Spirally Sinusoidal Stress Distributions in Elastic Helicoidal Shells[1])

By R. L. Mallett, Dept. of Applied Mechanics, Stanford University, Stanford, Cal., and F. Y. M. Wan, Dept. of Mathematics, M.I.T., Cambridge, Mass., USA

1. Introduction

In cylindrical coordinates, the middle surface of a helicoidal shell is described by the equation $z = a\theta$ where $2\pi a$ is the pitch of the helicoid. Physical problems for an elastic helicoidal shell involving stress distributions which depend on θ in the form of $\sin\theta$ and $\cos\theta$ have been investigated in [8, 9, 10]. It was shown in [8] that the stress distributions in a shell subject only to equal and opposite end bending moments are in fact of this type though the associated displacement field is multivalued in θ. An exact shallow shell solution of this *pure bending* problem for a shell with a small pitch-to-width ratio, a/r_0, was obtained in terms of elementary functions in [9]. For shells with a large pitch-to-width ratio, this same problem was solved in [3, 6] as a problem for shallow hyperbolic paraboloidal shells. For moderate values of a/r_0 (non-shallow shells), it was indicated in [8] that the problem of pure bending can be reduced to a two-point boundary value problem involving three simultaneous ordinary differential equations for three displacement variables, though the actual reduction was not carried out. An exact shallow shell solution for the *side force* problem of helicoidal shells obtained in [10] showed that the stress distributions in the shell is also sinusoidal in θ with a period of 2π.

In this paper, we reformulate the entire class of problems associated with stress distributions proportional to $\sin\theta$ and $\cos\theta$ as a boundary value problem for two coupled ordinary differential equations for a stress variable and a strain variable instead of three equations for three displacement variables as indicated in [8]. The following features of our formulation are noted:

1. By taking the shell equations in the form of equilibrium, constitutive and compatibility equations, our new formulation makes use of the first integrals of the equilibrium and compatibility equations known to exist for this class of problems and reduces the shell equations directly to two third order coupled equations (instead of an eighth order system [7, 8]). The reduction to two simultaneous equations and

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the derivation of the auxiliary equations expressing the stress resultants and couples in terms of the primary stress and strain variables are straightforward and can be carried out exactly.

2. It bypasses the difficulty associated with the determination of a displacement field (for the pure bending problem) and a stress function field (for the side force problem) which are multivalued in θ and which give rise to our particular sinusoidal stress distributions [7, 8, 9, 10].

3. It preserves the well-known static geometric duality of shell theory within the formulation so that the reduction of the shell equations as well as the solution for specific problems are simplified considerably. Except for terms of order h/R and h^2/l^2 (h is the shell thickness, R is the minimum radius of curvature and l is a characteristic length), the final two governing differential equations for a stress function ψ and a strain function ϕ are of the form

$$L(\phi)+a\psi=f, \qquad L^*(\psi)+a\phi=f^* \tag{1}$$

where f and f^* involve only load terms, and L and L^* are third order linear differential operators. They are very suitable for a perturbation, asymptotic or finite difference solution.

We then apply this new formulation to solve the problem of pure bending. We will show in particular that for shells with a small pitch, the leading term of a perturbation solution in powers of $(a/r_0)^2$ coincides with the exact shallow shell solution of [9]. A leading term perturbation solution in powers of $(r_0/a)^2$ is obtained for shells with a very large pitch. This solution indicates that the stress distributions in the shell differ qualitatively and quantitatively in different ranges of the dimensionless parameter

$$\mu^3=O\left(\frac{a}{h}\left[\frac{r_0^2}{a^2+r_0^2}\right]^{\frac{3}{2}}\right). \tag{2}$$

If $\mu \ll 1$, the solution coincides with the shallow shell solution of [3, 6]. But if μ is of order unity or larger, our solution differs significantly from that of [3, 6] throughout the shell even if r_0/a is small compared to unity. Finally, for a sufficiently thin shell so that $\mu \gg 1$, we obtained an asymptotic solution valid for all r_0/a. This solution is shown to agree with known results for $r_0/a \ll 1$ and $r_0/a \gg 1$ (with $\mu \gg 1$) and to supply new information for $r_0/a=O(1)$,

In a recently published work [7], the general equations for helicoidal shells were reduced to two coupled fourth order equations. Upon introducing appropriate multivalued stress and displacement functions, they were further reduced to two third order equations for our particular type of sinusoidal stress distributions. As it stands the more general formulation of [7] is not suitable for the analyses reported herein. For example, the counterpart of our independent variable r is $\zeta = \log\left[\frac{r}{a}+\sqrt{1+\left(\frac{r}{a}\right)^2}\right]$. Additional transformations of the governing equations are therefore necessary to get the limiting behavior as $r_0/a \to 0$ and $a/r_0 \to 0$. Furthermore, in view of the number of preliminary transformations of the original shell equations needed to get the more general results of [7], a more straightforward reduction for our particular class of problems in terms of natural surface coordinates, which bypasses the use of multivalued stress and displacement functions, should also be of independent interest.

2. Differential Equations

With stress resultants, N, stress couples, M, and components of surface load intensity, p, varying sinusoidally in the polar angle θ in the form[2])

$$\begin{aligned}(N_r, N_\theta, Q_\theta, M_{r\theta}=M_{\theta r}, p_r) &= (n_r, n_\theta, q_\theta, m, P_r)\sin\theta\\ (N_{r\theta}, N_{\theta r}, Q_r, M_r, M_\theta, p_\theta, p_n) &= (n_{r\theta}, n_{\theta r}, q_r, m_r, m_\theta, P_\theta, P_n)\cos\theta\end{aligned} \tag{1}$$

the six differential equations of equilibrium for a helicoidal shell become six equations for the r-dependent functions n_r, n_θ, etc. [8]

