Journal of Sound and Vibration (1973) **28**(1), 105–120

AN ALGEBRAIC METHOD FOR LINEAR DYNAMICAL SYSTEMS WITH STATIONARY EXCITATIONS

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(*Received* 4 *September* 1972, *and in revised form* 8 *February* 1973)

The paper describes an algebraic method for the second-order statistics of the response of multi-degree-of-freedom linear time-invariant dynamical systems to (zero mean) white noise or stationary filtered white noise excitation. The method is based on the observation that the steady-state covariance matrix $\mathbf{Y}$ of the response is the solution of the matrix equation $\mathbf{AY} + \mathbf{YA'} = \mathbf{C}$, where $\mathbf{A'}$ is the transpose of $\mathbf{A}$. A simple algorithm, which takes advantage of the special form of $\mathbf{A}$, is given for the solution of the matrix equation. The algorithm is particularly suitable for machine computation. The correlation matrix is simply the product of the impulse response matrix and the covariance matrix.

The above general method is applied to study the flapping response of a flexible lifting rotor blade in hovering or vertical flight to stationary random excitation. The results of this study show that the conventional rigid blade analysis does not in general give an adequate approximate description of the mean square response of the blade. They also suggest that an adequate approximate solution may be obtained by an uncoupled two-modes analysis if the Lock number of the blade is not too large.

An alternate method for the solution of the matrix equation is also given for systems with a large number of degrees of freedom. This alternate method is computationally less efficient but does allow us to keep the calculations strictly in-core for systems with up to 100 degrees of freedom.

1. INTRODUCTION

A general M-degree-of-freedom time-invariant dynamical system with damping may be characterized by†

$$\ddot{x}_m + \omega_m^2 x_m + \sum_{k=1}^{M} \zeta_{mk} \dot{x}_k = f_m(t), \quad m = 1, 2, \ldots, M. \tag{1}$$

If $f_m(t)$ are zero mean random functions with known statistics, the second-order correlation functions of the steady-state response are usually obtained by way of the impulse response matrix or the frequency response matrix [1]. A considerable amount of non-algebraic calculation is involved in either approach.

In practice, one often settles for the steady-state mean square properties of the response contained in the elements of the covariance matrix, $\langle x_j(t)\, x_k(t)\rangle$, $\langle x_j(t)\, \dot{x}_k(t)\rangle$ and $\langle \dot{x}_j(t)\, \dot{x}_k(t)\rangle$,

† A list of symbols is given in Appendix II.

where $\langle\cdots\rangle$ is the ensemble averaging operation. If the excitations are *temporally uncorrelated*, a third method described in reference [1] (see pp. 151–152) may be used to obtain the covariance matrix. The essential feature of the method is the formulation of a non-stochastic initial value problem for the covariance matrix itself. Though not mentioned in reference [1], it is not difficult to see that the steady-state solution of this initial value problem involves only the solution of a set of linear algebraic equations if the excitations, $f_m(t)$, are white noise processes. Moreover, the correlation matrix of the response, if needed, can be obtained from the covariance matrix and the impulse response matrix of the dynamical system.

The advantage of this algebraic method for time-invariant dynamical systems does not seem to have attracted much attention, possibly because of (i) the severe restriction imposed on the excitations, and (ii) the very awkward form in which the linear equations naturally appear, namely $\mathbf{AX}+\mathbf{XB}=\mathbf{C}$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$ and $\mathbf{C}$ are all square matrices with $\mathbf{X}$ being the unknown. In this paper, we will (i) describe an effective method for solving the relevant matrix equation suitable for automatic computation, (ii) make use of the technique of filtered white noise in control theory to remove the restriction of white noise excitations so that, for all practical purposes, the determination of the steady-state covariance matrix remains algebraic for all stationary excitations,† (iii) apply the method to obtain the steady-state mean square response of a flexible rotor blade in hovering to a zero mean stationary (randomly changing) pitch angle, and (iv) suggest an alternate strictly in-core method for the matrix equation when M is large.

The results for the rotor blade problem show that the conventional rigid blade analysis does not in general give an adequate approximate solution for the mean square response. They also suggest that an adequate approximate solution may be obtained by an uncoupled two-mode analysis if the Lock number of the blade is not too large.

2. THE COVARIANCE MATRIX FOR WHITE NOISE EXCITATION

Let

$$y_k = x_k, \qquad y_{M+k} = \dot{x}_k,$$
$$g_k = 0, \qquad g_{M+k} = f_k \quad (k = 1, 2, \ldots, M) \tag{2}$$

and write the system (1) as

$$\dot{\mathbf{y}} = \mathbf{Ay} + \mathbf{g}, \tag{3}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\boldsymbol{\Omega} & -\mathbf{Z} \end{bmatrix}, \qquad \boldsymbol{\Omega} = [\omega_j^2\,\delta_{ij}], \qquad \mathbf{Z} = [\zeta_{ij}], \tag{4}$$

with $\mathbf{I}$ being an $M \times M$ identity matrix. If the excitation is temporally uncorrelated so that

$$\langle f_m(t_2)\, f_n(t_1)\rangle = F_{mn}\,\delta(t_2 - t_1), \tag{5}$$

where $\langle\cdots\rangle$ is the ensemble averaging operation and where $\mathbf{F}=[F_{mn}]$ is a positive semi-definite symmetric constant matrix, the covariance matrix of $\mathbf{y}$, defined as

$$\mathbf{Y}(t) = [\langle \mathbf{y}(t)\,\mathbf{y}'(t)\rangle], \tag{6}$$

† A referee pointed out that this technique is also used in reference [1] in connection with the transition probability density function of a continuous Markov process.

where ( )′ denotes the transpose of ( ), satisfies the matrix **ODE** [1]

$$\mathbf{Y}^{\cdot} = \mathbf{AY} + \mathbf{YA}' + \mathbf{G}, \tag{7}$$

where

$$\mathbf{G} = [\mathbf{gg}'] = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{F} \end{bmatrix}. \tag{8}$$

Supplemented by suitable initial conditions (e.g., $\mathbf{Y}(0) = 0$ if the system is initially at rest), equation (7) determines $\mathbf{Y}$ completely.

If we are only interested in the steady-state solution of equation (7), we may get it simply by solving the system of $(2M)^2$ linear algebraic equations

$$\mathbf{AY} + \mathbf{YA}' = -\mathbf{G}. \tag{9}$$

With the help of Kronecker products [2], the matrix equation (9) may be put in the more convenient vector form

$$\mathbf{B}\boldsymbol{\xi} = \boldsymbol{\eta}, \tag{10}$$

where

$$\begin{aligned} \boldsymbol{\xi} &= (Y_{11}, Y_{12}, \ldots, Y_{1(2M)}, Y_{21}, \ldots, Y_{(2M)(2M)})', \\ \boldsymbol{\eta} &= (G_{11}, G_{12}, \ldots, G_{1(2M)}, G_{21}, \ldots, G_{(2M)(2M)})', \\ \mathbf{B} &= \mathbf{A} \times \mathbf{I} + \mathbf{I} \times \mathbf{A}, \end{aligned} \tag{11}$$

with $\mathbf{C} \times \mathbf{D} = [C_{ij} D]$. Equation (10) can now be solved by the conventional methods. The use of Kronecker products allows us to relegate all computations in the solution of equation (9) to an automatic computer.

