}{\alpha^2} + \psi a \frac{2a^2 - r^2}{\alpha^2} \right) \tag{22}$$

may be replaced by the abbreviated explicit relation

$$M_{r\theta} = M_{\theta r} = (1 - \nu_b) D \frac{ka + \psi a^2}{\alpha^2} \tag{22'}$$

which in turn implies $Q_\theta \equiv 0$ [8]. These simplifications enable us to reduce the governing differential equations to a single second order equation for the stress variable N_r [12].

The static geometric duality between the system of equations for the two distinct classes of problems suggests that analogous simplifications of the expression for $\varepsilon_{r\theta} = \varepsilon_{\theta r}$ and λ_r are also possible for the problems of shearing and bending and that the governing differential equations for this class of problems may be reduced to a second order equation for κ_θ.

3. Reduction of the problem of shearing and bending by a stress function

To obtain the simplification of the expression for $\varepsilon_{r\theta} = \varepsilon_{\theta r}$, we will discuss in this section a solution procedure for the problem of shearing and bending of helicoidal shells by means of a stress function.

The four inhomogeneous equilibrium equations (2.18) to (2.21) for five unknowns may be satisfied identically by expressing the five stress measures in terms of a stress function U in the form

$$
\begin{aligned}
M_\theta &= U' + \frac{3\alpha^2}{a}\int^r p_\theta\,dr - \frac{1}{a}\int^r p_\theta\alpha^2\,dr + \frac{\alpha^2}{r^2}\int^r p_n r\,dr \\
M_r &= \frac{r}{\alpha^2}U + \frac{\alpha^2}{a}\int^r p_\theta\,dr - \frac{1}{a}\int^r p_\theta\alpha^2\,dr \\
N_{r\theta} &= -\frac{ar}{\alpha^4}U - \int^r p_\theta\,dr \\
N_{\theta r} &= \frac{1}{R}\left(U' - \frac{2r}{\alpha^2}U\right) + \int^r p_\theta\,dr + \frac{a}{r^2}\int p_n r\,dr \\
Q_r &= \frac{U}{R^2} - \frac{1}{r}\int^r p_n r\,dr.
\end{aligned}
\tag{1}
$$

The terms involving U in (1) are suggested by (2.3) and the static geometric duality, with U being the dual of u. The choice of the inhomogeneous terms in (1) is not the only one but suffices for our purpose.

We note that the stress function solution (1) implies two algebraic relations among the stress measures of the form

$$
rQ_r + aN_{r\theta} = -\int^r (ap_\theta + rp_n)\,dr, \qquad \alpha^2 N_{r\theta} + aM_r = -\int^r p_\theta\alpha^2\,dr \tag{2}
$$

equivalent to the two first integrals of the equilibrium equations obtained in [9].

The strain measures can now be expressed in terms of U with the help of the stress strain relations (2.13) and (2.14) as

$$
\varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{(1+\nu_s)A}{2R}\left[U' - \frac{3r}{\alpha^2}U + \frac{\alpha^2}{r^2}\int^r p_n r\,dr\right] \tag{3}
$$

$$
\begin{aligned}
\kappa_r &= \frac{1}{D(1-\nu_b^2)}\left[-\nu_b U' + \frac{r}{\alpha^2}U - \frac{(1-\nu_b)}{a}\int^r p_\theta\alpha^2\,dr\right. \\
&\qquad \left. + \frac{(1-3\nu_b)\alpha^2}{a}\int^r p_\theta\,dr - \frac{\nu_b\alpha^2}{r^2}\int^r p_n r\,dr\right] \\
\kappa_\theta &= \frac{1}{D(1-\nu_b^2)}\left[U' - \frac{\nu_b r}{\alpha^2}U - \frac{(1-\nu_b)}{a}\int^r p_\theta\alpha^2\,dr\right. \\
&\qquad \left. + \frac{(3-\nu_b)\alpha^2}{a}\int^r p_\theta\,dr + \frac{\alpha^2}{r^2}\int^r p_n\,dr\right].
\end{aligned}
\tag{4}
$$

Equation (2) can be rewritten in the form

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = (1 + v_s)A\left[\frac{M_\theta - 3M_r}{2R} - \frac{1}{\alpha^2}\int^r p_\theta \alpha^2\, dr\right]. \tag{5}$$

Within the range of applicability of the stress strain relations (2.13) and (2.14), terms of the order of the bending moment divided by the radius of curvature are considered negligible in the expression for the in-plane shear strain [11]. It is therefore consistent to replace the above expression by the abbreviated explicit relation

$$\varepsilon_{r\theta} = \varepsilon_{\theta r} = -\frac{(1 + v_s)A}{\alpha^2}\int^r p_\theta \alpha^2\, dr \tag{3'}$$

analogous to the abbreviated explicit relation for $M_{r\theta} = M_{\theta r}$ first obtained in [8].

Introduction of (3′) into the compatibility equation (2.10) gives

$$\lambda_r = -(1 + v_s)A\alpha p_\theta \tag{6}$$

Therewith, the compatibility equation (2.9) becomes

$$(\alpha\kappa_\theta)' - \frac{r}{\alpha}\kappa_r = (1 + v_s)Aap_\theta \tag{7}$$

Introduction of (4) into (7) leads to the following second order linear ordinary differential equation for U

$$U'' + \frac{r}{\alpha^2}U' - \frac{v_b a^2 + r^2}{\alpha^4}U$$
$$= -\frac{2\alpha^2}{a}p_\theta - \frac{\alpha^2}{r}p_n - \frac{8r}{a}\int^r p_\theta\, dr - \frac{[(1 + v_b)r^2 - 2a^2]}{r^3}\int^r p_n r\, dr. \tag{8}$$

We note that the relation (6), which follows from the compatibility equation (2.10), depends on the step from (5) to (3′). Retention of the M/R terms in (5) would mean that an unknown λ_r, expressed in terms of U by (2.10), had to be carried along in (7) and elsewhere. On the other hand, the reduction to a second order differential equation for U does not depend on this simplification. The contribution of p_θ-term of (7) is negligible in (8) and will be omitted.