$$(\alpha n_r)' - n_{\theta r} - \frac{r}{\alpha} n_\theta + \frac{a}{\alpha} q_\theta + \alpha P_r = 0, \quad (\alpha n_{r\theta})' + n_\theta + \frac{r}{\alpha} n_{\theta r} + \frac{a}{\alpha} q_r + \alpha P_\theta = 0 \tag{2, 3}$$

$$(\alpha q_r)' + q_\theta - \frac{a}{\alpha}(n_{r\theta} + n_{\theta r}) + \alpha P_n = 0, \quad (\alpha m_r)' + m - \frac{r}{\alpha} m_\theta - \alpha q_r = 0 \tag{4, 5}$$

$$(\alpha m)' - m_\theta + \frac{r}{\alpha} m - \alpha q_\theta = 0, \qquad n_{r\theta} - n_{\theta r} + \frac{a}{\alpha^2}(m_\theta - m_r) = 0 \tag{6, 7}$$

where $\alpha = (a^2 + r^2)^{\frac{1}{2}}$ and primes indicate differentiation with respect to r.

The stress measures are related to the midsurface strain measures ε, and curvature change measures, κ, by the system of stress strain relations used in [8]. With the elastic moduli and shell thickness independent of θ, these stress strain relations imply that the strain and curvature change measures are also sinusoidal in θ in the form

$$\begin{aligned}(\varepsilon_r, \varepsilon_\theta, \kappa_{r\theta}, \kappa_{\theta r}) &= (e_r, e_\theta, k_{r\theta}, k_{\theta r})\sin\theta,\\ (\varepsilon_{r\theta} = \varepsilon_{\theta r}, \kappa_r, \kappa_\theta) &= (e, k_r, k_\theta)\cos\theta.\end{aligned} \tag{8}$$

In terms of the r-dependent quantities, the stress strain relations take the form

$$e_r = A(n_r - \nu_s n_\theta), \qquad e_\theta = A(n_\theta - \nu_s n_r), \qquad e = A_S(n_{r\theta} + n_{\theta r}) \tag{9, 10, 11}$$

$$m_\theta = D(k_\theta + \nu_b k_r), \qquad m_r = D(k_r + \nu_b k_\theta), \qquad m = D_S(k_{\theta r} + k_{r\theta}) \tag{12, 13, 14}$$

[2]) The relation $M_{r\theta} = M_{\theta r}$ anticipates the stress strain relation (14).

where $A_S=\frac{1}{2}A(1+\nu_s)$ and $D_S=\frac{1}{2}D(1-\nu_b)$, along with the condition of vanishing transverse shear strains. For a homogeneous isotropic shell, we have

$$A=\frac{1}{Eh}, \quad D=\frac{Eh^3}{12(1-\nu^2)}, \quad \nu_s=\nu_b=\nu \tag{15}$$

where E, ν, and h are Young's modulus, Poisson's ratio and the shell thickness, respectively.

The displacement components of the shell are related to the strain measures ε and κ by a system of strain displacement relations as stated in [8]. These strain displacement relations in turn require that the r-dependent portion of the strain measures satisfy the following compatibility conditions:

$$(\alpha k_\theta)'-k_{r\theta}-\frac{r}{\alpha}k_r+\frac{a}{\alpha}l_r+\alpha P_r^*=0, \quad (\alpha k_{\theta r})'+k_r+\frac{r}{\alpha}k_{r\theta}+\frac{a}{\alpha}l_\theta+\alpha P_\theta^*=0 \tag{16, 17}$$

$$(\alpha l_\theta)'+l_r-\frac{a}{\alpha}(k_{\theta r}+k_{r\theta})+\alpha P_n^*=0, \quad (\alpha e_\theta)'+e-\frac{r}{\alpha}e_r-\alpha l_\theta=0 \tag{18, 19}$$

$$(\alpha e)'-e_r+\frac{r}{\alpha}e-\alpha l_r=0, \qquad k_{\theta r}-k_{r\theta}+\frac{a}{\alpha^2}(e_r-e_\theta)=0 \tag{20, 21}$$

where (19) and (20) are taken to be the defining equations for l_θ and l_r and where the quantities P_r^*, P_θ^* and P_n^* are inserted so that the well known static geometric duality of shell theory is extended to include the surface load terms, but should be set equal to zero for the solution of specific problems[3]).

The sinusoidal stress and strain distributions (1) and (8) modify the general static geometric duality or analogy slightly with the quantities in the first line of the following table being now the duals of the quantities in the second line:

n_r	n_θ	$n_{r\theta}$	$n_{\theta r}$	q_r	q_θ	m_r	m_θ	m
k_θ	k_r	$k_{\theta r}$	$k_{r\theta}$	l_θ	l_r	e_θ	e_r	e

P_r	P_θ	P_n	D	D_S	ν_b
P_r^*	P_θ^*	P_n^*	A	A_S	$-\nu_s$

With these dualities, the equilibrium equations (2) to (7) and the stress strain relations (9) to (11) become the duals of the compatibility equations (16) to (21) and the stress strain relations (12) to (14). This complete static geometric duality will be used extensively later to reduce the amount of computation.

The six equilibrium equations, six stress strain relations and six compatibility equations are a system of eighteen equations for the determination of nine stress measures and nine strain measures appeared in these equations. However, overall

[3]) In the more general formulation of shell theory of [5], l_θ and l_r are the r-dependent portions of the normal component of the two curvature change vectors.

static equilibrium requires

$$\alpha n_r + r n_{r\theta} - a q_r = -P a, \qquad P a = \int^r (\alpha P_r + r P_\theta - a P_n)\, dr. \tag{22}$$

Equation (22) is evidently a first integral of the differential equations of equilibrium obtained by forming the combination $(2) + r\alpha^{-1}(3) - a\alpha^{-1}(4)$ and can be used to replace (2).