However, even for $M = 5$, $\mathbf{B}$ is already a 100×100 matrix. To avoid dealing with an extremely large $\mathbf{B}$, we will describe in the next section a more practical method of solution which takes advantage of the special form of $\mathbf{A}$.

3. AN EFFECTIVE METHOD FOR $\mathbf{AY} + \mathbf{YA}' = -\mathbf{G}$

Partition $\mathbf{Y}$ into four $M \times M$ sub-matrices,

$$\mathbf{Y} = \begin{bmatrix} \mathbf{U} & \mathbf{S} \\ \mathbf{T} & \mathbf{V} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned} \mathbf{U} &= [\langle x_i(t)\, x_j(t)\rangle], \quad \mathbf{S} = [\langle x_i(t)\, \dot{x}_j(t)\rangle], \\ \mathbf{T} &= [\langle \dot{x}_i(t)\, x_j(t)\rangle], \quad \mathbf{V} = [\langle \dot{x}_i(t)\, \dot{x}_j(t)\rangle]. \end{aligned} \tag{13}$$

We note parenthetically that $\mathbf{U}' = \mathbf{U}$, $\mathbf{V}' = \mathbf{V}$ and $\mathbf{S}' = \mathbf{T}$, so that there are at most $2M^2 + M$ unknowns.

With equation (12), equation (9) is equivalent to the four matrix equations

$$\begin{aligned} \mathbf{S} + \mathbf{T} = 0, &\qquad \boldsymbol{\Omega}\mathbf{U} + \mathbf{ZT} - \mathbf{V} = 0, \\ \mathbf{U}\boldsymbol{\Omega} + \mathbf{SZ}' - \mathbf{V} = 0, &\qquad \boldsymbol{\Omega}\mathbf{S} + \mathbf{T}\boldsymbol{\Omega} + \mathbf{ZV} + \mathbf{VZ}' = \mathbf{F}, \end{aligned} \tag{14}$$

where use has been made of the fact that $\boldsymbol{\Omega}$ is diagonal so that $\boldsymbol{\Omega}' = \boldsymbol{\Omega}$.† The first of equations (14) may be used to eliminate $\mathbf{T}$ from the other three. The results can be equivalently written as

$$\mathbf{ZS} + \mathbf{SZ}' = \boldsymbol{\Omega}\mathbf{U} - \mathbf{U}\boldsymbol{\Omega}, \qquad 2\mathbf{V} = \boldsymbol{\Omega}\mathbf{U} + \mathbf{U}\boldsymbol{\Omega} - (\mathbf{ZS} - \mathbf{SZ}'),$$
$$\boldsymbol{\Omega}\mathbf{S}' - \mathbf{S}\boldsymbol{\Omega} + \mathbf{ZV} + \mathbf{VZ}' = \mathbf{F}, \tag{15}$$

where the second and third of equations (14) have been replaced by their sum and difference.

Since $\boldsymbol{\Omega}$ is diagonal, the first of equations (15) can be written as

$$(\mathbf{Z} \times \mathbf{I} + \mathbf{I} \times \mathbf{Z})\,\mathbf{s} = (\boldsymbol{\Omega} \times \mathbf{I} - \mathbf{I} \times \boldsymbol{\Omega})\,\mathbf{u}, \tag{16}$$

where $\mathbf{s}$ and $\mathbf{u}$ are obtained from $\mathbf{S}$ and $\mathbf{U}$ in the same way as ξ from $\mathbf{Y}$. Equation (16) can be solved for $\mathbf{s}$ in terms of $\mathbf{u}$:

$$\mathbf{s} = (\mathbf{Z} \times \mathbf{I} + \mathbf{I} \times \mathbf{Z})^{-1}(\boldsymbol{\Omega} \times \mathbf{I} - \mathbf{I} \times \boldsymbol{\Omega})\,\mathbf{u} \equiv \mathbf{Z}_+^{-1}\,\boldsymbol{\Omega}_-\mathbf{u}. \tag{17}$$

The above expression for $\mathbf{s}$ can then be used to eliminate $\mathbf{S}$ from the second of equations (15) (written in terms of Kronecker products) so that

$$\mathbf{v} = \tfrac{1}{2}[(\boldsymbol{\Omega} \times \mathbf{I} + \mathbf{I} \times \boldsymbol{\Omega}) - (\mathbf{Z} \times \mathbf{I} - \mathbf{I} \times \mathbf{Z})\,\mathbf{Z}^{-1}\,\boldsymbol{\Omega}_-]\,\mathbf{u} \equiv \tfrac{1}{2}[\boldsymbol{\Omega}_+ - \mathbf{Z}_-\mathbf{Z}_+^{-1}\,\boldsymbol{\Omega}]\,\mathbf{u}. \tag{18}$$

Finally, the last of equations (15) may be written as an equation for $\mathbf{u}$ alone with the help of equations (17) and (18):

$$[\boldsymbol{\Omega}_-\mathbf{Z}_+^{-1}\,\boldsymbol{\Omega}_- + \tfrac{1}{2}\mathbf{Z}_+\boldsymbol{\Omega}_+ - \tfrac{1}{2}\mathbf{Z}_+\mathbf{Z}_-\mathbf{Z}_+^{-1}\,\boldsymbol{\Omega}_-]\,\mathbf{u} = \mathbf{f}. \tag{19}$$

We now solve equation (19) for $\mathbf{u}$ (or $\mathbf{U}$). Equations (18) and (19) then give $\mathbf{v}$ and $\mathbf{s}$ (and therefore $\mathbf{V}$ and $\mathbf{S}$), respectively. The above method of solution depends only on the $M^2 \times M^2$ matrices $\boldsymbol{\Omega}_\pm$ and $\mathbf{Z}_\pm$, with $\boldsymbol{\Omega}_\pm$ being diagonal. In contrast to the straightforward method of section 2, which requires the inversion of a $(2M)^2 \times (2M)^2$ matrix $\mathbf{B}$, only the three $M^2 \times M^2$ matrices, $\mathbf{Z}_+$, $\mathbf{Z}_-$ and $\mathbf{Z}_+^{-1}$ constitute the bulk of the storage requirement (while $\boldsymbol{\Omega}_\pm$ are effectively $M^2 \times 1$ vectors). The algorithm is not the most efficient one available, but it is suitable for machine computation.

4. A FLEXIBLE LIFTING ROTOR SUBJECT TO SHAPED WHITE NOISE EXCITATION

As an application of the general results of sections 2 and 3, we consider a very flexible uniform rotor blade hinged at the axis of revolution (Figure 1). If the effect of bending stiffness can be neglected,‡ the transverse motion of the blade in hover may be adequately described by the dimensionless equation (see reference [3] for a derivation of and the basic assumptions inherent in this equation)

$$u_{tt} + \gamma_0 x u_t - \tfrac{1}{2}[(1 - x^2)\,u_x]_x = \gamma_0 x^2 n(t) \qquad (0 < x < 1,\ t > 1), \tag{20}$$

where u is the dimensionless transverse displacement of the blade (normalized by the blade length). The Lock number $\gamma = 6\gamma_0$ is a measure of the aerodynamic effects. $n(t)$ is the randomly changing pitch angle assumed to be uniform over the blade length. Equation (20) is supplemented by the conditions $u = 0$ at $x = 0$ and u_x bounded at $x = 1$, and by suitable initial conditions which we will take to be $u(x,0) = u_t(x,0) = 0$, $0 \leqslant x \leqslant 1$.

