Studies in Applied Mathematics. Vol. LII No. 2, June 1973. Copyright © 1973 by The Massachusetts Institute of Technology

An In-Core Finite Difference Method for Separable Boundary Value Problems on a Rectangle

By Frederic Y. M. Wan

1. The forced random vibration of a rotating beam

In a model for lifting rotor blades developed in [1], the dimensionless transverse displacement of a uniform flexible blade (normalized by the blade length l) satisfies the dimensionless equation

$$L_{x\tau}[w] + \gamma_0|x + \mu \sin \tau|w_\tau + w_{\tau\tau} = \gamma_0|x + \mu \sin \tau|^2\theta(x, \tau)$$
$$(0 < x < 1, \quad \tau > 0) \tag{1.1}$$

where

$$L_{x\tau}[\ ] = \zeta^4[\ ]_{xxxx} - \tfrac{1}{2}(1 - x^2)[\ ]_{xx} + [x + \gamma_0\mu \cos \tau|x + \mu \sin \tau|][\ ]_x \tag{1.2}$$

and where the known constant $\gamma = 6\gamma_0$ and μ are the Lock number and the advance ratio, respectively. The former characterizes the aerodynamic effect and the latter is the ratio of the forward speed to the tip rotating speed, Ωl, of the blade. The effective bending stiffness factor ζ^4 is related to the bending stiffness of the uniform beam, EI, by the relation $\zeta^4 = EI/ml^4\Omega^2$ where m is the linear mass density of the beam. Also, x is the distance along the blade normalized by l and τ is a dimensionless time with τ/Ω being the real time.

For the purpose of this report, we will take the initial conditions of the blade to be

$$w(x, 0) = w_\tau(x, 0) = 0 \qquad (0 \le x \le 1) \tag{1.3}$$

and will consider a blade hinged at the axis of rotation so that

$$w(0, \tau) = w_{xx}(0, \tau) = 0 \qquad (\tau > 0). \tag{1.4}$$

The blade is of course free at the outer end so that

$$w_{xx}(1, \tau) = w_{xxx}(1, \tau) = 0 \qquad (\tau > 0). \tag{1.5}$$

Equations (1.1)–(1.5) define a well-posed initial-boundary value problem. We are interested here in the case where the pitch angle, $\theta(x, \tau)$, is a random function with known statistics. It was pointed out in [2] and [3] that conventional methods in random vibration (see [4] and [5], for example) are either inapplicable or impractical for the determination of the statistical response of the blade. Therefore, a new efficient method was developed there [2, 3] suitable for a numerical solution of the second order statistics of the blade response. The essential feature of this new method is to formulate a nonstochastic initial-boundary value problem for the *spatial correlation functions*

$$\begin{aligned} u(x, y, \tau) &= \langle w(x, \tau) w(y, \tau) \rangle, \qquad & s(x, y, \tau) &= \langle w(x, \tau) w_\tau(y, \tau) \rangle \\ t(x, y, \tau) &= \langle w_\tau(x, \tau) w(y, \tau) \rangle, \qquad & v(x, y, \tau) &= \langle w_\tau(x, \tau) w_\tau(y, \tau) \rangle \end{aligned} \tag{1.6}$$

where $\langle \cdots \rangle$ is the ensemble-averaging operation and $0 < x, y < 1$. On the one hand, these spatial correlation functions contain the mean square response of the blade as a special case. On the other hand, they serve as the initial conditions for another non-stochastic initial-boundary value problem for the determination of the (spatial-temporal) correlation functions of the response. Since the latter does not pose any difficulty as far as a numerical solution is concerned, further discussions of the general method can be confined to the solution of the spatial correlation functions.

For the class of stationary, temporally uncorrelated $\theta(x, \tau)$ with an autocorrelation function of the form

$$\langle \theta(x_1, \tau_1) \theta(x_2, \tau_2) \rangle = R_S(x_1, x_2) \delta(\tau_2 - \tau_1), \tag{1.7}$$

it is not difficult to show, either by specializing the general result of [3] or by extending the result of [2], that the spatial correlation functions u, s, t and v are the solution of the initial-boundary value problem

$$\begin{aligned} u_\tau &= s + t \\ s_\tau &= v - L_{y\tau}[u] - \gamma_0 |y + \mu \sin \tau| s \\ t_\tau &= v - L_{x\tau}[u] - \gamma_0 |x + \mu \sin \tau| t \\ v_\tau &= -\gamma_0 [|x + \mu \sin \tau| + |y + \mu \sin \tau|] v - L_{x\tau}[s] - L_{y\tau}[t] \\ &\quad + \gamma_0^2 |x + \mu \sin \tau|^2 |y + \mu \sin \tau|^2 R_S(x, y) \end{aligned} \tag{1.8}$$

with

$$\hat{w}(x, y, 0) = 0 \qquad (0 \le x, y \le 1) \tag{1.9}$$

and*

$$\begin{aligned} \hat{w}(0, y, \tau) &= \hat{w}_{xx}(0, y, \tau) = 0 \qquad & (0 < y < 1, \tau > 0) \\ \hat{w}(x, 0, \tau) &= \hat{w}_{yy}(x, 0, \tau) = 0 \qquad & (0 < x < 1, \tau > 0) \\ \hat{w}_{xx}(1, y, \tau) &= \hat{w}_{xxx}(1, y, \tau) = 0 \qquad & (0 < y < 1, \tau > 0) \\ \hat{w}_{yy}(x, 1, \tau) &= \hat{w}_{yyy}(x, 1, \tau) = 0 \qquad & (0 < x < 1, \tau > 0) \end{aligned} \tag{1.10}$$

where

$$\hat{w} = \begin{bmatrix} u & s \\ t & v \end{bmatrix}. \tag{1.11}$$

* Some of the boundary conditions in (1.10) are redundant, but are included to take advantage of the compact matrix notation.

