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On Rotationally Symmetric Stress and Strain in Anisotropic Shells of Revolution†

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1. Introduction

The present paper complements our recent work [1] on rotationally symmetric stress and strain in the linear theory of elastic shells of revolution. Our departure here is from the two uncoupled systems of equilibrium and compatibility differential equations [1] for what we have called the problems of bending and of twisting, respectively. Developments in our earlier paper which were based on these two uncoupled systems were limited by the assumption that the material of the shell was polar-orthotropic in such a way that uncoupling persisted for the complete system of differential equations consisting of equilibrium, compatibility and constitutive relations.

In what follows we consider a quite general class of anisotropic, or spirally orthotropic, shells. The assumption of anisotropy results, as already indicated in [1], in a coupling of the two systems of differential equations for bending and twisting. Our object here is to reduce the complete set of equations once more to a system of two simultaneous second-order differential equations, of the same type as the well-known equations for the problem of bending alone of the polar-orthotropic shell.

In order to save space no attempt has been made to write this paper in such a way that it may be read without reference to the developments in [1].

2. Constitutive equations

With stress resultants N and Q, stress couples M and P, strain resultants ε and γ and strain couples $\varkappa$ and λ, as in [1], the most general linear (homogeneous) system of constitutive equations for shells of revolution of the type considered in [1] can be written in the form

$$\{N_{\xi\xi}\, N_{\theta\theta}\, N_{\xi\theta}\, N_{\theta\xi}\, Q_{\xi}\, Q_{\theta}\, M_{\xi\xi}\, M_{\theta\theta}\, M_{\xi\theta}\, M_{\theta\xi}\, P_{\xi}\, P_{\theta}\}$$
$$= A_{12}\{\varepsilon_{\xi\xi}\, \varepsilon_{\theta\theta}\, \varepsilon_{\xi\theta}\, \varepsilon_{\theta\xi}\, \gamma_{\xi}\, \gamma_{\theta}\, \varkappa_{\xi\xi}\, \varkappa_{\theta\theta}\, \varkappa_{\xi\theta}\, \varkappa_{\theta\xi}\, \lambda_{\xi}\, \lambda_{\theta}\}. \tag{1}$$

In this A_{12} is a twelve-by-twelve matrix the elements of which are given functions of the meridional coordinate ξ.

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Application of (1) can be shown to allow reduction of the rotationally symmetric problem of the shell of revolution to a system of eight simultaneous first-order differential equations (with four additional such equations being of zeroth-order as a consequence of the four known first integrals of the equilibrium and compatibility differential equations).

In what follows we consider a subclass of the system (1) consisting of six equations

$$P_\xi = 0, \qquad P_\theta = 0, \qquad M_{\xi\theta} - M_{\theta\xi} = 0, \tag{2a}$$

$$\gamma_\xi = 0, \qquad \gamma_\theta = 0, \qquad \varepsilon_{\xi\theta} - \varepsilon_{\theta\xi} = 0, \tag{2b}$$

together with six equations of the form

$$\{N_\xi\, N_\theta\, N_S\, M_\xi\, M_\theta\, M_S\} = A_6\{\varepsilon_\xi\, \varepsilon_\theta\, \varepsilon_S\, \varkappa_\xi\, \varkappa_\theta\, \varkappa_S\} \tag{2c}$$

where

$$N_S = \tfrac{1}{2}(N_{\xi\theta} + N_{\theta\xi}), \qquad M_S = M_{\xi\theta} = M_{\theta\xi}, \tag{3a}$$

$$\varepsilon_S = 2\varepsilon_{\xi\theta} = 2\varepsilon_{\theta\xi}, \qquad \varkappa_S = \varkappa_{\xi\theta} + \varkappa_{\theta\xi}, \tag{3b}$$

and $N_\xi \equiv N_{\xi\xi}$, $N_\theta \equiv N_{\theta\theta}$, etc.

In conjunction with the constitutive equations (of constraint) in (2b) we have that Q_ξ, Q_θ and $N_{\xi\theta} - N_{\theta\xi}$ are reactive quantities, while in conjunction with the constitutive equations (2a), which express the medium's incapability to support moments with axes normal to the middle surface of the shell, as well as a less easily explained property of equal twisting moments $M_{\xi\theta}$ and $M_{\theta\xi}$, we have that λ_ξ, λ_θ and $\varkappa_{\xi\theta} - \varkappa_{\theta\xi}$ are auxiliary kinematic variables.