† If we wish, we can now use $\mathbf{S} + \mathbf{T} = 0$ and $\mathbf{S}' = \mathbf{T}$ to get $\mathbf{S}' = -\mathbf{S}$ so that $S_{jj} = 0$ and $S_{ij} = -S_{ji}$, leaving us with $\tfrac{1}{2}(3M^2 + M)$ unknowns (the elements of the symmetric matrices $\mathbf{U}$ and $\mathbf{V}$ and the skew symmetric matrix $\mathbf{S}$). For them, we have $\boldsymbol{\Omega}\mathbf{U} - \mathbf{ZS} - \mathbf{V} = 0$ and $\boldsymbol{\Omega}\mathbf{S} - \mathbf{S}\boldsymbol{\Omega} + \mathbf{VZ}' = \mathbf{F}$ which form a set of $\tfrac{1}{2}(3M^2 + M)$ equations since both sides of the second matrix equation are symmetric. However, we will proceed in a different direction leading to a simpler algorithm.

‡ The effective bending stiffness factor is about 0·06 for many existing blades.

To obtain the mean square response of the blade, we write

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) P_{2n-1}(x), \tag{21}$$

where $P_k(x)$ is the Legendre polynomial of degree k. Equation (20) requires the coefficients $a_m(t)$ to satisfy the coupled **ODE**

$$\ddot{a}_m + \omega_m^2 a_m + \gamma_0(4m-1) \sum_{k=1}^{\infty} \sigma_{mk} \dot{a}_k(t) = \gamma_0(4m-1)\tau_m n(t) \qquad (m = 1, 2, \ldots), \tag{22}$$

where

$$\omega_m^2 = m(2m-1), \quad \sigma_{mk} = \int_0^1 x P_{2m-1}(x) P_{2k-1}(x)\,\mathrm{d}x = \sigma_{km},\ \tau_m = \int_0^1 x^2 P_{2m-1}(x)\,\mathrm{d}x = \sigma_{1m} \tag{23}$$

and the initial conditions $a_m(0) = \dot{a}_m(0) = 0$, $m = 1, 2, 3, \ldots$. The constants $\sigma_{mk} = \sigma_{km}$ are given in Table 1 for $m, k = 1, 2, 3, \ldots, 10$.

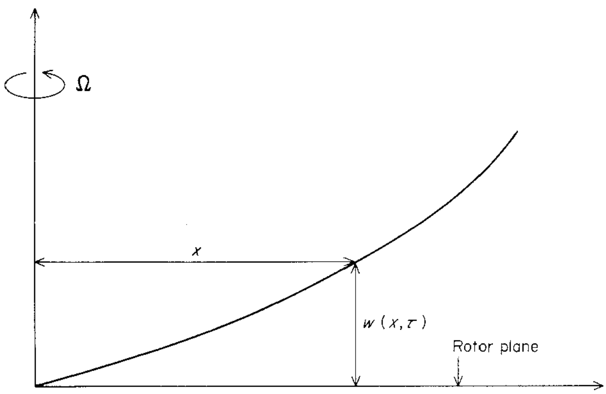

Figure 1. A flexible rotor blade.

The mean square properties of the blade are given by

$$\langle u^2(x,t)\rangle = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \langle a_m(t) a_k(t)\rangle P_{2m-1}(x) P_{2k-1}(x),$$

$$\langle u(x,t)\, u_t(x,t)\rangle = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \langle a_m(t) \dot{a}_k(t)\rangle P_{2m-1}(x) P_{2k-1}(x),$$

$$\langle u_t^2(x,t)\rangle = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \langle \dot{a}_m(t) \dot{a}_k(t)\rangle P_{2m-1}(x) P_{2k-1}(x). \tag{24}$$

Our problem now is to determine $\langle a_m(t) a_k(t)\rangle$, $\langle a_m(t)\dot{a}_k(t)\rangle$ and $\langle \dot{a}_m(t)\dot{a}_k(t)\rangle$ for all m and k from equation (22). An approximate solution of this problem can be obtained by retaining only the first M terms in the expansion (21). In that case, equation (22) becomes a system of M equations of the form (1), with $\zeta_{mk} = \gamma_0(4m-1)\sigma_{mk} (\neq \zeta_{km})$ and $f_m(t) = \gamma_0(4m-1)\sigma_{1m} n(t)$. The results of sections 2 and 3 are directly applicable for the determination of the $(M \times M)$ matrices $\mathbf{U} \equiv [\langle a_m(t) a_k(t)\rangle]$, $\mathbf{S} \equiv [\langle a_m(t)\dot{a}_k(t)\rangle]$ and $\mathbf{V} \equiv [\langle \dot{a}_m(t)\dot{a}_k(t)\rangle]$ for large t when $n(t)$ is a white noise process with zero mean and normalized spectral density. For this problem, we have $F_{mk} = \gamma_0^2(4m-1)(4k-1)\sigma_{1m}\sigma_{1k}$.

Just how large M must be for a good approximate solution depends of course on the rate of convergence of the series in equations (24). For all the cases investigated, we found that no more than five terms in the expansion (21) are needed to get convergence to at least four significant figures in all the quantities calculated. Our solution also agrees with the solution

TABLE 1

Values of the constants $\sigma_{nk} = \int_0^1 x P_{2n-1}(x) P_{2k-1}(x)\,dx = \sigma_{kn}$

n/k	1	2	3	4	5	6	7	8	9	10
1	0·250000	0·041667	−0·005208	0·001563	−0·000651	0·000326	−0·000183	0·000112	−0·000073	0·000050
2		0·093750	0·023438	−0·003646	0·001256	−0·000575	0·000309	−0·000183	0·000117	−0·000079
3			0·058594	0·016276	−0·002747	0·001007	−0·000485	0·000271	−0·000166	0·000109
4				0·042725	0·012461	−0·002193	0·000833	−0·000413	0·000236	−0·000148
5					0·033646	0·010094	−0·001822	0·000708	−0·000358	0·000209
6						0·027758	0·008482	−0·001558	0·000615	−0·000315
7							0·023627	0·007313	−0·001360	0·000543
8								0·020568	0·006428	−0·001206
9									0·018211	0·005733
10										0·016340

for the same problem obtained in reference [4] by the spatial correlation function method developed recently in references [5] and [6]. The present Fourier Legendre expansion method is equivalent to and (in terms of computing time) somewhat more efficient than the method of reference [4] for th^ problem considered herein. In the event that the random excitation in equation (20) also depends on the spatial variable x, the present expansion method no longer applies and the spatial correlation function method seems to be the most efficient approach available.