The computing time required to determine the covariance matrix $\hat{w}(x, x, \tau)$ of the blade response by solving the above problem is several hundred folds less than that required by the more conventional methods (e.g., the auto-correlation function method [2, 3, 6]). Nevertheless, it still takes about 5 minutes on a UNIVAC 1106 to get the time history of $\hat{w}$ over four blade revolutions (which is the time it takes to reach a steady state solution for $\gamma = 4$) for a fixed set of γ_0, μ, etc. While this time requirement is relatively modest, a further reduction is still desirable since solutions over a wide range of values of the various blade and load parameters are necessary for design purposes. Such a reduction seems possible in the hovering case, i.e., $\mu = 0$, if we are only interested in the steady state solution. This steady state solution will be independent of τ and can be obtained by solving the boundary value problem

$$s + t = 0, \qquad v - L_x[u] - \gamma_0 x t = 0$$

$$v - L_y[u] - \gamma_0 y s = 0, \qquad \gamma_0(x + y)v + L_x[s] + L_y[t] = \gamma_0^2 x^2 y^2 R_S(x, y) \tag{1.12}$$

where

$$L_z[\ ] = \zeta^4[\ ]_{zzzz} - \tfrac{1}{2}(1 - z^2)[\ ]_{zz} + z[\ ]_z \tag{1.13}$$

with the boundary conditions (1.10). Unfortunately, the gain in computing time associated with a time-independent solution is offset by the fact that a finite difference solution of (1.12) and (1.10) increases the storage requirement considerably. For example, if an $(M + 1) \times (M \times 1)$ grid is used on the unit square, there would be $4M^2$ unknowns.* With these unknowns arranged as one dimensional array, the relevant coefficient matrix of the finite difference analogue of (1.12) and (1.10) (in the form $C\vec{\xi} = \vec{\eta}$) would be a $4M^2 \times 4M^2$ matrix. A solution by Gaussian elimination would require the storage of $O[(4M)^3]$ words. Even with $M = 20$, the problem could not be done completely in core. To get in and out of core during the solution process not only increases the computing time, but also makes the programming much more intricate.

When confronted by the unpalatable prospect of an in-and-out-of-core algorithm, this author noticed that the storage requirement may be reduced considerably if we could treat the unknowns as elements of a matrix instead of a vector. For example, we may use a matrix U whose element U_{ij} denotes the value of u at the mesh point (x_i, y_j), etc. Assuming that it is possible to write the difference analogue of (1.12) and (1.10) in terms of U, S, T and V, the difference analogue would be in the form of one or more matrix equations and a method of solution for the matrix equation(s) would still have to be devised.

In the next section, we will use a scalar Poisson's equation to bring out the form of, and the method of solution for the matrix equation(s). Subsequent sections extend the method to more general boundary value problems and use it to obtain extensive numerical results for the rotor blade problem. Aside from providing us with a better understanding of the statistical response of a rotating blade in hovering to random load, these results also enable us to delimit the applicability of available approximate solutions.

* For the system (1.12), we can of course eliminate some of the unknowns to reduce the storage requirement. But this may not be easily accomplished for other systems. Moreover, it turns out (as we shall see) to be an advantage to work with (1.12) as it stands even for the blade problem.

2. Poisson's equation

Consider Poisson's equation

$$w_{xx} + w_{yy} = f(x, y) \qquad (0 < x, y < 1) \tag{2.1}$$

with w vanishing on the boundary of the unit square. Take a mesh with the same equal spacing, Δ, in both the x and y direction on the unit square so that we have $M \times M$ interior mesh points. With a second order central difference scheme and with the element W_{ij} of the matrix W denoting the value of w at the mesh point (x_i, y_j), it is not difficult to see that the difference analogue of (2.1) and of the homogeneous boundary condition $w = 0$ may be written as

$$AW + WA = F \tag{2.2}$$

where A is a tri-diagonal matrix with -2 as the diagonal elements and with unit adjacent elements, and where $F_{ij} = \Delta^2 f(x_i, y_j)$.

The solution of (2.2) may be written as

$$W = -\int_0^\infty e^{At}F\, e^{At}\, dt. \tag{2.3}$$

This can be seen from the fact that

$$\begin{aligned} AW + WA &= -\int_0^\infty (A\, e^{At}F\, e^{At} + e^{At}F\, e^{At}A)\, dt \\ &= -\int_0^\infty \frac{d}{dt}(e^{At}F\, e^{At})\, dt \end{aligned} \tag{2.4}$$

and that A has only negative eigenvalues (see also (2.5)–(2.7)).