We have previously found [1] that for the case that the system (2c) can be separated in the form

$$\{N_\xi\, N_\theta\, M_\xi\, M_\theta\} = A_4\{\varepsilon_\xi\, \varepsilon_\theta\, \varkappa_\xi\, \varkappa_\theta\}, \tag{4a}$$

$$\{N_S\, M_S\} = A_2\{\varepsilon_S\, \varkappa_S\}, \tag{4b}$$

the problem associated with (4a) can be reduced to two simultaneous second-order differential equations, in agreement with previously known results, while the problem associated with (4b) can be reduced to two *zeroth*-order differential equations, this being a result for which no earlier reference is known to the authors.

In what follows we intend to show that use of (2c) in place of (4a, b) also allows reduction of the problem to two simultaneous second-order differential equations, of the same type as the equations for the problem of bending associated with the constitutive relations (4a).

3. Reduction of equilibrium and compatibility equations of twisting

We take these in the form of the four differential equations (3.7, 8), with $q_\theta = q_n = 0$, and the two first-integral equations (4.3, 4) in [1]. With the assumptions (2a, b) and the notation (3a, b) we have then

$$Q_\theta = \frac{(r^2 M_S)'}{r^2\alpha}, \qquad \lambda_\xi = -\frac{(r^2 \varepsilon_S)'}{2r^2\alpha}, \tag{5}$$

and

$$N_{\theta\xi} - N_{\xi\theta} = \left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right) M_S, \qquad \varkappa_{\theta\xi} - \varkappa_{\xi\theta} = \left(\frac{1}{R_\xi} - \frac{1}{R_\theta}\right) \frac{\varepsilon_S}{2}. \tag{6}$$

Introduction of (6) into equations (4.3, 4) in [1] leaves the important relations

$$N_S + \frac{1}{2}\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right) M_S + \frac{T}{r^2} = 0, \tag{7a}$$

and

$$\varkappa_S - \frac{1}{2}\left(\frac{3}{R_\theta} - \frac{1}{R_\xi}\right) \varepsilon_S + \frac{2c_T}{r^2} = 0 \tag{7b}$$

(see also (6.1) of [1]).

4. Reduction of constitutive equations

We first combine equations (7a, b) with *part* of the constitutive equations (2c), written in the form

$$\{N_S M_S\} = A_2(\varepsilon_S \varkappa_S\} + A_{24}\{\varepsilon_\xi \varepsilon_\theta \varkappa_\xi \varkappa_\theta\}. \tag{8}$$

Evidently, (7a, b) together with (8) are a system of four simultaneous linear equations for the four quantities N_S, M_S, ε_S and $\varkappa_S$. Its solution may be written in the form

$$\{N_S M_S\} = B_{24}\{\varepsilon_\xi \varepsilon_\theta \varkappa_\xi \varkappa_\theta\} + B_2\{c_T T\}, \tag{9a}$$

$$\{\varepsilon_S \varkappa_S\} = C_{24}\{\varepsilon_\xi \varepsilon_\theta \varkappa_\xi \varkappa_\theta\} + C_2(c_T T\}. \tag{9b}$$

Equations (9a, b) include, in somewhat simpler and more general form, our earlier results for the case of *no* coupling between bending and twisting, upon setting $A_{24} = 0$ and therewith $B_{24} = C_{24} = 0$.

Having the results (9a, b) we now consider the remaining part of the constitutive equations (2c) in the form

$$\{N_\xi N_\theta M_\xi M_\theta\} = A_4\{\varepsilon_\xi \varepsilon_\theta \varkappa_\xi \varkappa_\theta\} + A_{42}\{\varepsilon_S \varkappa_S\}. \tag{10}$$

Introduction of (9b) into (10) leaves a system of relations of the form

$$\{N_\xi N_\theta M_\xi M_\theta\} = A_4^*\{\varepsilon_\xi \varepsilon_\theta \varkappa_\xi \varkappa_\theta\} + A_{42}^*\{c_T T\} \tag{11}$$

where $A_4^* = A_4 + A_{42}C_{24}$ and $A_{42}^* = A_{42}C_2$.

Equation (11) represents the result to be established in this note. It follows from this result that the two simultaneous differential equations for the bending stress function variable $\chi = r(N_\xi \cos\psi + Q_\xi \sin\psi)$ and the bending angular displacement variable $\varphi = -r(\varkappa_\theta \cos\psi + \lambda_\theta \sin\psi)$ are of the same structure as for the case for which the uncoupled constitutive equations (4a) apply, except for the appearance of additive terms involving c_T and T on the right side of these equations.

Reference

1. E. Reissner and F. Y. M. Wan, "Rotationally Symmetric Stress and Strain in Shells of Revolution," Studies in Applied Mathematics **48**, 1–17, 1969.

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