5. NUMERICAL RESULTS FOR ROTOR BLADES WITH WHITE NOISE EXCITATION

If only the first term is retained in the expansion (21), it follows immediately from equation (14) (or from equations (17)–(19)) that

$$\langle a_1(t)\,\dot{a}_1(t)\rangle = 0,$$
$$\langle a_1^2(t)\rangle = \langle \dot{a}_1^2(t)\rangle = 6\gamma_0\,\sigma_{11}^2 = \frac{3\gamma_0}{8}. \qquad (25)$$

They correspond to the steady-state mean square response of a hinged uniform, rigidly flapping blade subject to a white noise excitation [7]. Note that the displacement and velocity are statistically independent as they should be.

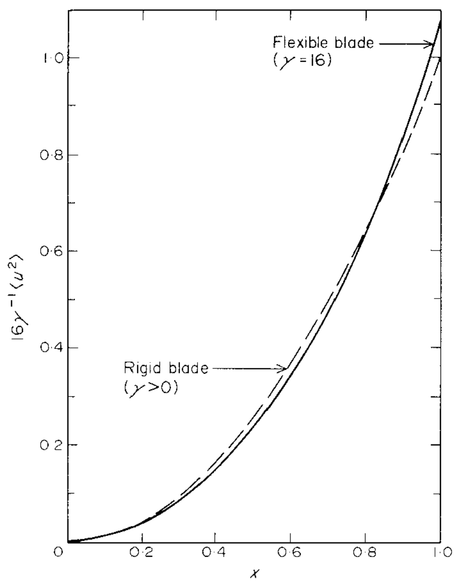

Figure 2. Distribution of the mean square displacement along the blade span.

The "multi-mode" solution (24) shows that the steady-state mean square displacement at the blade tip differs from the rigid blade value by no more than 5% for all realistic values of γ, namely $2 \leqslant \gamma \leqslant 16$. The distribution of the normalized mean square displacement over the blade span is shown in Figure 2 for the extreme value $\gamma = 16$, with the normalized rigid blade solution also plotted as a reference. The tip mean square displacement is larger than the corresponding rigid blade value and tends toward the latter for smaller values of γ.

From Figure 3, we see that the discrepancy in the mean square velocity between a flexible and a rigid blade is much larger, by as much as 28% at the tip for $\gamma = 16$. The distribution of $\langle u_t^2\rangle$ is closer to that of the rigid blade solution for smaller values of γ. However, there is still about a 20% difference between the flexible and rigid blade solution at the blade tip for $\gamma = 2$. Again, the rigid blade tip value of $\langle u_t^2 \rangle$ is a lower bound for the flexible blade tip mean square velocity.

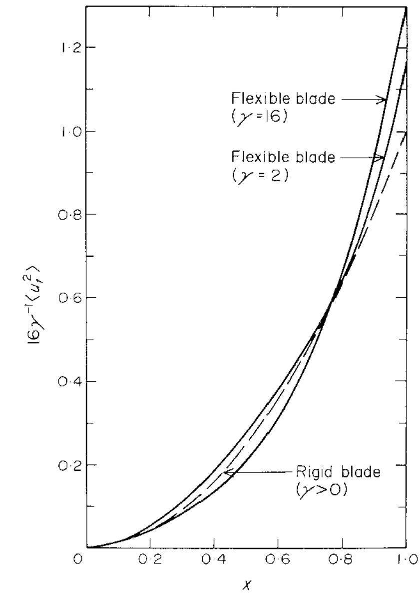

Figure 3. Distribution of the mean square velocity along the blade span.

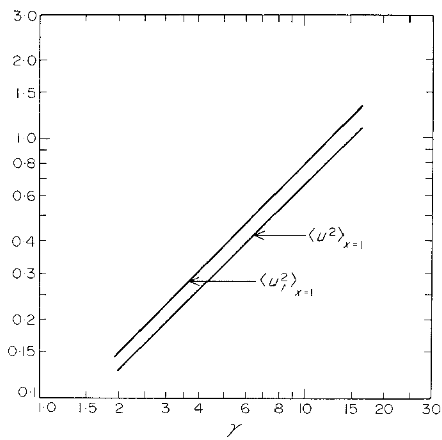

Figure 4. Effect of the Lock number, γ, on the mean square properties at blade tip.

The actual steady-state values of $\langle u^2 \rangle$ and $\langle u_t^2 \rangle$ at the tip of the flexible blade are plotted as functions of γ in Figure 4. The fact that these values are nearly proportional to γ suggests the possibility of an approximate solution to be described in Appendix I.

It should be emphasized that the good agreement between the rigid blade and flexible blade mean square displacement distribution for the present problem does not necessarily mean that a rigid blade model is always adequate as far as the mean square displacement is concerned. For example, if the factor x^2 on the right side of equation (20) is replaced by x (corresponding to the case of a random inflow), the rigid blade mean square displacement can be off by as much as 10%. Later, in section 8, we will see that even for the case of a randomly changing pitch angle the same discrepancy appears if $n(t)$ is exponentially correlated with a correlation time much longer than one blade revolution.

6. THE COVARIANCE MATRIX FOR A GENERAL STATIONARY EXCITATION

The derivation of equation (7) from equation (3) as given in reference [1] requires that $\mathbf{g}(t)$ be temporally uncorrelated. If $\mathbf{g}(t)$ is a general stationary process of zero mean, we can reduce the problem to one with a vector white noise excitation by associating $\mathbf{g}(t)$ with the response of some (fictitious) linear time-invariant dynamical system characterized by

$$\mathbf{g}^{\cdot} = \mathbf{C}\mathbf{g} + \mathbf{D}\mathbf{w}, \tag{26}$$

where $\mathbf{w}(t)$ is a zero mean vector white noise process and $\mathbf{D}$ is some constant matrix. By this, we mean that $\mathbf{g}(t)$ and the steady-state output of the supplementary dynamical system have the same (first- and) second-order statistics.† Treating $\mathbf{g}(t)$ as a vector unknown, the system of **ODE** for the augmented column vector $(\mathbf{x}', \mathbf{x}^{\cdot\prime}, \mathbf{g}')'$ is one with temporally uncorrelated excitation and the results of section 2 apply to this augmented system. The construction of an appropriate supplementary dynamical system has been discussed in references [8, 9, 10] and elsewhere. In particular, a supplementary linear time-invariant system can always be found for a stationary process with a rational power spectral density, and the use of rational functions to approximate general functions is well documented. We should also keep in mind that the statistics of the actual forcing are usually obtained only approximately from a few sample histories.

7. A FLEXIBLE LIFTING ROTOR SUBJECT TO EXPONENTIALLY CORRELATED EXCITATION

The above reduction is a known technique in control theory [8, 10] but not often used in random vibration (see references [9, 11, 12] and references therein). To illustrate the procedure, we consider again the string model of a flexible rotor blade of section 4 but now take the scalar random function $n(t)$ to be of zero mean and exponentially correlated with

$$\langle n(t_1)\, n(t_2)\rangle = \mathrm{e}^{-\alpha|t_2 - t_1|}, \tag{27}$$

where α is a positive constant. For the purpose of obtaining the second-order statistics of $u(x,t)$, we may associate $n(t)$ with the steady-state response of

$$n^{\cdot} + \alpha n = \sqrt{2\alpha}\, w(t), \tag{28}$$

where $w(t)$ is a zero mean white noise process with a normalized spectral density.