The static geometric duality immediately suggests that there is also a first integral of the compatibility equations in the form

$$\alpha k_\theta + r k_{\theta r} - a l_\theta = -P^* a, \qquad P^* a = \int^r (\alpha P_r^* + r P_\theta^* - a P_n^*)\, dr \tag{23}$$

with P^* being the dual of P. Equation (23) is also an immediate consequence of the strain distributions associated with the multivalued displacement field obtained in [8], and can be used to replace (16). A more complete analysis of first integrals can be found in [2][4]).

3. Reduction of Differential Equations

With the first integrals (2.22) and (2.23), the shell equations stated in section (2) can be reduced to a sixth order system of two simultaneous third order differential equations for a stress variable and a strain variable. Motivated by the result of [11], we will take the variables to be $n_{r\theta}$ and $k_{\theta r}$.

We begin the reduction by writing the first integrals (2.22) and (2.23) in the form

$$n_r = \alpha^{-1}(a q_r - r n_{r\theta} - P a), \qquad k_\theta = \alpha^{-1}(a l_\theta - r k_{\theta r} - P^* a). \tag{1, 2}$$

Upon expressing m_r and m_θ in terms of k_r and k_θ by means of the stress strain relations (2.12) and (2.13) and using (2) to eliminate k_θ, the sixth equilibrium equation (2.7) can be rewritten as

$$n_{\theta r} + \alpha^{-2} a D(1-\nu_b) k_r = n_{r\theta} + \alpha^{-3} a D(1-\nu_b)(a l_\theta - r k_{\theta r} - P^* a). \tag{3}$$

The dual operations transform the sixth compatibility equation (2.21) into

$$k_{r\theta} + \alpha^{-2} a A(1+\nu_s) n_\theta = k_{\theta r} + \alpha^{-3} a A(1+\nu_s)(a q_r - r n_{r\theta} - P a). \tag{4}$$

The equilibrium equation (2.3) and the dual compatibility equation (2.17) can be written as

$$n_\theta + r\alpha^{-1} n_{\theta r} = -(\alpha n_{r\theta})' - a\alpha^{-1} q_r - \alpha P_\theta \tag{5}$$

and

$$k_r + r\alpha^{-1} k_{r\theta} = -(\alpha k_{\theta r})' - a\alpha^{-1} l_\theta - \alpha P_\theta^*, \tag{6}$$

respectively.

[4]) In a recent private communication, Professor J. G. Simmonds brought to our attention the less complete analysis of first integrals in [1].

We now solve the four simultaneous equations (3) to (6) for the stress variables n_θ and $n_{\theta r}$ and their dual strain variables k_r and $k_{r\theta}$ in terms of q_r, $n_{r\theta}$, l_θ and $k_{\theta r}$ and loads [5]).

$$\{n_\theta\, n_{\theta r}\, k_r\, k_{r\theta}\} = A_{46}\{q_r\, n_{r\theta}\, n'_{r\theta}\, l_\theta\, k_{\theta r}\, k'_{\theta r}\} + \{f_1\, f_2\, f_1^*\, f_2^*\} \tag{7}$$

where A_{46} is a 4×6 matrix and the load terms f_1^* and f_2^* are the static geometric duals of f_1 and f_2.

The equilibrium equation (2.4) and the dual compatibility equation (2.18) then give q_θ and l_r in terms of $n_{r\theta}$, q_r, $k_{\theta r}$, l_θ and the load terms

$$q_\theta = -(\alpha q_r)' - \alpha P_n + a\alpha^{-2}[2\alpha n_{r\theta} + a\alpha^{-2} D(1-v_b)(a l_\theta - r k_{\theta r} - \alpha k_r - P^* a)], \tag{8}$$

$$l_r = -(\alpha l_\theta)' - \alpha P_n^* + a\alpha^{-2}[2\alpha k_{\theta r} + a\alpha^{-2} A(1+v_s)(a q_r - r n_{r\theta} - \alpha n_\theta - P a)] \tag{9}$$

where k_r and n_θ are as given by (7).

With the help of the stress strain relations (2.9) to (2.14), the remaining two moment equilibrium equations (2.5) and (2.6) and the dual compatibility equations (2.19) and (2.20) can be written as

$$[D\alpha(k_r + v_b k_\theta)]' + D_S(k_{\theta r} + k_{r\theta}) - D r\alpha^{-1}(k_\theta + v_b k_r) - \alpha q_r = 0, \tag{10}$$

$$\begin{aligned}&[D_S\alpha(k_{\theta r} + k_{r\theta})]' - D(k_\theta + v_b k_r) + D_S r\alpha^{-1}(k_{\theta r} + k_{r\theta})\\ &\quad + \alpha(\alpha q_r)' - a(n_{r\theta} + n_{\theta r}) + \alpha^2 P_n = 0,\end{aligned} \tag{11}$$

$$[A\alpha(n_\theta - v_s n_r)]' + A_S(n_{r\theta} + n_{\theta r}) - A r\alpha^{-1}(n_r - v_s n_\theta) - \alpha l_\theta = 0, \tag{12}$$

$$\begin{aligned}&[A_S\alpha(n_{r\theta} + n_{\theta r})]' - A(n_r - v_s n_\theta) + A_S r\alpha^{-1}(n_{r\theta} + n_{\theta r})\\ &\quad + \alpha(\alpha l_\theta)' - a(k_{\theta r} + k_{r\theta}) + \alpha^2 P_n^* = 0.\end{aligned} \tag{13}$$

Upon using (1), (2) and (7), to eliminate k_r, k_θ, $k_{r\theta}$ and their duals, equations (10) to (13) are effectively four linear algebraic equations for q'_r, q_r, $n''_{r\theta}$, $n'_{r\theta}$, $n_{r\theta}$ and their dual strain measures. We can solve them for q'_r, q_r, l'_θ and l_θ in terms of $n_{r\theta}$ and $k_{\theta r}$ and their derivatives up to second order (and load terms) to get

$$\{q_r, q'_r, l_\theta, l'_\theta\} = B_{46}\{n_{r\theta}, n'_{r\theta}, n''_{r\theta}, k_{\theta r}, k'_{\theta r}, k''_{\theta r}\} + \{f_3, f_4, f_3^*, f_4^*\} \tag{14}$$

where B_{46} is another 4×6 matrix and f_3 and f_4 and their duals involve load terms only.