4. An alternate reduction

Motivated by the results of [12], we take κ_θ instead of U as our primary dependent variable for an alternate reduction. We use the simplified compatability equation (3.7) to express κ_r in terms of κ_θ (with p_θ-term omitted)

$$\kappa_r = \frac{\alpha}{r}(\alpha\kappa_\theta)'. \tag{1}$$

With (1), the two stress strain relations in (2.14) become

$$M_\theta = D\left[\kappa_\theta + v_b\frac{\alpha}{r}(\alpha\kappa_\theta)'\right], \qquad M_r = D\left[\frac{\alpha}{r}(\alpha\kappa_\theta)' + v_b\kappa_\theta\right]. \tag{2}$$

The first two stress couple-stress function relations of (3.1) give rise to an equilibrium equation

$$\left(\frac{\alpha^2}{r}M_r\right)' - M_\theta = \frac{a}{r^2}\int^r \alpha^2 p_\theta \, dr - \frac{\alpha^2}{r^2}\int^r (ap_\theta + rp_n)\, dr \tag{3}$$

or

$$M_r' + \frac{r}{\alpha^2}(M_r - M_\theta) - \frac{a^2}{r\alpha^2}M_r = -\frac{1}{r}\int^r (ap_\theta + rp_n)\, dr + \frac{a}{r\alpha^2}\int^r p_\theta \alpha^2 \, dr. \tag{4}$$

Alternatively, (4) can be obtained by using (3.2) to eliminate Q_r from the (2.20).

Introduction of (2) into (4) leads to a differential equation for κ_θ. For constant D and ν_b, it takes the form

$$\kappa_\theta'' - \frac{2a^2 - 3r^2}{r(a^2 + r^2)}\kappa_\theta' - \frac{(1 + \nu_b)a^2}{(a^2 + r^2)^2}\kappa_\theta = -\frac{1}{D\alpha^2}\int^r (ap_\theta + rp_n)\, dr + \frac{a}{D\alpha^4}\int^r p_\theta\alpha^2 \, dr. \tag{5}$$

Equation (5), together with appropriate boundary conditions, determines κ_θ completely.

We can then use (2) to calculate M_r and M_θ. The two first integrals (3.2) of the equilibrium equations, written in the form

$$N_{r\theta} = -\frac{M_r}{R} - \frac{1}{\alpha^2}\int^r p_\theta\alpha^2 \, dr, \qquad Q_r = -\frac{aN_{r\theta}}{r} - \frac{1}{r}\int^r (ap_\theta + rp_n)\, dr \tag{6}$$

determine $N_{r\theta}$ and Q_r. Finally, we calculate $N_{\theta r}$ from the relation (3.3′) written in the form

$$N_{\theta r} = -N_{r\theta} - \frac{2}{\alpha^2}\int^r p_\theta\alpha^2 \, dr = \frac{M_r}{R} - \frac{1}{\alpha^2}\int^r p_\theta\alpha^2 \, dr. \tag{7}$$

It should be mentioned that while there is no similarity between the two reductions of the present paper and those of earlier work on the same class of problems, the quantity κ_θ turns out to be proportional to the primary dependent variable used in [1].

5. The problem of relative axial and circumferential rigid body edge displacements at a helical edge

As an application of the results of Section 4 and an illustration of the computational use of the concept of the static geometric duality, we consider a helicoidal shell clamped to a fixed rigid cylindrical wall at its outer helical edge $r = r_0$, and clamped to a rigid movable cylinder at its inner helical edge $r = r_i$. The movable cylinder is subject to an axial force and an axial torque resulting in an axial displacement δ and an axial rotation Ω, both in the positive z-direction. We are interested in the stress distribution in the shell as well as the overall load-deformation relations between the applied force and torque on the one hand and the axial displacement and rotation of the inner helical edge on the other hand. We will not prescribe the boundary conditions at the radial edges $\theta = \pm\theta_0$, but rather, appealing to St. Venant's principle, take them as they come out to be.

Previous studies of this problem can be found in [7, 4]. Our objective here is to utilize the formulation of section (4) with an additional analysis of the boundary conditions to establish a complete static geometric duality between this problem and the problem of axial torsion and extension of helicoidal shells studied in [12]. Aside from its theoretical interest, this complete duality also allows us to use the computer program developed in [12] (without any modification) to obtain results for the present problem which were not obtained by the power series solution of [4].

We begin by noting that $p_\theta = p_n = 0$ for our problem and write (3.2) as

$$rQ_r + aN_{r\theta} = -\int^r (ap_\theta + rp_n)\,dr = -Pa$$
$$d^2N_{r\theta} + aM_r = -\int^r p_\theta\alpha^2\,dr = -(Ma + Pa^2) \tag{1}$$

where the constants P and M are the axial force and torque applied to the movable rigid cylinder per unit winding of the shell.

The differential equation (4.5) now becomes

$$\kappa_\theta'' - \frac{2a^2 - 3r^2}{r(a^2 + r^2)}\kappa_\theta' - \frac{(1 + \nu_b)a^2}{(a^2 + r^2)^2}\kappa_\theta = \frac{Ma^2 - Par^2}{D(a^2 + r^2)^2} \tag{2}$$

which is the static geometrical dual of the governing differential equation (24) of [12] including the load terms with $-P$ and $-M$ being the duals of ψ and k respectively.

The outer helical edge of the shell is clamped to a fixed rigid cylindrical wall; this requires the satisfaction of the homogeneous displacement conditions

$$\phi_r(r_0) = w(r_0) = v(r_0) = 0^* \tag{3}$$

where we have from (1), with $\psi = \kappa = c = 0$, $\phi_r = -w' + v/R$.

The inner helical edge is subject to a rigid body axial displacement δ and axial rotation Ω, both in the positive z-direction; this requires the satisfaction of the inhomogeneous displacement conditions [4]

$$\phi_r(r_i) = -w' + \left.\frac{av}{\alpha^2}\right|_{r=r_i} = \frac{a\Omega}{\alpha_i}, \qquad v_z(r_i) = \left.\frac{av + rw}{\alpha}\right|_{r=r_i} = \delta \tag{4}$$
$$v_\theta(r_i) = \left.\frac{rv - aw}{\alpha}\right|_{r=r_i} = r_i\Omega$$

where $\alpha_i = (a^2 + r_i^2)^{1/2}$ and v_z and v_θ are the axial and circumferential displacement components of the middle surface.

We will show that the boundary conditions (3) and (4) are completely equivalent to another set of conditions which are the static geometric duals of the boundary conditions for the problem of axial torsion and extension as studied in [12]. To show this, we first observe that the sum of the expressions for κ_r and κ_θ in (2.4) gives

$$\alpha\phi_r = \alpha(-w' + v/R) = -\int_r^{r_0} (\kappa_r + \kappa_\theta)\alpha\,dr \tag{5}$$

where the limits of integration are chosen to make $\phi_r(r_0) = 0$.