If only M terms are retained in the expansion (21), and if we set

$$\mathbf{y} = (a_1, a_2, \ldots, a_M, a_1^{\cdot}, \ldots, a_M^{\cdot}, n(t))', \tag{29}$$

then the truncated version of equation (22) may be written as

$$\mathbf{y}^{\cdot} = \hat{\mathbf{A}}\mathbf{y} + \mathbf{g}, \tag{30}$$

where $\mathbf{g} = (0, \ldots, 0, \sqrt{2\alpha}w)'$ is a $(2M+1)$-dimensional vector, $\mathbf{f} = \{\gamma_0(4m-1)\sigma_{1m}\}$ is an $(M \times 1)$ matrix, and

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & \mathbf{I} & 0 \\ -\boldsymbol{\Omega} & -\mathbf{Z} & \mathbf{f} \\ 0 & 0 & -\alpha \end{bmatrix}. \tag{31}$$

† No essential difficulty arises even if the supplementary system is of order higher than $2M$.

Since equation (30) is with a temporally uncorrelated excitation, the covariance matrix

$$\hat{\mathbf{Y}} = [\langle \mathbf{y}(t)\,\mathbf{y}'(t)\rangle = \begin{bmatrix} \mathbf{U} & \mathbf{S} & \mathbf{P} \\ \mathbf{T} & \mathbf{V} & \mathbf{Q} \\ \mathbf{P}' & \mathbf{Q}' & \mathbf{N} \end{bmatrix}, \tag{32}$$

where $\mathbf{P} = \{\langle a_m(t)\,n(t)\rangle\}$ and $\mathbf{Q} = \{\langle \dot{a_m}(t)\,n(t)\rangle\}$ are $M \times 1$ matrices and $N(t) = \langle n^2(t)\rangle$ is a scalar, satisfies the matrix **ODE**

$$\hat{\mathbf{Y}}^{\cdot} = \hat{\mathbf{A}}\hat{\mathbf{Y}} + \hat{\mathbf{Y}}\hat{\mathbf{A}}' + \mathbf{G}, \tag{33}$$

where $G_{(2M+1)(2M+1)} = 2\alpha$ and $G_{mk} = 0$ otherwise.

Again, for the purpose of obtaining a steady-state solution of the stationary response, we need only to solve the system of linear algebraic equations

$$\hat{\mathbf{A}}\hat{\mathbf{Y}} + \hat{\mathbf{Y}}\hat{\mathbf{A}}' = -\hat{\mathbf{G}}, \tag{34}$$

or, equivalently, the two subsystems

$$\mathbf{Q} = \alpha\mathbf{P}, \qquad (\mathbf{Z} + \alpha\mathbf{I})\,\mathbf{Q} + \boldsymbol{\Omega}\mathbf{P} = \mathbf{f} \tag{35}$$

and

$$\begin{gathered} \mathbf{T} + \mathbf{S} = 0, \qquad \boldsymbol{\Omega}\mathbf{U} + \mathbf{Z}\mathbf{T} - \mathbf{V} = [\mathbf{f}\mathbf{P}'], \\ \mathbf{U}\boldsymbol{\Omega} + \mathbf{S}\mathbf{Z}' - \mathbf{V} = [\mathbf{P}\mathbf{f}'], \qquad \boldsymbol{\Omega}\mathbf{S} + \mathbf{T}\boldsymbol{\Omega} + \mathbf{Z}\mathbf{V} + \mathbf{V}\mathbf{Z}' = [\mathbf{f}\mathbf{Q}' + \mathbf{Q}\mathbf{f}']. \end{gathered} \tag{36}$$

The first of these systems, equations (35), can be solved immediately for **P** and **Q**:

$$(\boldsymbol{\Omega} + \alpha\mathbf{Z} + \alpha^2\,\mathbf{I})\,\mathbf{P} = \mathbf{f}, \quad \mathbf{Q} = \alpha\mathbf{P}. \tag{37}$$

The second system, equations (36), are then four matrix equations for **U**, **S**, **T** and **V** which differ from equations (14) only in their right-hand members. Therefore, the algorithm developed in section 3 is again appropriate. For all cases considered, no more than five terms in the expansion (24) are needed for an accurate solution, based on convergence to at least four significant figures and on agreement with the solution obtained in reference [4] by the spatial correlation method. We emphasize that for the special case where the excitation is uniform across the blade length, the present approach requires less computing time than the method of reference [4] which allows for excitations to vary along the blade length.

8. NUMERICAL RESULTS FOR ROTOR BLADE WITH CORRELATED EXCITATION

For $M = 1$, we have immediately from equations (37) the steady-state solution

$$\langle a_1(t)\,n(t)\rangle = \frac{12\gamma_0\,\sigma_{11}}{4(1+\alpha^2) + 3\alpha\gamma_0} = \frac{3\gamma_0}{4(1+\alpha^2) + 3\alpha\gamma_0} \tag{38}$$

and from equations (36)

$$\langle a_1^2(t)\rangle = \frac{12\gamma_0\,\sigma_{11}(4\alpha + 3\gamma_0)}{4(1+\alpha^2) + 3\alpha\gamma_0} = \frac{3\gamma_0}{4}\,\frac{4\alpha + 3\gamma_0}{4(1+\alpha^2) + 3\alpha\gamma_0},$$

$$\langle \dot{a_1}^2(t)\rangle = \frac{48\alpha\gamma_0\,\sigma_{11}^2}{4(1+\alpha^2) + 3\alpha\gamma_0} = \frac{3\alpha\gamma_0}{4(1+\alpha^2) + 3\alpha\gamma_0},$$

$$\langle \dot{a_1}(t)\,a_1(t)\rangle = 0, \tag{39}$$

which are just the steady-state mean square properties of a rigidly flapping blade in hover.

For a fixed α, our multi-mode solution shows the dependence of the mean square response of the blade on γ to be similar to that of the white noise case. For a fixed γ, the difference between the flexible blade and the rigid blade mean square tip displacement is about 10% of the former for $\alpha = 0{\cdot}1$. This difference decreases as α increases. Upon appropriate normalization, both tip values tend to the corresponding values for a white noise $n(t)$ as $\alpha \to \infty$. Evidently, a greater portion of the bending energy is distributed among the "flexural modes" at the lower range of values of α. On the other hand, the mean square tip velocity for a flexible blade differs more and more from the corresponding rigid blade value as α increases. Again, both approach the corresponding values for a white noise $n(t)$.