At this point, we have effectively transformed, algebraically and exactly, all six equilibrium and six compatibility equations into twelve equivalent relations, (14), (1), (2), (7), (8) and (9) giving q_r, q'_r, n_r, n_θ, $n_{\theta r}$, q_θ and their dual strain measures in terms of $n_{r\theta}$ and $k_{\theta r}$ and their derivatives up to second order as well as load terms. The six stress strain relations (2.9) to (2.14) give the remaining three stress and three strain measures in terms of the same derivatives of $n_{r\theta}$ and $k_{\theta r}$.

To get two more equations for the determination of $n_{r\theta}$ and $k_{\theta r}$, we simply differentiate both sides of the expression for q_r and l_θ in (14) and equate the right

[5]) We adopt the convention here that braces transpose a row vector into a column vector.

hand side of the differentiated expressions to the right hand side of the expression for q_r' and l_θ' in (14), respectively. This gives us two third order differential equations for $n_{r\theta}$ and $k_{\theta r}$. These equations are evidently the static geometric duals of each other, load terms included.

We emphasize that the final two differential equations for $n_{r\theta}$ and $k_{\theta r}$ as well as the auxiliary equations giving the remaining stress and strain measures in terms of $n_{r\theta}$ and $k_{\theta r}$ and their derivatives are exact consequences of the original shell equations. The reduction described above involves no approximation whatsoever.

4. A Set of Simplified Equations

The two differential equations for the two primary dependent variables $n_{r\theta}$ and $k_{\theta r}$ contain small terms of order $\sqrt{DA}/R$ or DA/l^2 compared to the other terms in the same equations. We will limit our discussion to shells with uniform properties, $v_s = v_b$ and $\sqrt{DA} = O(h)$ in this report. Upon omitting all these negligible terms and upon setting $P_r^* = P_\theta^* = P_n^* = 0$ (so that P^* is a constant)[6], we have the following two third order differential equations for $n_{r\theta}$ and $k_{\theta r}$:

$$L(k_{\theta r}) + \frac{2a}{D}\, n_{r\theta} = f, \qquad L^*(n_{r\theta}) + \frac{2a}{A}\, k_{\theta r} = g \tag{1, 2}$$

where

$$L(\;) = \alpha^3 (\;)''' + 6 r \alpha (\;)'' + \left[3 + (4 - v_b)\frac{a^2}{\alpha^2}\right] \alpha (\;)' - \left[3 + (5 + v_b)\frac{a^2}{\alpha^2}\right] \frac{r}{\alpha} (\;), \tag{3}$$

$$f = (3 - v_b)\frac{a^3}{\alpha^3} P^* + \frac{\alpha^2}{D} P_n, \tag{4}$$

$$\begin{aligned} g = {} & (3 + v_s)\frac{a^3}{\alpha^3} P + v_s \alpha^2 P_r' + (1 + v_s) r P_r - \alpha^3 P_\theta'' - 4 r \alpha P_\theta' \\ & - \left[1 + v_s + (2 + v_s)\frac{a^2}{\alpha^2}\right] \alpha P_\theta + a\left(\alpha P_n' - \frac{r}{\alpha} P_n\right). \end{aligned} \tag{5}$$

Upon using (3.14) to eliminate q_r and l_θ from (3.1), (3.2) and (3.7), we get n_r, n_θ, $n_{\theta r}$, and their duals explicitly in terms of $n_{r\theta}$ and $k_{\theta r}$. These results along with (3.8) and (3.9) can be used to express q_θ and l_r in terms of $n_{r\theta}$ and $k_{\theta r}$ alone. The stress strain relations (2.9) to (2.14) can then be used to get the remaining stress and strain measures in terms of $n_{r\theta}$ and $k_{\theta r}$.

In terms of two primary dependent variables, the exact expressions for all stress and strain measures obtained in [2] contain many terms which are $O(DA/l^2)$ or $O(\sqrt{DA}/R)$ compared to the other terms of the same expressions and can therefore be omitted. We will defer listing the simplified expressions for the stress resultants and couples until the next section.

[6]) Recall that these quantities were inserted to get a more complete static geometric duality and should be set to zero in applications.

5. The Problem of Pure Bending

The formulation of sections (3) and (4) is particularly appropriate for the solution of the problem of pure bending. We consider here a homogeneous isotropic shell bounded by the helical edges $r=r_i$ and $r=r_0$ $(r_i<r_0)$ and the radial edges $\theta=\pm\theta_0$. The shell is free of surface loads and free of edge loads at the helical edges. It is subject to only equal and opposite end bending moments turning about the x-axis at the edges $\theta=\pm\theta_0$.

For such a shell, we have $P=0$ and $P^*=k$ where k is a constant to be determined. The stress free conditions at the helical edges require the satisfaction of the homogeneous Kirchhoff-Bassett conditions

$$r=r_i, r_0:\ N_r+\frac{M_{r\theta}}{R}=N_{r\theta}=M_r=Q_r+\frac{1}{\alpha}\frac{\partial M_{r\theta}}{\partial\theta}=0 \tag{1}$$

which, in view of (2.1) and (2.22), are only three independent conditions

$$r=r_i, r_0:\ n_{r\theta}=m_r=q_r+\alpha^{-1}m=0. \tag{2}$$

The two third order differential equations (4.1) and (4.2), supplemented by the three boundary conditions (2) at each of the two helical edges, determine $n_{r\theta}$ and $k_{\theta r}$ in terms of k. The parameter k is then related to the applied moment M_p by the integrated condition [8]

$$M_p=-\int_{r_i}^{r_0} m_\theta\, dr. \tag{3}$$

To obtain a solution of this problem, we first put the relevant equations in dimensionless form by setting

$$\begin{gathered}\rho=r/r_0,\quad \rho_i=r_i/r_0,\quad \lambda=r_0/a,\quad \mu^3=\frac{2a\,r_0^3}{\sqrt{DA}\,(a^2+r_0^2)^{\frac{3}{2}}},\\ K=r_0\,k_{\theta r},\quad N=\sqrt{A/D}\,r_0\,n_{r\theta}.\end{gathered} \tag{4}$$