* The remaining condition of vanishing radial displacement for a clamped edge is associated with the first class of problems and in effect makes $u(r) \equiv 0$ throughout the shell.

We then rewrite the expression for v_z' obtained in [4] as

$$v_z' = \alpha(R^{-1}\varepsilon_{r\theta} - \kappa_\theta). \tag{6}$$

From this follows

$$v_z = \int_r^{r_0} \left(\kappa_\theta - \frac{\varepsilon_{r\theta}}{R}\right)\alpha\, dr \tag{7}$$

again with limits of integration chosen to make $v_z(r_0) = 0$.

From (2.4), and (4), we have also

$$\kappa_\theta + \frac{\varepsilon_{\theta r}}{R} = \frac{aw - rv}{\alpha^2 R} + \frac{r\phi_r}{\alpha^2} = \frac{r\phi_r}{\alpha^2} - \frac{rv_\theta}{\alpha R} \tag{8}$$

so that all three homogeneous conditions (3) at the outer helical edge will be satisfied if we have

$$\kappa_\theta + \frac{\varepsilon_{\theta r}}{R} = 0 \tag{9}$$

at $r = r_0$ along with (5) and (7).

Of the three inhomogeneous conditions (4) at the inner helical edge, the first and second require

$$\Omega = -\frac{1}{a}\int_{r_i}^{r_0} (\kappa_\theta + \kappa_r)\alpha\, dr, \qquad \delta = \int_{r_i}^{r_0} \left(\kappa_\theta - \frac{\varepsilon_{r\theta}}{R}\right)\alpha\, dr \tag{10}$$

respectively. In view of (8), the last condition in (4) will be satisfied also if (9) is satisfied at the inner helical edge $r = r_i$ as well.

Altogether, the displacement boundary conditions (3) and (4) may now be replaced by the single condition (9) at both the inner and outer helical edge of the shell, together with two integrated conditions (10). From (3.3′) and (1) we have $\varepsilon_{r\theta} = \varepsilon_{\theta r}$ explicitly in terms of P and M.

$$\varepsilon_{\theta r} = \varepsilon_{r\theta} = -\frac{(1 + v_s)A}{\alpha^2}(Ma + Pa^2). \tag{11}$$

In view of (11), the differential equation (2) and the boundary conditions $\kappa_\theta + \varepsilon_{\theta r}/R = 0$ at $r = r_i$ and $r = r_0$ form a two point boundary value problem which determines κ_θ in terms of the axial force P and the axial torque M applied to the inner cylinder. P and M are then related to δ and Ω by means of the two linear flexibility relations (10).

It can be shown through use of the compatibility equations (2.9) and (2.10) and the boundary conditions (9) at both helical edges that the two flexibility relations (10) can be written as

$$\begin{aligned} \Omega &= -\int_{r_i}^{r_0} \left[\frac{a}{\alpha}\kappa_r + \frac{r}{\alpha}\lambda_r + \left(\frac{r}{\alpha}\varepsilon_{r\theta}\right)'\right] dr + \left[\frac{r}{\alpha}(\varepsilon_{r\theta} + \varepsilon_{\theta r})\right]_{r_i}^{r_0} \\ \delta &= -\int_{r_i}^{r_0} \left[\frac{r}{\alpha}\kappa_r - \frac{a}{\alpha}\lambda_r - \left(\frac{a}{\alpha}\varepsilon_{r\theta}\right)'\right] r\, dr - \left[\frac{ar}{\alpha}(\varepsilon_{r\theta} + \varepsilon_{\theta r})\right]_{r_i}^{r_0}. \end{aligned} \tag{12}$$

In the first relation of (12), we have through the use of the equation (2.9)

$$-\int_{r_i}^{r_0}\left[\frac{a}{\alpha}\kappa_r+\frac{r}{\alpha}\lambda_r+\left(\frac{r}{\alpha}\varepsilon_{r\theta}\right)'\right]dr+\left[\frac{r}{\alpha}(\varepsilon_{r\theta}+\varepsilon_{\theta r})\right]_{r_i}^{r_0}$$
$$=-\int_{r_i}^{r_0}\left[\frac{a}{\alpha}\kappa_r+\frac{r}{a}\left\{\frac{r}{\alpha}\kappa_r-(\alpha\kappa_\theta)'\right\}\right]dr+\left[\frac{r}{\alpha}\varepsilon_{\theta r}\right]_{r_i}^{r_0}$$
$$=-\int_{r_i}^{r_0}(\kappa_r+\kappa_\theta)\frac{\alpha}{a}\,dr+\left[\frac{r\alpha}{a}\left(\kappa_\theta+\frac{\varepsilon_{\theta r}}{R}\right)\right]_{r_i}^{r_0}. \tag{13}$$

As the term outside of the integral sign in the last step vanishes because of (9), we have proved the equivalence of the two expressions for Ω. The validity of the second equation of (12) can be ascertained similarly.

We now note that the two conditions in (12) are the static geometric duals of the only two nontrivial overall equilibrium conditions for the problem of axial torsion and extension (see equation (9) of [12]) with Ω and δ being the duals of F and T of [12], and that the boundary condition (9) at the two helical edges is the static geometric dual of the only nontrivial homogeneous Kirchoff–Bassett stress boundary condition for the same edge in [12] (see equation (19) of [12]). Together with the static geometric duality between the differential equations for the two problems, as noted earlier, we have now established a complete static geometric duality between the present problem and the problem studied in [12]. We can therefore in part confirm and in part extend the earlier results of [4] by simply translating the results of [12] in accordance with the rules of the static geometric duality.

The boundary condition (9) may be simplified somewhat if we write it as

$$\kappa_\theta+\frac{\varepsilon_{\theta r}}{R}=\frac{M_\theta-\nu_b M_r}{D(1-\nu_b^2)}+\frac{\varepsilon_{\theta r}}{R}. \tag{14}$$

Since all terms of the form ε/R have been neglected in the derivation of the stress strain relations (2.14) for the bending moments [11], it is consistent to replace (9) by

$$\kappa_\theta(r_i)=\kappa_\theta(r_0)=0 \tag{15}$$

Making use of the simplifying relation (3.6), the two integrated conditions (12) may be written as

$$\begin{aligned}\Omega&=-\int_{r_i}^{r_0}\frac{a}{\alpha}\kappa_r\,dr+\left[\frac{r}{\alpha}\varepsilon_{\theta r}\right]_{r_i}^{r_0}\\ \delta&=-\int_{r_i}^{r_0}\frac{r^2}{\alpha}\kappa_r\,dr-\left[\frac{r(2a^2+r^2)}{a\alpha}\varepsilon_{\theta r}\right]_{r_i}^{r_0}\end{aligned} \tag{16}$$

where $\varepsilon_{\theta r}$ is given explicitly in terms of P and M by (11).