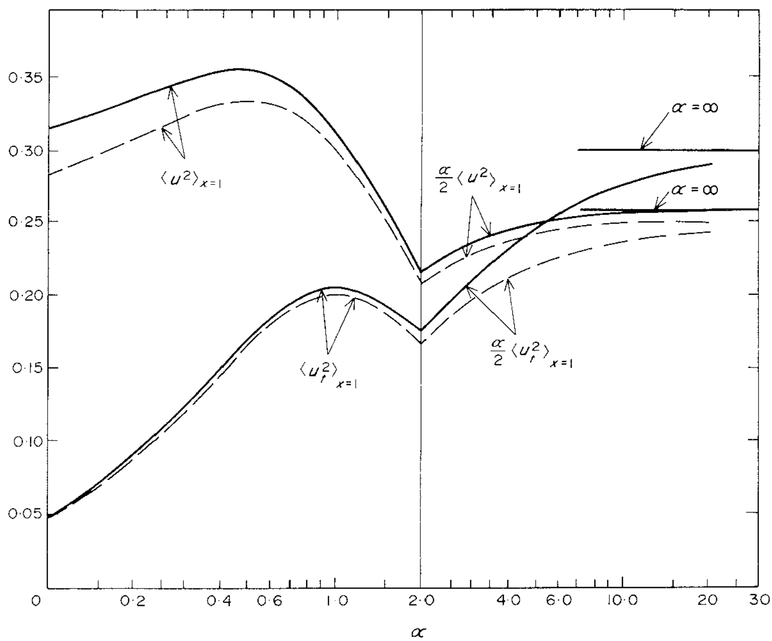

Figure 5. Variation of the mean square properties at the blade tip with load correlation time, α^{-1}. $\gamma = 4$. – – –, Rigid blade.

The actual mean square tip displacement and velocity are plotted as functions of α for $\gamma = 4$ in Figure 5. We note in particular that the (normalized) results for large α approach from below the corresponding results for a flexible blade with a white noise $n(t)$.

9. THE CORRELATION MATRIX

Once the covariance matrix of the response of the dynamical system, equation (3), has been obtained it is a straightforward matter to calculate the correlation matrix of $\mathbf{y}$ defined by

$$\mathbf{R}(t, \tau) = \langle \mathbf{y}(t)\, \mathbf{y}'(\tau) \rangle. \tag{40}$$

Upon postmultiplying equation (3) through by $\mathbf{y}'(\tau)$ for a fixed $\tau < t$ and ensemble averaging the result, we get

$$\frac{d\mathbf{R}}{dt} = \mathbf{AR}, \tag{41}$$

since, for a vector white noise process $\mathbf{g}$, we have $\langle \mathbf{g}(t)\mathbf{y}'(\tau)\rangle = 0$ for $t > \tau$. From the definition of $\mathbf{R}$,

$$\mathbf{R}(\tau, \tau) = \mathbf{Y}(\tau). \tag{42}$$

The solution of the initial value problem, equations (41) and (42), is

$$\mathbf{R}(t, \tau) = \mathbf{h}(t-\tau)\,\mathbf{Y}(\tau) \qquad (t \geqslant \tau), \tag{43}$$

where $\mathbf{h}(t)$ is the fundamental matrix of the vector equation $\mathbf{y}^{\cdot} = \mathbf{A}\mathbf{y}$. This fundamental matrix can be obtained once for all.

From equation (43), we see that for a white noise excitation, the second-order statistics of the response is completely specified by the covariance matrix $\mathbf{Y}(t)$ (the steady state of which is a constant matrix) and the properties of the particular dynamical system characterized by the impulse response matrix. If in addition the excitation is a zero mean Gaussian process so that the response is also a zero mean Gaussian process, then the covariance matrix $\mathbf{Y}$ (together with the system properties) completely specifies the response $\mathbf{y}(t)$. On the other hand, unless the excitation is a (filtered) shot noise, the response cannot be completely specified by the covariance matrix alone even if the response is Gaussian. However, in view of the remarks in section 6, the question whether $\mathbf{y}(t)$ is completely specified by its covariance matrix when $\mathbf{f}$ is a zero mean stationary Gaussian process is almost academic.

10. ALTERNATE METHODS FOR $\mathbf{AY} + \mathbf{YA}' = -\mathbf{G}$

The reasonably straightforward method of solution for the matrix equation (9) described in section 3 requires much less storage than the brute force method of section 2 (*via* equation (10)) and is perfectly adequate for the analysis of our particular rotor blade problem. However, it is still necessary to store three $M^2 \times M^2$ matrices where M is the number of degrees of freedom. We have $M \leqslant 5$ for the rotor blade problem. For $M > 13$, we can no longer do the problem entirely in the core of most machines. To get in and out of core during the solution process increases the computing time considerably. More importantly, it makes the programming much more intricate. Therefore, it is desirable to have alternate methods of solution which require less storage.

We can eliminate the storage problem completely for $M \leqslant 100$ if we use a method described in reference [13]. The method is a synthesis of the results of Krylov, Franklin and Kalman (see reference [13] for specific references).† We prefer however a conceptually simpler method recently used in reference [14] in a different connection. This method is based on the observation that the solution of equation (9) can be given in the form

$$\mathbf{Y} = \int_0^\infty \mathrm{e}^{\mathbf{A}t}\,\mathbf{G}\,\mathrm{e}^{\mathbf{A}'t}\,\mathrm{d}t, \tag{44}$$

provided that the real parts of all the eigenvalues of $\mathbf{A}$ are negative. (Work with $-\mathbf{AY} - \mathbf{YA}' = \mathbf{G}$ if they are all positive.) Note that, with equation (44), we have

$$\mathbf{AY} + \mathbf{YA}' = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\mathbf{A}t}\,\mathbf{G}\,\mathrm{e}^{\mathbf{A}'t})\,\mathrm{d}t. \tag{45}$$

If $\mathbf{A}$ can be diagonalized and $\mathbf{P}$ is the relevant similarity matrix whose columns are the eigenvectors of $\mathbf{A}$, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \Lambda \equiv [\lambda_k \delta_{jk}]$ where λ_j are the eigenvalues of $\mathbf{A}$. Moreover, we have

$$\mathbf{P}^{-1}\,\mathrm{e}^{\mathbf{A}t}\,\mathbf{P} = \mathrm{e}^{\Lambda t}, \quad \mathbf{Q}^{-1}\,\mathrm{e}^{\mathbf{A}'t}\,\mathbf{Q} = \mathrm{e}^{\Lambda t}, \tag{46}$$

† Reference [13] and a related paper by the same author were brought to our attention by a referee.

where $\mathbf{Q}' = \mathbf{P}^{-1}$. Now form $\mathbf{P}^{-1}\mathbf{Y}\mathbf{Q} \equiv \bar{\mathbf{Y}}$ to get

$$\bar{\mathbf{Y}} = \int_0^\infty e^{\Lambda t}\bar{\mathbf{G}}e^{\Lambda t}\,dt = -\left[\frac{\bar{G}_{ij}}{\lambda_i + \lambda_j}\right], \qquad (47)$$

where $\bar{\mathbf{G}} = \mathbf{P}^{-1}\mathbf{G}\mathbf{Q}$.