In terms of these dimensionless quantities, the two differential equations (4.1) and (4.2) become

$$\begin{aligned}&K^{\cdots}+\frac{6\lambda^2\rho}{1+\lambda^2\rho^2}K^{\cdot\cdot}+\frac{(7-\nu)\lambda^2+3\lambda^4\rho^2}{(1+\lambda^2\rho^2)^2}K^{\cdot}-\frac{(8+\nu)\lambda^4\rho+3\lambda^6\rho^3}{(1+\lambda^2\rho^2)^3}K\\&\quad+\mu^3\left(\frac{1+\lambda^2}{1+\lambda^2\rho^2}\right)^{\frac{3}{2}}N=\frac{(3-\nu)\,k\,r_0\,\lambda^3}{(1+\lambda^2\rho^2)^3},\end{aligned} \tag{5}$$

$$\begin{aligned}&N^{\cdots}+\frac{6\lambda^2\rho}{1+\lambda^2\rho^2}N^{\cdot\cdot}+\frac{(7+\nu)\lambda^2+3\lambda^4\rho^2}{(1+\lambda^2\rho^2)^2}N^{\cdot}-\frac{(8-\nu)\lambda^4\rho+3\lambda^6\rho^3}{(1+\lambda^2\rho^2)^3}N\\&\quad+\mu^3\left(\frac{1+\lambda^2}{1+\lambda^2\rho^2}\right)^{\frac{3}{2}}K=0\end{aligned} \tag{6}$$

where dots indicate differentiation with respect to ρ.

Upon writing the simplified expressions for the stress resultants and couples, previously discussed but not listed in section (4), in terms of the dimensionless quantities in (4) and upon omitting all terms in these simplified expressions unessential for the present problem, we have

$$\sqrt{\frac{A}{D}}\, r_0\, n_\theta = -\frac{\sqrt{1+\lambda^2\rho^2}}{\lambda}\left(N^{\cdot} + \frac{2\lambda^2\rho}{1+\lambda^2\rho^2}N\right), \quad \sqrt{\frac{A}{D}}\, r_0\, n_r = -\frac{\lambda\rho}{\sqrt{1+\lambda^2\rho^2}}N,$$

$$\sqrt{\frac{A}{D}}\, r_0\, n_{\theta r} = N, \quad \frac{r_0}{D} m = (1-\nu)K,$$

$$\frac{r_0}{D} m_r = -\frac{\sqrt{1+\lambda^2\rho^2}}{\lambda}\left[K^{\cdot} + \frac{(2+\nu)\lambda^2\rho}{1+\lambda^2\rho^2}K + \frac{\nu k r_0 \lambda}{1+\lambda^2\rho^2}\right], \tag{7}$$

$$\frac{r_0}{D} m_\theta = -\frac{\sqrt{1+\lambda^2\rho^2}}{\lambda}\left[\nu K^{\cdot} + \frac{(1+2\nu)\lambda^2\rho}{1+\lambda^2\rho^2}K + \frac{k r_0 \lambda}{1+\lambda^2\rho^2}\right],$$

$$\frac{r_0^2}{D} q_r = -\frac{\sqrt{1+\lambda^2\rho^2}}{\lambda}\left[K^{\cdot\cdot} + \frac{4\lambda^2\rho}{1+\lambda^2\rho^2}K^{\cdot} + \frac{3\lambda^2}{(1+\lambda^2\rho^2)^2}K - \frac{k r_0 \lambda^3\rho}{(1+\lambda^2\rho^2)^2}\right],$$

$$\frac{r_0^2}{D} q_\theta = K^{\cdot} + \frac{3\lambda^2\rho}{1+\lambda^2\rho^2}K + \frac{k r_0\lambda}{1+\lambda^2\rho^2}\left[1 - \frac{2(3-\nu)}{1+\lambda^2\rho^2}\right].$$

With the help of (7), the boundary conditions (2) become

$$\rho = \rho_i, 1:\ N = 0, \quad K^{\cdot} + \frac{(2+\nu)\lambda^2\rho}{1+\lambda^2\rho^2}K + \frac{\nu k r_0\lambda}{1+\lambda^2\rho^2} = 0,$$
$$K^{\cdot\cdot} + \frac{4\lambda^2\rho}{1+\lambda^2\rho^2}K^{\cdot} + \frac{(2+\nu)\lambda^2 - (1-\nu)\lambda^4\rho^2}{(1+\lambda^2\rho^2)^2}K - \frac{k r_0\lambda^3\rho}{(1+\lambda^2\rho^2)^2} = 0. \tag{8}$$

Finally, the expression for the end bending moment, (3), become

$$\frac{M_p}{D} = \frac{1}{\lambda}\int_{\rho_i}^{1}\left[\nu K^{\cdot} + \frac{(1+2\nu)\lambda^2\rho}{1+\lambda^2\rho^2}K + \frac{k r_0\lambda}{1+\lambda^2\rho^2}\right]\sqrt{1+\lambda^2\rho^2}\, d\rho. \tag{9}$$

In terms of λ, we have for a homogeneous and isotropic shell,

$$\mu^3 = \frac{2a}{\sqrt{DA}}\frac{r_0^3}{(a^2+r_0^2)^{\frac{3}{2}}} = O\left(\frac{r_0}{h}\frac{\lambda^2}{(1+\lambda^2)^{\frac{3}{2}}}\right). \tag{10}$$

For a fixed h, μ may be small or large compared to unity depending on the pitch-to-width ratio, a/r_0. In particular, we have $\mu \gg 1$ whenever λ is $O(1)$ in which case the contribution of N and K to the stresses in the shell is significant only in a narrow region near the helical edges. On the other hand, we have for a fixed value of h/r_0, $\mu \to 0$ as $a \to 0$ and as $a \to \infty$.