The simplified form of the boundary condition at the helical edges (15) and the integrated conditions (16) are the static geometric duals of the analogous conditions (20a) and (25) of [12] respectively. The latter conditions were those actually used in [12] (instead of (9) and (19)) for the solution of the problem of axial extension and torsion of helicoidal shells.

6. Application of static geometric duality

In order to translate the results of [12] as well as to make direct use of the computer program developed there for the problem formulated in Section 5 for homogeneous and isotropic shells, we set

$$\rho = r/r_0, \qquad \rho_i = r_i/r_0, \qquad \lambda = r_0/a \tag{1}$$

$$\begin{Bmatrix} K_r \\ K_\theta \end{Bmatrix} = D \begin{Bmatrix} \kappa_r \\ \kappa_\theta \end{Bmatrix}, \qquad \mathscr{E} = \frac{r_0 \varepsilon_{r\theta}}{A(1+\nu)} = \frac{r_0 \varepsilon_{\theta r}}{A(1+\nu)} \tag{2}$$

Introducing (1) and (2) into the differential equation (5.2), we have

$$K_\theta^{\cdot\cdot} - \frac{2 - 3\lambda^2\rho^2}{\rho(1+\lambda^2\rho^2)} K_\theta - \frac{\lambda^2(1+\nu)}{(1+\lambda^2\rho^2)^2} K_\theta = -\frac{Pr_0\lambda^3\rho^2 - M\lambda^2}{(1+\lambda^2\rho^2)^2} \tag{3}$$

where dots indicate differentiation with respect to ρ. The boundary conditions (5.15) become

$$K_\theta(\rho_i) = K_\theta(1) = 0. \tag{3b}$$

In terms of ρ and λ, we have further from (4.1), (5.11), (4.2), (4.6) and (4.7)

$$\begin{aligned}
K_r &= \frac{1+\lambda^2\rho^2}{\lambda^2\rho} K_\theta^{\cdot} + K_\theta, \qquad \mathscr{E} = -\frac{M\lambda + Pr_0}{1+\lambda^2\rho^2} \\
r_0 N_{r\theta} &= -\frac{1}{\lambda\rho} K_\theta^{\cdot} - \frac{\lambda(1+\nu)}{1+\lambda^2\rho^2} K_\theta - \frac{Pr_0 + M\lambda}{1+\lambda^2\rho^2} \\
r_0 N_{\theta r} &= \frac{1}{\lambda\rho} K_\theta^{\cdot} + \frac{\lambda(1+\nu)}{1+\lambda^2\rho^2} K_\theta - \frac{Pr_0 + M\lambda}{1+\lambda^2\rho^2} \\
r_0 Q_r &= \frac{1}{\lambda\rho}\left[\frac{1}{\lambda\rho} K_\theta^{\cdot} + \frac{\lambda(1+\nu)}{1+\lambda^2\rho^2} K_\theta + \frac{M\lambda - Pr_0\lambda^2\rho^2}{1+\lambda^2\rho^2}\right].
\end{aligned} \tag{4}$$

For a dimensionless representation of δ and Ω, we introduce the additional parameter

$$\mu^2 = \frac{12DA(1-\nu^2)}{r_0^2} \tag{5}$$

with $\mu^2 = h^2/r_0^2$ for a homogeneous shell. The two integrated conditions (5.16) can now be written as

$$\begin{aligned}
Dr_0^{-1}\Omega &= -\int_{\rho_i}^{1} \frac{K_r}{\sqrt{1+\lambda^2\rho^2}} d\rho + \frac{\mu^2}{12(1-\nu)} \left[\frac{\lambda\rho\mathscr{E}}{\sqrt{1+\lambda^2\rho^2}}\right]_{\rho_i}^{1} \\
Dr_0^{-2}\delta &= -\int_{\rho_i}^{1} \frac{\lambda\rho^2 K_r}{\sqrt{1+\lambda^2\rho^2}} d\rho - \frac{\mu^2}{12(1-\nu)} \left[\frac{\rho(2+\lambda^2\rho^2)\mathscr{E}}{\sqrt{1+\lambda^2\rho^2}}\right]_{\rho_i}^{1}.
\end{aligned} \tag{6}$$

The form of the differential equation (3a) suggests that we write the K_θ and K_r in the form

$$K_\theta = -MK_{\theta M} - Pr_0 K_{\theta P}, \qquad K_r = -MK_{rM} - Pr_0 K_{rP} \tag{7, 8}$$

where quantities with subscripts M and P, respectively, correspond to the solution for the case of *no force* $(-M = 1, P = 0)$ and the case of *no torque* $(M = 0, -Pr_0 = 1)$.

With (8), we can now write the integrated relations (6) as two linear flexibility relations

$$\Omega = MC_{\Omega M} + PC_{\Omega P}, \qquad \delta = MC_{\delta M} + PC_{\delta P} \tag{9}$$

where

$$\begin{aligned}
\tfrac{1}{2}Dr_0^{-1}C_{\Omega M} &= C_{\Omega M}^{B} + \mu^2 C_{\Omega M}^{D} \\
&= \left[\int_{\rho_i}^{1} \frac{K_{rM}}{2\sqrt{1+\lambda^2\rho^2}}\,d\rho\right] + \mu^2\left[\frac{-\lambda^2\rho}{24(1-\nu)(1+\lambda^2\rho^2)^{3/2}}\right]_{\rho_i}^{1} \\
\tfrac{1}{2}Dr_0^{-2}C_{\Omega P} &= C_{\Omega P}^{B} = \int_{\rho_i}^{1} \frac{K_{rP}}{2\sqrt{1+\lambda^2\rho^2}}\,d\rho \\
&= \int_{\rho_i}^{1} \frac{\lambda\rho^2 K_{rM}}{2\sqrt{1+\lambda^2\rho^2}}\,d\rho = C_{\delta M}^{B} = \tfrac{1}{2}Dr_0^{-2}C_{\delta M} \\
\tfrac{1}{2}Dr_0^{-3}C_{\delta P} &= C_{\delta P}^{B} + \mu^2 C_{\delta P}^{D} \\
&= \left[\int_{\rho_i}^{1} \frac{\lambda\rho^2 K_{rP}}{2\sqrt{1+\lambda^2\rho^2}}\,d\rho\right] + \mu^2\left[\frac{\rho(2+\lambda^2\rho^2)}{24(1-\nu)(1+\lambda^2\rho^2)^{3/2}}\right]_{\rho_i}^{1}.
\end{aligned} \tag{10}$$