The problem of finding the solution of equation (9) is therefore reduced to finding the eigenvalues and eigenvectors of **A**. This can be accomplished accurately by reducing **A** to a Hessenberg matrix and then applying the QR method. The subroutine CMPXQR used in reference [14] was written to carry out this task. The routine uses double precision complex arithmetics since the eigenvalues and eigenvectors are in general complex when **A** is not symmetric. We note further that the method also applies to the more general equation $\mathbf{AY} + \mathbf{YB} = -\mathbf{G}$ where **Y** is not necessarily a square matrix.

Evidently, the above method (as well as the method of reference [13]) involves considerably more computation than the method of section 3. For systems with a small number of degrees of freedom, the simpler method of section 3 seems preferable.

REFERENCES

1. Y. K. Lin 1967 *Probabilistic Theory of Structural Dynamics*. New York: McGraw-Hill Book Company.
2. R. Bellman 1960 *Introduction to Matrix Analysis*. New York: McGraw-Hill Book Company.
3. C. Lakshmikantham and C. V. Joga Rao 1972 *Aeronautical Quarterly* **23**, 276–284. Response of helicopter rotor blades to random loads near hover.
4. F. Y. M. Wan and C. Lakshmikantham 1973 *AIAA-ASME 14th SDM Conference (Williamsburg), AIAA Paper No.* 73-406. Spatial correlation method and a time varying flexible structure.
5. F. Y. M. Wan 1972 *Studies in Applied Mathematics* **51**, 163–178. Linear partial differential equations with random forcing.
6. F. Y. M. Wan 1972 *Presented at the XIIIth International Congress on Applied Mechanics, Moscow, USSR, August* 1972. A method for linear dynamical problems in continuum mechanics with random loads.
7. G. H. Gaonkar 1971 *Journal of Sound and Vibration* **18**, 381–389. Interpolation of aerodynamic damping of lifting rotors in forward flight from measured response variance.
8. H. L. van Trees 1968 *Detection, Estimation and Modulation Theory, Part I*. New York: John Wiley and Sons. See pp. 516–526.
9. Y. K. Lin 1963 *Journal of Applied Mechanics* **30**, 555–558. Application of nonstationary shot noise in the study of system response to a class of nonstationary excitations.
10. A. Bryson and Y. C. Ho 1969 *Applied Optimal Control*. Waltham, Massachusetts: Ginn & Co.
11. F. Y. M. Wan and C. Lakshmikantham 1972 *Presented at the American Institute of Aeronautics and Astronautics 10th Aerospace Science Meeting, San Diego, January* 1972. Rotor blade response to random loads: a direct time domain approach. (1973 *American Institute of Aeronautics and Astronautics Journal* **11**, 24–28.)
12. F. Y. M. Wan 1973 To appear *Journal of Applied Mechanics* **40**. Nonstationary response of linear time-varying dynamical systems to random excitation.
13. W. Gersch 1970 *Journal of the Acoustical Society of America* **48**, 403–413. Meansquare responses in structural systems.
14. F. Y. M. Wan 1973 To appear *Studies in Applied Mathematics* **52**. An in-core finite difference method for separable boundary value problems on a rectangle.

APPENDIX I

AN APPROXIMATE SOLUTION FOR THE ROTOR BLADE PROBLEM

From Table 1, we see that the ratio $\zeta_{mk}/\zeta_{mm} = \sigma_{mk}/\sigma_{mm}$ is considerably less than unity for all $k \neq m$ and less than 0·1 if $|k - m| \geqslant 2$. We may therefore attempt an approximate solution of the rotor blade problem by omitting all the coupling terms in equation (22). It should be noted that while the M-degree-of-freedom system decouples into M single-degree-of-freedom systems, the motions of the different "modes" are not statistically independent since all subsystems experience the same random forcing $n(t)$.

With the coupling coefficients ζ_{mk}, $k \neq m$, set equal to zero, and with $\bar{\mathbf{Y}}$ denoting the corresponding approximate steady-state covariance matrix, we get from equation (9) (which determines the steady-state covariance matrix for the case of zero mean white noise excitations)

$$\bar{S}_{mk} + \bar{T}_{mk} = 0, \quad \omega_m^2 \bar{U}_{mk} + \zeta_m \bar{T}_{mk} - \bar{V}_{mk} = 0,$$
$$\omega_k^2 \bar{U}_{mk} + \zeta_k \bar{S}_{mk} - \bar{V}_{mk} = 0, \quad \omega_m^2 \bar{S}_{mk} + \omega_k^2 \bar{T}_{mk} + (\zeta_m + \zeta_k) \bar{V}_{mk} = F_{mk}, \tag{A1}$$

where $\zeta_k \equiv \zeta_{kk} = \gamma_0(4k-1)\sigma_{kk}$. The solution of the system (A1) is

$$\bar{U}_{mk} = \frac{1}{\Delta_{mk}}(\zeta_m + \zeta_k) F_{mk}, \quad \bar{S}_{mk} = \frac{1}{\Delta_{mk}}(\omega_m^2 - \omega_k^2) F_{mk},$$

$$\bar{V}_{mk} = \frac{1}{\Delta_{mk}}(\zeta_m \omega_k^2 + \zeta_k \omega_m^2) F_{mk} \tag{A2}$$

with

$$\Delta_{mk} = (\omega_k^2 - \omega_m^2)^2 + (\zeta_m + \zeta_k)(\zeta_m \omega_k^2 + \zeta_k \omega_m^2),$$
$$F_{mk} = \gamma_0^2(4m-1)(4k-1)\sigma_{1m}\sigma_{1k}, \qquad \omega_m^2 = m(2m-1). \tag{A3}$$

From equations (A2) and (A3), we see that the diagonal terms of the matrices $\bar{\mathbf{U}}$, $\bar{\mathbf{V}}$ and $\bar{\mathbf{S}}$ are

$$\bar{V}_{kk} = \frac{F_{kk}}{2\zeta_k} = \frac{\gamma_0(4k-1)\sigma_{1k}^2}{2\sigma_{kk}},$$

$$\bar{U}_{kk} = \frac{\bar{V}_{kk}}{\omega_k^2}, \qquad \bar{S}_{kk} = 0. \tag{A4}$$

The solution $\bar{S}_{kk} = 0$ is the same as the exact solution found earlier. To assess the contribution of the diagonal terms associated with the higher modes, we form

$$\frac{\bar{V}_{kk}}{\bar{V}_{11}} = \frac{(4k-1)\sigma_{1k}^2}{3\sigma_{11}\sigma_{kk}} \qquad \frac{\bar{U}_{kk}}{\bar{U}_{11}} = \frac{1}{\omega_k^2}\frac{\bar{V}_{kk}}{\bar{V}_{11}}. \tag{A5}$$

These quantities are independent of γ_0. With the help of Table 1, it is not difficult to see that

$$\frac{\bar{V}_{22}}{\bar{V}_{11}} \cong 17\%, \qquad \frac{\bar{U}_{22}}{\bar{U}_{11}} \cong 3\%$$

$$\frac{\bar{V}_{33}}{\bar{V}_{11}} < 0{\cdot}7\%, \qquad \frac{\bar{U}_{33}}{\bar{U}_{11}} < 0{\cdot}05\%. \tag{A6}$$

Ratios with $k > 3$ are much smaller still. In view of the exact solution given in Figures 1 and 2, an approximate two-mode solution with the coupling effect due to both damping and loading neglected, i.e.,

$$\langle u^2(x,t)\rangle \simeq \bar{U}_{11}[P_1(x)]^2 + \bar{U}_{22}[P_3(x)]^2,$$
$$\langle u_t^2(x,t)\rangle \simeq \bar{V}_{11}[P_1(x)]^2 + \bar{V}_{22}[P_3(x)]^2 \qquad (t \gg 1), \tag{A7}$$

accounts for almost all the discrepancies between the rigid blade and flexible blade steady-state mean square response for the particular type of random forcing considered, provided

γ is not too large (say $\gamma_0 < 1$). Such an approximate solution gives within its range of applicability mean square displacement and velocity which vary linearly with γ, a property already exhibited by the exact solution.