6. A λ^{-2}-Perturbation Solution for Shells with a Small Pitch

For shells with a small pitch to width ratio so that $\lambda^{-2} \ll 1$ a perturbation solution in powers of λ^{-2} is appropriate provided that $(\lambda\rho)^2 \ll 1$ for all $\rho_i \leq \rho \leq 1$ [7]). The form of the differential equations (5.5) and (5.6) and boundary conditions (5.8) implies

$$K = ka \sum_{m=0}^{\infty} K_m(\rho)\,\lambda^{-2m}, \qquad N = ka \sum_{m=0}^{\infty} N_m(\rho)\,\lambda^{-2m}. \tag{1}$$

With $\mu_0^3 = 2a/\sqrt{DA}$ and $\mu^3 = \mu_0^3[1 - 3/2\lambda^2 + \cdots]$, the coefficients K_0 and N_0 are determined by the boundary value problem

$$\begin{cases} K_0^{\cdots} + 6\rho^{-1}K_0^{\cdot\cdot} + 3\rho^{-2}K_0^{\cdot} - 3\rho^{-3}K_0 + \mu_0^3\rho^{-3}N_0 = 0 \\ N_0^{\cdots} + 6\rho^{-1}N_0^{\cdot\cdot} + 3\rho^{-2}N_0^{\cdot} - 3\rho^{-3}N_0 + \mu_0^3\rho^{-3}K_0 = 0, \end{cases} \tag{2}$$

$$\rho = \rho_i, 1: \begin{cases} N_0 = 0, \quad K_0^{\cdot} + (2+\nu)\rho^{-1}K_0 + \nu\rho^{-2} = 0 \\ K_0^{\cdot\cdot} + 4\rho^{-1}K_0^{\cdot} - (1-\nu)\rho^{-2}K_0 - \rho^{-3} = 0. \end{cases} \tag{3}$$

The corresponding expression for the end bending moment becomes

$$\frac{M_p}{Dka} = \int_{\rho_i}^{1} \{\nu\rho K_0^{\cdot} + (1+2\nu)K_0 + \rho^{-1}\}\,d\rho. \tag{4}$$

Upon setting $K_0 = (\rho^{-1}W)^{\cdot}$ and $N_0 = -(\rho^{-1}f)^{\cdot}$, the zeroth order problem (equations (2), (3) and (4)) is identical with the dimensionless form of the shallow shell formulation in [9]. An exact solution of this system as well as the corresponding stress distributions and overall load deformation relation were given there. We note in particular that these results tend to the results for a flat circular ring sector plate as $a \to 0$ ($\mu \to 0$) with ka kept finite. On the other hand, for a fixed value of λ and for μ large compared to unity, the contribution of K_0 and N_0 to the stresses in the shell is of the nature of edge effects.

For moderately small values of λ^{-2}, the formulation of this section enables us to obtain a more accurate solution by adding to the leading term (shallow shell) solution higher order correction terms.

7. A λ^2-Perturbation Solution for a Slightly Pretwisted Strip

For a shell with a large pitch to width ratio, a perturbation solution in powers of λ^2 is appropriate. The form of the differential equations (5.5) and (5.6) and the

[7]) For a shell with $\rho_i \leq 0$, it is necessary to use a more elaborate perturbation solution including an expansion of ρ in powers of λ^{-2} as done in [4] for problems involving rotationally symmetric stress distributions.

boundary conditions (5.8) implies

$$K = k\, r_0\, \lambda \sum_{m=0}^{\infty} k_m(\rho)\, \lambda^{2m}, \qquad N = k\, r_0\, \lambda \sum_{m=0}^{\infty} n_m(\rho)\, \lambda^{2m}. \tag{1}$$

With $\hat{\mu}^3 = 2 r_0^3 / a^2 \sqrt{DA}$ and $\mu^3 = \hat{\mu}^3 [1 - 3\lambda^2/2 + \cdots]$, the leading terms, k_0 and n_0, are determined by the boundary value problem

$$\begin{aligned} &\dddot{k}_0 + \hat{\mu}^3 n_0 = 0, \quad \dddot{n}_0 + \hat{\mu}^3 k_0 = 0, \\ &\rho = \rho_i, 1: \quad n_0 = 0, \quad \dot{k}_0 + \nu = 0, \quad \ddot{k}_0 = 0. \end{aligned} \tag{2}$$

Note that the differential equations for the determination of n_0 and k_0 (as well as those for n_m and k_m) are with constant coefficients.

We will be concerned here only with the leading term solution for the case $\rho_i = -1$. For this case, the solution of (2) is

$$\begin{aligned} k_0 &= c_1 \sinh(\hat{\mu}\rho) + c_2 \sinh\left(\frac{1}{2}\hat{\mu}\rho\right) \cos\left(\frac{\sqrt{3}}{2}\hat{\mu}\rho\right) + c_3 \cosh\left(\frac{1}{2}\hat{\mu}\rho\right) \sin\left(\frac{\sqrt{3}}{2}\hat{\mu}\rho\right), \\ n_0 &= -c_1 \cosh(\hat{\mu}\rho) - c_2 \cosh\left(\frac{1}{2}\hat{\mu}\rho\right) \cos\left(\frac{\sqrt{3}}{2}\hat{\mu}\rho\right) - c_3 \sinh\left(\frac{1}{2}\hat{\mu}\rho\right) \sin\left(\frac{\sqrt{3}}{2}\hat{\mu}\rho\right) \end{aligned} \tag{3}$$