Evidently, the dimensionless flexibility coefficients $C_{\Omega M}^{B}$, $C_{\Omega M}^{D}$, $C_{\Omega P}^{B} = C_{\delta M}^{B}$, $C_{\delta P}^{B}$ and $C_{\delta P}^{D}$ are the static geometric duals of the dimensionless stiffness coefficients C_{Fk}^{M}, C_{Fk}^{B}, $C_{F\psi}^{M} = C_{Tk}^{M}$, $C_{T\psi}^{M}$ and $C_{T\psi}^{B}$ (see equation (34) of [12]). In (10), we have omitted terms multiplied by μ^2 in $C_{\Omega P} = C_{\delta M}$ as they can be shown to be negligible for all λ in a manner analogous to what has been done in [12] to the corresponding terms in $C_{F\psi} = C_{Tk}$.

The boundary value problem defined by (3a, b), the auxiliary equations (4) and the flexibility coefficients (10) are exactly the same as equations (29a, b), (30) and (34) of [12] with K_θ, K_r, $\mathscr{E}$, $-P$, $-M$ and $-\nu$ taking the place of n_r, n_θ, m, ψ, k and ν in the latter. Perturbation solutions of the boundary value problem and the stress and strain measures as well as the influence coefficients for the present problem can therefore be obtained without further calculations by translating the results of [12] according to the rules of the static geometric duality. We will not list these analytical results here.

Beyond the direct translation of the analytic results, the computer program developed in [12] which generates the finite difference solution for the problem of axial torsion and extension can be used to generate the corresponding finite difference solution for the present problem without any modification of the program. To do this, we simply use the negative values of ν as the input for ν in the program (since D and A do not appear explicitly) and interpret all outputs as the results for the dual quantities.

It was pointed out in [4] that the power series solution used there is not practical for $\lambda > 4$ because of the slow convergence of the series. The finite difference solution on the other hand is practical for a wide range of values of λ (at least for $0 \leq \lambda \leq 100$) while the time needed for the computation remains relatively constant. We have used the computer program of [12] to extend the results of [4] for the influence coefficients and to supply the stress distributions which had not been obtained previously.

7. Numerical results

In presenting numerical results for stress distributions and the influence coefficients for the problem formulated in Sections 5 and 6 for homogeneous and isotropic shells of constant thickness, we note that the limiting case of a shell with an infinite pitch ($a = \infty, \lambda = 0$) is a flat rectangular strip. The solution of this problem can be found from plate theory and from the theory of generalized plane stress as follows:

$$M_r = -M\left[1 - \frac{3}{2}\frac{1-\rho_i^2}{1-\rho_i^3}\rho\right], \qquad M_\theta = -\nu M\left[1 - \frac{3}{2}\frac{1-\rho_i^2}{1-\rho_i^3}\rho\right],$$
$$Q_r = \frac{3}{2}\frac{M}{r_0}\frac{1-\rho_i^2}{1-\rho_i^3} \tag{1}$$

$$\Omega = \frac{Mr_0}{D}\left[1 - \rho_i - \frac{3}{4}\frac{(1-\rho_i^2)^2}{1-\rho_i^3}\right] = \frac{2Mr_0}{D}C^B_{\Omega M}\bigg|_{\lambda=0} \equiv \frac{2Mr_0}{D}C^0_{\Omega M} \tag{2}$$

and

$$N_{r\theta} = N_{\theta r} = -\mathrm{P}, \tag{3}$$

$$\delta = 2Pr_0A(1+\nu)(1-\rho_i) = 24P_0r_0A(1-\nu^2)[a^{-1}C^D_{\delta P}]_{\lambda=0}. \tag{4}$$

We see that for this extreme value of λ, the effects of axial force and torque uncouple. The shallow shell solution of [5], for $\lambda \ll 1$, shows that a small amount of pretwist in the strip will give rise to a coupling between the shearing and bending of the structure when $\lambda \neq 0$.

At the other end of the spectrum, the shell becomes a circular ring plate sector as a tends to zero. The effects of the axial force and torque again uncouple in the limit. For $a = 0$ ($\lambda = \infty$), we must keep $P_0 = Pa$ and $M_0 = Ma$ finite in order that equation (5.1) be meaningful. We have for this limiting case

$$N_{r\theta} = N_{\theta r} = -M_0 r^{-2} \tag{5}$$

$$\Omega = M_0 r_i^{-2}A(1+\nu)(1-\rho_i^2) = 24M_0Ar_0^{-1}(1-\nu^2)[a^{-1}C^D_{\Omega M}]_{\lambda=\infty} \tag{6}$$

and

$$M_r = -P_0\left\{A_0\left[1+\nu+\frac{1-\nu}{\rho^2}\right] + \tfrac{1}{2}[1+(1+\nu)\ln\rho]\right\}$$
$$M_\theta = -P_0\left\{A_0\left[1+\nu-\frac{1-\nu}{\rho^2}\right] + \tfrac{1}{2}[\nu+(1+\nu)\ln\rho]\right\} \tag{7}$$
$$Q_r = P_0 r^{-1}$$

$$\delta = \frac{P_0 r_0^2}{8D}(1-\rho_i^2)\left[1-\left(\frac{2\rho_i \ln \rho_i}{1-\rho_i^2}\right)^2\right] = \frac{2r_0^3 P_0}{D}\left[\frac{C_{\delta P}^D}{a}\right]_{\lambda=\infty} \equiv \frac{2Pr_0^3}{D}C_{\delta P}^{\infty} \tag{8}$$

with

$$A_0 = \frac{1}{2}\frac{\rho_i \ln \rho_i}{1-\rho_i^2}. \tag{9}$$

The shallow shell solution of [7] which is valid when $\lambda \gg 1$ shows that a small but nonvanishing pitch again gives rise to a coupling between the bending and shearing actions.