For larger values of γ, the contribution from $\bar{V}_{22}$ is by itself not enough to bring the mean square velocity to within 5% of the true solution. In view of the magnitude of the ratios $\bar{V}_{kk}/\bar{V}_{11}$ (which are independent of γ), retaining more diagonal terms alone cannot improve the situation. This means that the covariances $V_{jk}, j \neq k$, must contribute in a significant way to the mean square velocity for large γ. To assess the actual contribution of V_{12}, for instance, we may be tempted to consider the ratio $\bar{V}_{12}/\bar{V}_{11}$. Unfortunately, $\bar{V}_{12}$, obtained by omitting the coupling coefficients ζ_{jk} $(j \neq k)$ from the matrix $\mathbf{Z}$, is not an adequate approximation of V_{12}. That this is so can be seen from the following exact two-mode solution of equation (9): i.e., with $M = 2$,

$$V_{12} = V_{21} = \frac{\zeta_1(1 + r_2 r_\omega^2)}{2\omega_2^2 \Delta}[2F_{12} - r_{12} F_{22} - r_{21} F_{11}],$$

$$S_{21} = -S_{12} = \frac{1 - r_\omega^2}{1 + r_2 r_\omega^2}\frac{V_{12}}{\zeta_1}, \qquad U_{12} = U_{21} = \frac{1 + r_2}{1 + r_2 r_\omega^2}\frac{V_{12}}{\omega_2^2},$$

$$V_{11} = \frac{1}{2\zeta_1}(F_{11} - 2\zeta_{12} V_{12}), \qquad V_{22} = \frac{1}{2\zeta_2}(F_{22} - 2\zeta_{21} V_{12}),$$

$$U_{11} = \frac{1}{2\zeta_1 \omega_1^2}\left[F_{11} - \frac{r_\omega^2(1 + r_2)}{1 + r_2 r_\omega^2} 2\zeta_{12} V_{12}\right],$$

$$U_{22} = \frac{1}{2\zeta_2 \omega_2^2}\left[F_{22} - \frac{1 + r_2}{1 + r_2 r_\omega^2} 2\zeta_{21} V_{12}\right], \tag{A8}$$

where

$$r_{12} = \frac{\zeta_{12}}{\zeta_{22}}, \quad r_{21} = \frac{\zeta_{21}}{\zeta_{11}}, \quad r_2 = \frac{\zeta_{22}}{\zeta_{11}}, \quad r_\omega^2 = \frac{\omega_1^2}{\omega_2^2},$$

$$\Delta = (1 - r_\omega^2)^2 + \frac{\zeta_1^2}{\omega_2^2}(1 + r_2)(1 + r_2 r_\omega^2)(1 - r_{12} r_{21}). \tag{A9}$$

With $\zeta_{12} = \zeta_{21} = 0$, the above solution reduces to equation (A2) for $m, k \leqslant 1, 2$. Having equation (A8), we now see that the difference between $\bar{V}_{12}$ and V_{12} is always of the same order of magnitude as V_{12}, itself, whatever γ_0 may be. If the covariance V_{12} contributes significantly to the mean square velocity, V_{12} itself (or a corresponding solution for $M > 2$) and not $\bar{V}_{12}$ must be used. In contrast, the difference between V_{kk} and $\bar{V}_{kk}$ is $0\,[(\zeta_{12}\zeta_1)/(\omega_2^2 \Delta)]$ compared to V_{11}. For γ sufficiently small (say, $\gamma_0 \leqslant 2/3$) we have $(\zeta_{12}\zeta_1)/(\omega_2^2 \Delta) = 0(\gamma_0^2/\omega_2^2)$; $\bar{V}_{kk}$ is therefore an adequate approximation of V_{kk} for $\gamma_0 \leqslant 2/3$. The same statement is also applicable to U_{kk} and $\bar{U}_{kk}$.

Furthermore, from the magnitude of the ratio $\bar{V}_{13}/\bar{V}_{11}$ (which is of the same order of magnitude as V_{13}/V_{11} even if it may be substantially different from the latter), we see that V_{13} may contribute more than 5% of the mean square velocity for $\gamma = 16$. Therefore a two-mode solution may not be adequate for $\gamma_0 > 2/3$.

Altogether, we have that a two-mode solution without any modal coupling should be within 5% of the true mean square displacement and velocity for $\gamma_0 \leqslant 2/3$. For larger values of γ_0, we should retain the coupling due to damping as well as the load terms. More than two terms in the expansion (21) will generally be needed. On the other hand, the exact results obtained earlier indicate that no more than five terms will be needed for all realistic values of $\gamma (2 \leqslant \gamma \leqslant 16)$.

A similar analysis can be carried out for an exponentially correlated $n(t)$. The conclusions are essentially the same as the white noise case. In particular, a two-mode solution (without any modal coupling) in the case of $\gamma = 4$ brings the mean square displacement and velocity to within 5% of the "exact solution".

APPENDIX II

LIST OF SYMBOLS

x_m	displacement or the *m*th generalized coordinate
$(\ )^{\cdot}$	time derivative of ( )
ω_m	natural frequency of the *m*th degree of freedom if the system is uncoupled
ζ_{mk}	coefficient of viscous damping associated with $\dot{x}_k$ in the *m*th equation
$f_m(t)$	zero mean random excitations
$\mathbf{y}$	the phase space vector $\{x_1, \ldots, x_m, \dot{x}_1, \ldots, \dot{x}_m\}$
$\mathbf{g}$	the vector $\{0, \ldots, 0, f_1, \ldots, f_m\}$
$\mathbf{Z}$	the $M \times M$ matrix $[\zeta_{ij}]$
$\mathbf{\Omega}$	the $M \times M$ matrix $[\omega_j^2 \delta_{ij}]$
$(\ )'$	the transpose of ( )
$\mathbf{Y}$	the $2M \times 2M$ matrix $\langle yy' \rangle$
$\mathbf{G}$	the $2M \times 2M$ matrix $\langle gg' \rangle$
u	dimensionless transverse displacement of the rotor blade
x	dimensionless coordinate along the blade span
γ	the Lock number $(=6\gamma_0)$
$\mathbf{h}$	the fundamental solution matrix
n	randomly changing pitch angle
R	autocorrelation function
w	a white noise process of zero mean and normalized spectral density
α	reciprocal of the input correlation time