where

$$\begin{aligned} c_1 &= \frac{\nu}{\hat{\mu}} \frac{\sinh(\hat{\mu}) - \frac{1}{\sqrt{3}} \sin(\sqrt{3}\,\hat{\mu})}{\sinh(\hat{\mu})\,[\cosh(\hat{\mu}) + \cos(\sqrt{3}\,\hat{\mu}] + \frac{2}{\sqrt{3}} \sin(\sqrt{3}\,\hat{\mu}) \cosh(\hat{\mu})}, \\ c_2 &= \frac{\nu}{\hat{\mu}\Delta} \left[2 \tanh(\hat{\mu}) \sinh\left(\frac{1}{2}\hat{\mu}\right) \sin\left(\frac{\sqrt{3}}{2}\hat{\mu}\right) + \cosh\left(\frac{1}{2}\hat{\mu}\right) \sin\left(\frac{\sqrt{3}}{2}\hat{\mu}\right) \right. \\ &\qquad \left. - \sqrt{3} \sinh\left(\frac{1}{2}\hat{\mu}\right) \cos\left(\frac{\sqrt{3}}{2}\hat{\mu}\right) \right], \\ c_3 &= -\frac{\nu}{\hat{\mu}\Delta} \left[2 \tanh(\hat{\mu}) \cosh\left(\frac{1}{2}\hat{\mu}\right) \cos\left(\frac{\sqrt{3}}{2}\hat{\mu}\right) + \sinh\left(\frac{1}{2}\hat{\mu}\right) \cos\left(\frac{\sqrt{3}}{2}\hat{\mu}\right) \right. \\ &\qquad \left. + \sqrt{3} \cosh\left(\frac{1}{2}\hat{\mu}\right) \sin\left(\frac{\sqrt{3}}{2}\hat{\mu}\right) \right], \\ \Delta &= \frac{\sqrt{3}}{2} \tanh(\hat{\mu})\,[\cosh(\hat{\mu}) + \cos(\sqrt{3}\,\hat{\mu})] + \sin(\sqrt{3}\,\hat{\mu}). \end{aligned} \tag{4}$$

The leading term λ^2-perturbation solution for the stress resultants and couples are

$$
\begin{aligned}
&\sqrt{\frac{A}{D}}\, r_0 N_\theta = -(k\lambda n_0^{\cdot})\, a \sin\theta\,, \qquad \sqrt{\frac{A}{D}}\, r_0 N_r = -(k\lambda^3 \rho\, n_0)\, a \sin\theta\,,\\
&\sqrt{\frac{A}{D}}\, r_0 N_{r\theta} = \sqrt{\frac{A}{D}}\, r_0 N_{\theta r} = (k\lambda n_0)\, r_0 \cos\theta\,, \qquad \frac{r_0}{D} M_{r\theta} = (1-\nu)(k\lambda^2 k_0)\, a \sin\theta\,,\\
&\frac{r_0}{D} M_r = -k(k_0^{\cdot}+\nu)\, r_0 \cos\theta\,, \qquad \frac{r_0}{D} M_\theta = -k(\nu k_0^{\cdot}+1)\, r_0 \cos\theta\,,\\
&\frac{r_0^2}{D} Q_r = -(k k_0^{\cdot\cdot})\, r_0 \cos\theta\,, \qquad \frac{r_0^2}{D} Q_\theta = (k\lambda^2 k_0^{\cdot})\, a \sin\theta\,.
\end{aligned} \tag{5}
$$

The corresponding expression for the end moment is

$$
\frac{M_p}{D k r_0} = \int_{\rho_i}^{1} [\nu k_0^{\cdot} + 1]\, d\rho = [\nu k_0 + \rho]_{\rho_i}^{1}\,. \tag{6}
$$

For $\hat{\mu} \ll 1$, we must have $\lambda^2 \ll h/r_0$ (see equation (5.10)). In this case, we have from (3) and (4)

$$
\begin{aligned}
&k_0 \simeq -\nu\rho\,, \quad k_0^{\cdot} \simeq -\nu\,, \quad k_0^{\cdot\cdot} \simeq -\tfrac{1}{6}\nu\hat{\mu}^4 \rho(1-2\rho^2)\,,\\
&n_0 \simeq -\tfrac{1}{2}\nu\hat{\mu}(1-\rho^2)\,, \quad n_0^{\cdot} \simeq \nu\hat{\mu}\rho\,, \quad n_0^{\cdot\cdot} \simeq \nu\hat{\mu}\,.
\end{aligned} \tag{7}
$$

In the limit as $\lambda \to 0$ ($\hat{\mu} \to 0$), we have from (5) and (6)[8])

$$
M_\theta = -\frac{M_p}{2 r_0} \tag{8}
$$

while all other stress resultants and couples tend to zero. Thus, in the limiting case of a flat strip, the solution given by (3), (5) and (6) reduces to that for the pure bending of a rectangular plate.

For small but finite values of λ and $\hat{\mu}$, we have from (5) and (6), with the help of (7),

$$
\sigma_\theta^B = \frac{6 M_\theta}{h^2} = -\frac{3 M_p}{r_0 h^2} \tag{9}
$$

except for terms of higher order in $\hat{\mu}$ and λ. All other direct and bending stresses are at most $O(\lambda\hat{\mu})$ compared to σ_θ^B. Therefore, the shell behaves effectively as a flat plate whenever $\lambda^2 \ll h/r_0$ as predicted by the shallow shell solution for 'plate bending' of [3, 6].

On the other hand, when $h/r_0 \ll \lambda^2 \ll 1$, so that $\hat{\mu} \gg 1$, the contribution of n_0 and k_0 terms to all stress resultants and couples given by (4) are evidently confined to a narrow region near the helical edges. Away from the edges, we have, except for terms of higher order in λ and $\hat{\mu}^{-1}$,

$$\sigma_\theta^B = \frac{6 M_\theta}{h^2} = -\frac{3 M_p}{r_0 h^2}, \quad \sigma_r^B = \frac{6 M_r}{h^2} = -\frac{3 \nu M_p}{r_0 h^2}. \tag{10}$$

All other stresses are negligibly small compared to σ_θ^B. We note that while σ_θ^B remains the same as in the case $\hat{\mu} \ll 1$, σ_r^B is now of the same order of magnitude as σ_θ^B in the interior of the shell.