Having the results for the limiting cases, we now investigate the in-between range of pitch values. The flexibility coefficients $C_{\Omega M}$, $C_{\Omega P} = C_{\delta M}$ and $C_{\delta P}$ for homogeneous isotropic shells of constant thickness were obtained in [4] for $0 \leq \lambda \leq 4$ and several values of r_i/r_0 with $h/r_0 = 0.04$ and $\nu = 0.3$. The corresponding results generated by the computer program of [12] show very good agreement with the results of [4].* Since the effect of h/r_0 is always negligible in

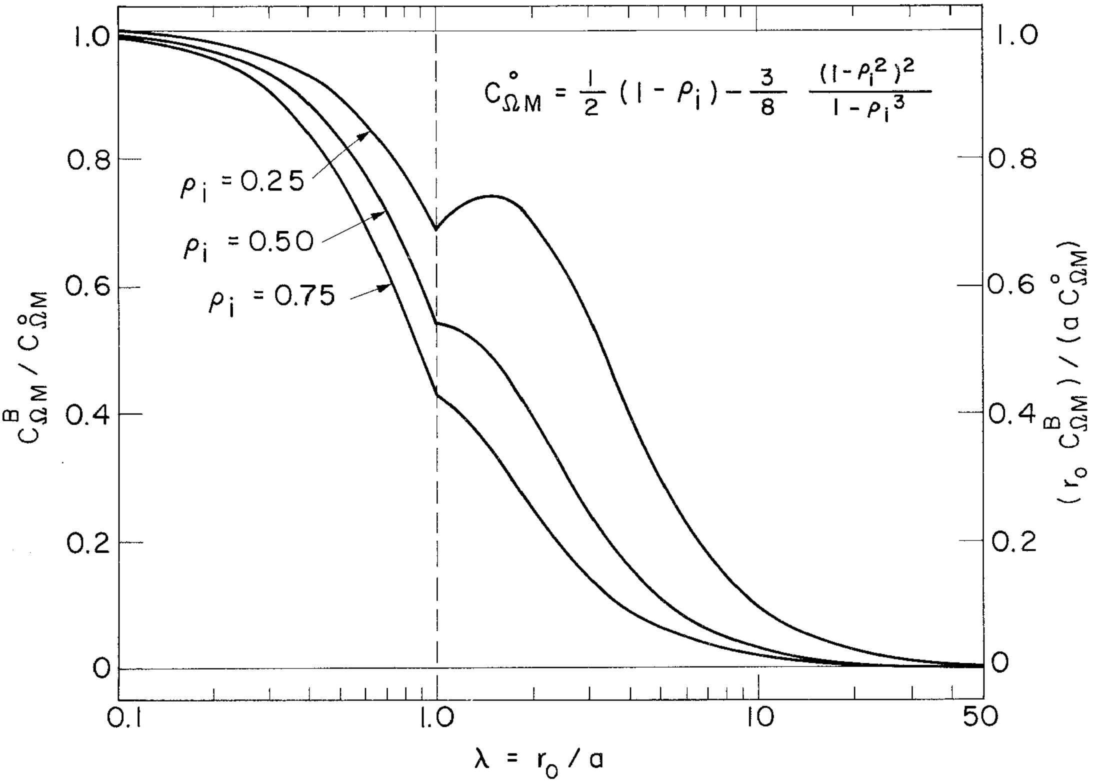

Figure 1. Bending contribution to the flexibility coefficient $C_{\Omega M}$ for $\nu = 0.3$.

* The plots for $u_{ij}/2\pi$ in [4] are actually plots for u_{ij} with μ, η and ε corresponding to our $\varepsilon = \lambda^{-1}$, ρ_i and μ respectively. They contain both the membrane and bending contributions and are therefore only valid for a fixed value of h/r_0. There is also a difference in sign convention between the present work and [4].

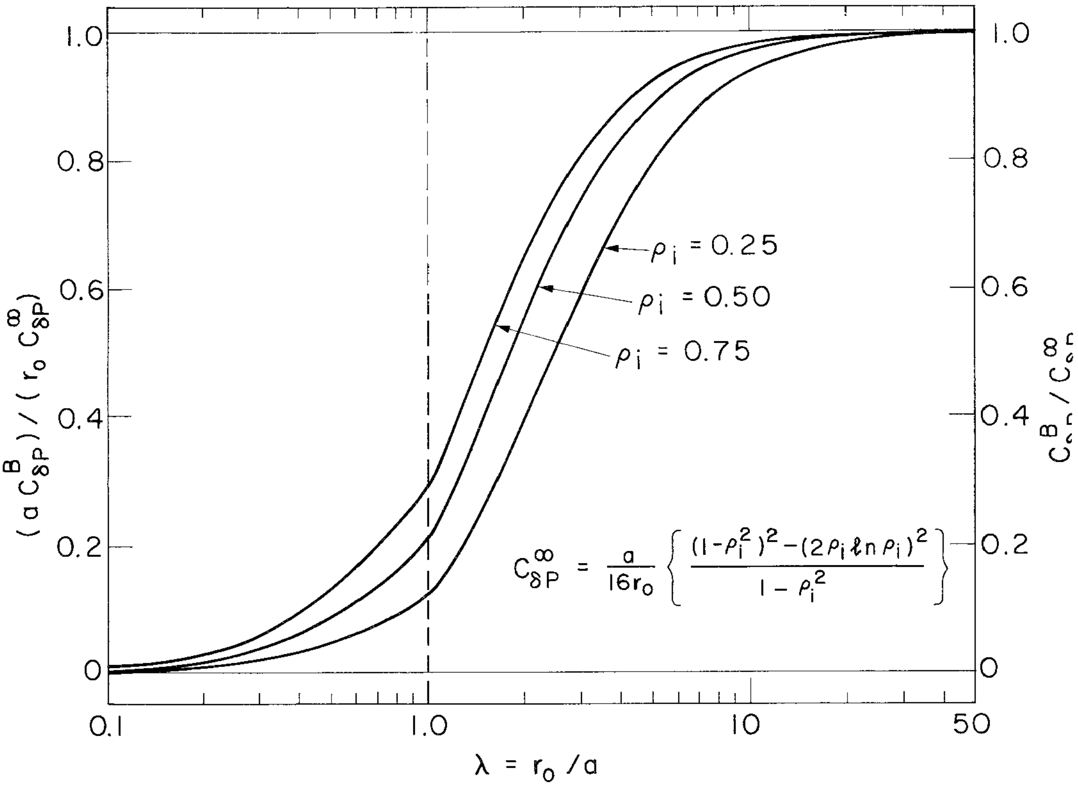

Figure 2. Bending contribution to the flexibility coefficient $C_{\delta P}$ for $\nu = 0.3$.

the coupling coefficient $\frac{1}{2}Dr_0^{-2}C_{\Omega P} = \frac{1}{2}Dr_0^{-2}C_{\delta M}$ and since the dependence of $\frac{1}{2}Dr_0^{-1}C_{\Omega M}$ and $\frac{1}{2}Dr_0^{-3}C_{\delta P}$ on h/r_0 is explicitly determined in terms of elementary functions (see equations (6.10)), we give in Figures 1 and 2 only plots of $C_{\Omega M}^{B}/C_{\Omega M}^{0}$ (see also equation (2)) and $C_{\delta P}^{B}/C_{\delta P}^{\infty}$ (see also equation (8)) for $0 < \lambda < 100$ and $\nu = 0.3$. Together with (6.10), we have in effect the flexibility coefficients for all h/r_0, $\nu = 0.3$, $0 < \lambda < 100$ and $\rho_i = 0.25, 0.5$, and 0.75 with $C_{\delta M}^{B} = C_{\Omega P}^{B}$ as given in [4].