The above results indicate that, for $\lambda \ll 1$, the stress distribution in the shell is significantly different in the range $\hat{\mu} \ll 1$ and $\hat{\mu} \gg 1$.

8. Asymptotic Solution for Large μ

For a sufficiently thin shell, the parameter (see also (5.4))

$$\mu = O\left(\left[\frac{a}{h}\right]^{\frac{1}{3}} \left[\frac{r_0^2}{a^2 + r_0^2}\right]^{\frac{1}{2}}\right) \tag{1}$$

is large compared to unity. We may then seek complementary solutions of the system (5.5) and (5.6) in the form

$$N_c \sim e^{\mu \xi} \sum_{m=0}^{\infty} N_{cm}(\rho)\, \mu^{-m}, \qquad K_c \sim e^{\mu \xi} \sum_{m=0}^{\infty} K_{cm}(\rho)\, \mu^{-m} \tag{2}$$

where $\xi = \xi(\rho)$, and particular solutions of the same system in the form

$$N_p \sim \sum_{m=0}^{\infty} N_{pm}(\rho)\, \mu^{-3m}, \qquad K_p \sim \sum_{m=0}^{\infty} K_{pm}(\rho)\, \mu^{-3m}. \tag{3}$$

A straight forward application of the method of asymptotic integration gives, except for higher order terms in $1/\mu$,

$$\begin{aligned} K \sim \frac{\lambda/\mu}{\sqrt{1+\lambda^2 \rho^2}} &\{c_1 (\varphi_i)^{2\mu} + (\varphi_i)^{\mu} [c_2 \cos(\sqrt{3}\, \mu \ln \varphi_i) + c_3 \sin(\sqrt{3}\, \mu \ln \varphi_i)] \\ &+ c_4 (\varphi_0)^{2\mu} + (\varphi_0)^{\mu} [c_5 \cos(\sqrt{3}\, \mu \ln \varphi_0) + c_6 \sin(\sqrt{3}\, \mu \ln \varphi_0)]\}, \\ N \sim \frac{\lambda/\mu}{\sqrt{1+\lambda^2 \rho^2}} &\{c_1 (\varphi_i)^{2\mu} - (\varphi_i)^{\mu} [c_2 \cos(\sqrt{3}\, \mu \ln \varphi_i) + c_3 \sin(\sqrt{3}\, \mu \ln \varphi_i)] \\ &- c_4 (\varphi_0)^{2\mu} + (\varphi_0)^{\mu} [c_5 \cos(\sqrt{3}\, \mu \ln \varphi_0) + c_6 \sin(\sqrt{3}\, \mu \ln \varphi_0)]\} \end{aligned} \tag{4}$$

where

$$\varphi_i(\rho)=\left[\frac{\sqrt{1+\lambda^2\rho_i^2}+\lambda\rho_i}{\sqrt{1+\lambda^2\rho^2}+\lambda\rho}\right]^{\sqrt{1+\lambda^2}/2\lambda},$$
$$\varphi_0(\rho)=\left[\frac{\sqrt{1+\lambda^2\rho^2}+\lambda\rho}{\sqrt{1+\lambda^2}+\lambda}\right]^{\sqrt{1+\lambda^2}/2\lambda} \tag{5}$$

and where c_1-c_6 are six real constants of integration to be determined by the six boundary conditions (5.8). These boundary conditions imply that the c_j's are $O(1)$.

Having determined K and N, we can calculate m_θ by way of (5.7). Upon introducing the result into the expression for the resultant end moment (5.9), we get

$$\frac{M_p}{r_0 D k}\sim\ln\left[\frac{\sqrt{1+\lambda^2}+\lambda}{\sqrt{1+\lambda^2\rho_i^2}+\lambda\rho_i}\right]\left\{1+O\left(\frac{1}{\mu}\right)\right\}. \tag{6}$$

For $(\lambda\rho_i)^2\gg 1$, we have $M_p/r_0 Dk\approx -\ln\rho_i$ which is the limiting value obtained by a shallow shell theory for very large μ [9]. On the other hand, when $\lambda^2\ll 1$, we have $M_p/r_0 Dk\approx\lambda(1-\rho_i)$ which agrees with (7.7) since $k_0=O(1/\mu)$ for $\mu\gg 1$.

The stiffness relation (6) therefore bridges the known results for the two extreme ranges of λ and provides the previously unknown overall bending stiffness in the range $\lambda=O(1)$ when the shell is sufficiently thin so that terms of order $1/\mu$ are negligible compared to unity.

While we can also calculate the stress distributions by way of (5.7), we will merely note that, for the case of highly pretwisted strips ($\rho_i=-1$ and $\lambda\gg 1$), these expressions suggest the possibility of a stress concentration phenomenon. For example, it is not difficult to see from the expression for m_θ that $m_\theta(\rho=0)/m_\theta(\rho=1)=O(\lambda)$ for $\lambda\gg 1$. This conclusion was further confirmed by our finite difference solution [2]. A more detailed discussion of stress distributions and overall bending stiffness for a wide range of values of λ and μ (including the case where λ and μ are both $O(1)$ not discussed herein) will be reported in a future publication in connection with a staggered-mesh finite difference scheme for our sixth order system (5.5) and (5.6).

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Zusammenfassung

Die Klasse von Problemen der linearen Theorie für schraubenförmige Schalen mit Spannungen und Verzerrungen von der Form $e^{i\theta} f(r)$ wird auf ein Randwertproblem für ein System von zwei gewöhnlichen Differentialgleichungen dritter Ordnung reduziert. Für Schalen unter dem Einfluß von Endbiegemomenten werden Lösungen mit Hilfe von Störungsrechnung sowie asymptotische Lösungen bestimmt.

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