The stress distributions across the width of the shell for the present problem have not been obtained previously for nonshallow shells, and therefore we consider here the representative direct and bending stress quantities

$$\sigma_r^B = \frac{6M_r}{h^2}, \qquad \tau_{r\theta}^D = \frac{N_{r\theta}}{h}. \tag{10}$$

A λ^2-perturbation solution (see [12]) shows that for $\lambda^2 \ll 1$, the direct stresses are significant only in the case of *no torque* and only in the range $\lambda = 0(\mu)$. For this range of values of λ, we have to within the accuracy of shell theory a uniformly distributed shearing stress (see equation (3) and also [5])

$$\tau_{r\theta}^D = -\frac{P}{h}. \tag{11}$$

For $\varepsilon^2 = \lambda^{-2} \ll 1$, and ε^2-perturbation solution (see [12]) shows that the direct stresses are significant only for the case of *no force* and only if $\varepsilon = 0(\mu)$. If $\varepsilon^2/\rho_i^2 = 0(\mu)$ also, then we have effectively

$$\tau_{r\theta}^D = -\frac{M_0}{r^2} = -\frac{M\varepsilon r_0}{r^0} \tag{12}$$

which is just the shearing stress distribution for the limiting case of a ring sector plate (see equation (5)) and also that obtained by a shallow shell theory [7]. If $\mu \ll \varepsilon^2/\rho_i^2 \ll 1$, correction term(s) of second (and higher) order in the ε^2-perturbation series may have to be retained for an accurate determination of the shearing stresses.

Our finite difference solution shows that $\lambda = 0(\mu)$ and $\varepsilon = 0(\mu)$ are in fact the only two ranges of values of λ for which the membrane shearing stresses are significant in comparison with the bending stresses.

Motivated by the results for $\lambda = 0$ and $\lambda = \infty$, we present in Figures 3 through 6 plots of the relevant dimensionless bending stresses $(\sigma_r^B/\sigma_0)_{P=0}$, $(h\sigma_r^B/6r_0\tau_0)_{M=0}$, $(\sigma_r^B/\sigma_\infty)_{M=0}$ and $(h\sigma_r^B/6r_0\tau_\infty)_{P=0}$ for $\rho_i = 0.5$ and $\nu = 0.3$ where

$$\tau_0 = -\frac{P}{h}, \qquad \sigma_0 = -\frac{3M}{h^2}\frac{(1+\rho_i)(2-\rho_i)}{1+\rho_i+\rho_i^2} \tag{13}$$

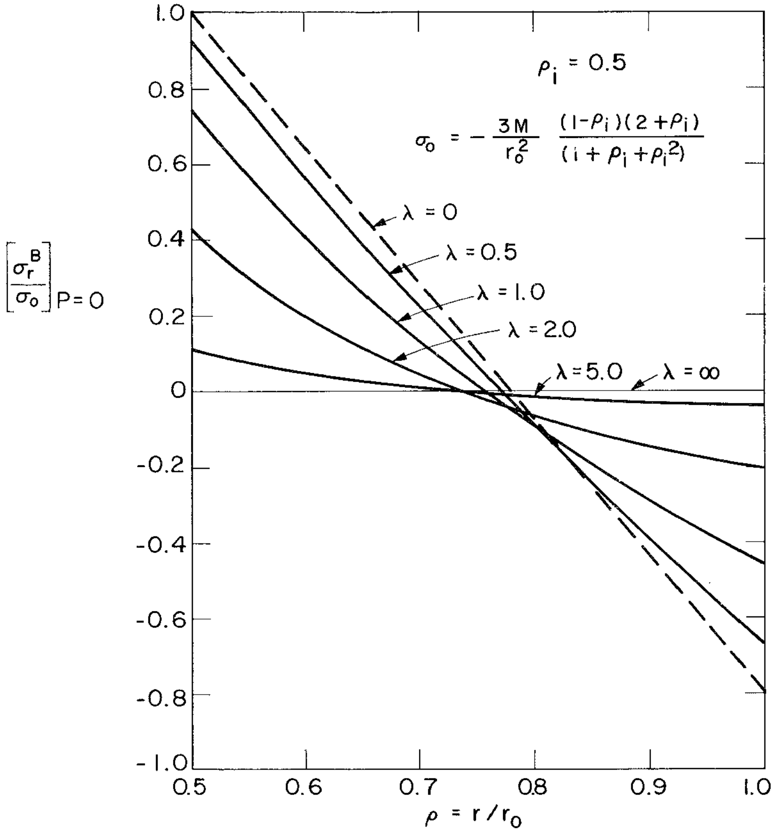

Figure 3. Distribution of bending stresses σ_r^B for the case of *No Force* with $\rho_i = 0.5$, $\nu = 0.3$ and $\lambda \leq 5.0$, normalized by σ_0.

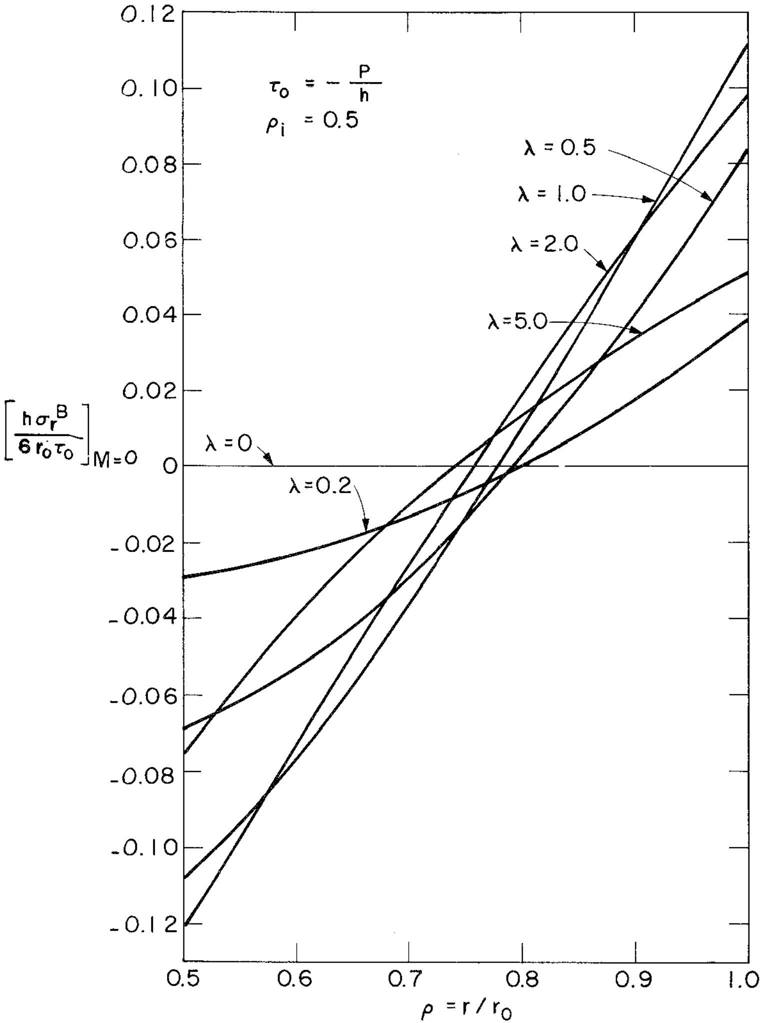

Figure 4. Distribution of bending stresses σ_r^B for the case of *No Torque* with $\rho_i = 0.5$, $\nu = 0.3$ and $\lambda \leq 5.0$, normalized by τ_0.

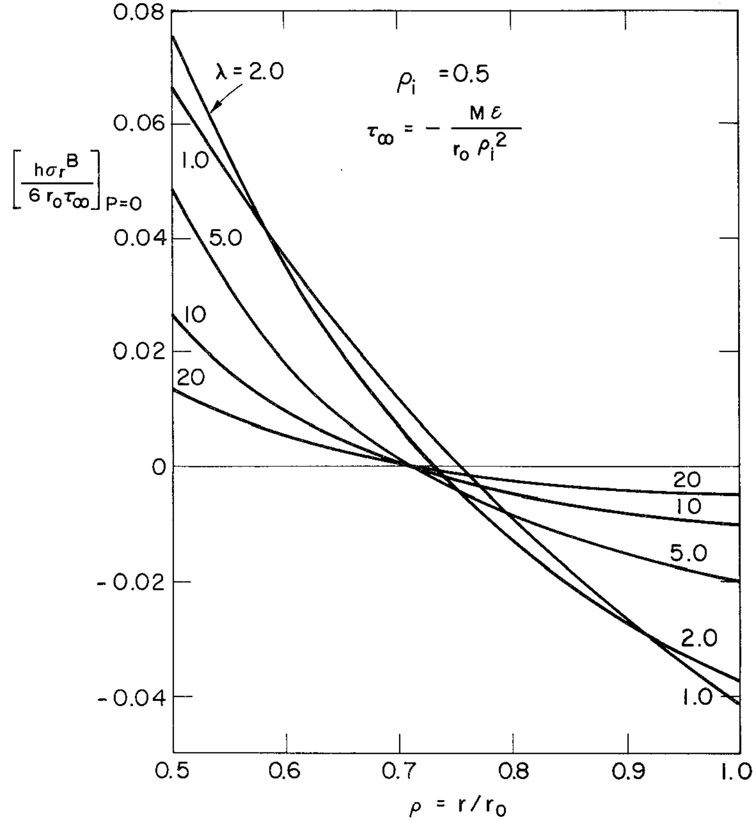

Figure 5. Distribution of bending stresses σ_r^B for the case of *No Force* with $\rho_i = 0.5$, $\nu = 0.3$ and $\lambda \geq 1.0$, normalized by τ_∞.

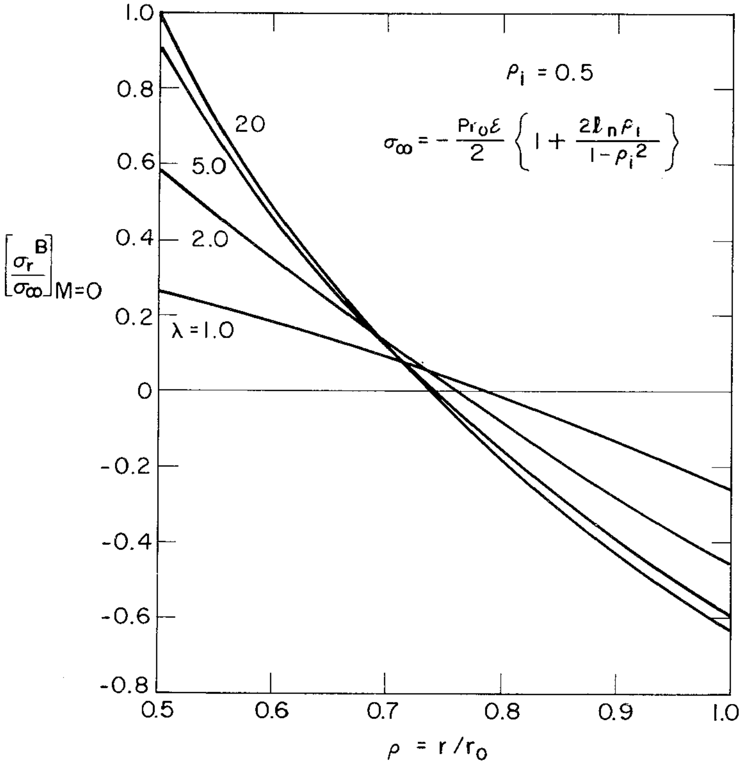

Figure 6. Distribution of bending stresses σ_r^B for the case of *No Torque* with $\rho_i = 0.5$, $\nu = 0.3$ and $\lambda \geq 1.0$, normalized by σ_∞.

and

$$\tau_\infty = -\frac{M\varepsilon}{r_0 \rho_i^2}, \qquad \sigma_\infty = -\frac{P r_0 \varepsilon}{2}\left(1 + \frac{2 \ln \rho_i}{1 - \rho_i^2}\right) \tag{14}$$

are the maximum values of $\tau_{r\theta}^D$ and σ_r^B for the limiting cases of $\lambda = 0$ and $\lambda = \infty$, respectively (see equations (1), (3), (5) and (7)).

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Massachusetts Institute of Technology

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