Now the exact evaluation of the right hand side of (2.3) can be made algebraic. Let P be the orthonormal modal matrix of A, i.e., the columns of P are the mutually orthogonal unit eigenvectors of A; then $P^T AP = \Lambda$ where P^T is the transpose of P and Λ is the diagonal matrix with the eigenvalues of A along the main diagonal. Since we have also $P^T e^{At}P = e^{\Lambda t}$, we get from (2.3)

$$P^T WP = -\int_0^\infty e^{\Lambda t}\bar{F}\, e^{\Lambda t}\, dt \tag{2.5}$$

where $\bar{F} = P^T FP$. But $e^{\Lambda t}$ is a diagonal matrix so that

$$e^{\Lambda t}\bar{F}\, e^{\Lambda t} = [e^{(\lambda_i + \lambda_j)t}\bar{F}_{ij}]. \tag{2.6}$$

Therefore, we have

$$W = P\left[\frac{\bar{F}_{ij}}{\lambda_i + \lambda_j}\right]P^T. \tag{2.7}$$

For our simple A, it is known that $\lambda_k = -2[1 - \cos(k\pi/(M + 1))]$ and $P = [P_{mn}] = [\sin(mn\pi/M + 1)]$.

In summary, our method of solution for the matrix equation (2.2) amounts to finding the eigenvalues and eigenvectors of A. Of course for the above simple problem, this solution scheme has no significant advantage (if any) over a more conventional method such as the Successive Over Relaxation (SOR) method or the Alternating Direction Implicit (ADI) method* since there is no storage problem here. But as we shall see, for a more complex problem such as (1.12) and (1.10), a considerable saving in storage can be achieved by our method.

3. A single scalar separable equation

For a linear separable equation of the form

$$\mathscr{L}[w] \equiv (\mathscr{L}_x + \mathscr{L}_y)[w] = f(x, y), \qquad (0 < x < a, 0 < y < b) \tag{3.1}$$

where $\mathscr{L}_z$ is a linear operator involving only z-derivatives and z-dependent coefficients, with separable boundary conditions on the boundary of the rectangle, the finite difference analogue (with an equally spaced mesh, etc.) can evidently be written as

$$AW + WB = F. \tag{3.2}$$

For $a \neq b$ and/or different mesh size in the x and y direction, W and F are in general $M \times N$ matrixes with $M \neq N$. A necessary and sufficient condition for the existence of a unique solution of (3.2) is that $\lambda_i + \omega_j \neq 0$ where λ_i and ω_j are the eigenvalues of A and B, respectively [8].

In view of the development of the last section, it is not difficult to see that the unique solution of (3.2) may be given in the form

$$W = -\int_0^\infty e^{At}F\, e^{Bt}\, dt \tag{3.3}$$

provided that the improper integral exists. (Note that we can always multiply (3.2) through by -1 if necessary.) If in addition, both A and B are non-defective, i.e., both have a full set of eigenvectors, and can be diagonalized, then we have

$$W = P\left[\frac{\bar{F}_{ij}}{\lambda_i + \omega_j}\right]Q^{-1}, \qquad \bar{F} = P^{-1}FQ \tag{3.4}$$

where P and Q are the modal matrix of A and B, respectively. The result is less simple if A and/or B cannot be diagonalized.

4. The rotating beam problem—A separable system

With an equal spacing $\Delta = 1/M$ in both the x and y direction and with second order central difference approximations for the various derivatives, the finite difference analogue of (1.12) and (1.10) can be written as

$$S + T = 0, \qquad V - UB^T - SD = 0,$$

$$V - BU - DT = 0, \qquad -DV - VD - BS - TB^T = -G \tag{4.1}$$

* While the method was being applied to several test cases, his colleague, W. G. Strang brought Ref. [7] to the author's attention. In [7], Lynch et al. work with the conventional form of the difference analogue, $C\vec{\xi} = \vec{\eta}$, for a scalar linear separable equation and separable boundary conditions, and obtain two main results by way of tensor products (*TP*) of matrices. One of these is a commutativity theorem for the *ADI* method; the other is a *TP* solution for $\vec{\xi}$. An alternate form of the *TP* solution similar (but not identical) to (3.4) is then stated without proof. In this second form, the solution seems to be equivalent to (3.4) only for self-adjoint problems. For a scalar Poisson's equation, they find that, in terms of computing time, the *TP* method is more efficient than *SOR* but not as efficient as *ADI*.

where B is the difference analogue of the operator L_z, and

$$D = \left[\frac{\gamma_0 m}{M}\delta_{mn}\right], \qquad G = \left[\frac{\gamma_0^2 m^2 n^2}{M^4} R_S\left(\frac{m}{M}, \frac{n}{M}\right)\right]. \tag{4.2}$$

The difference analogue of the boundary conditions (1.10) has also been incorporated into B. All matrices involved are $M \times M$ matrices, since we know u, s, t and v along $x = 0$ and $y = 0$.

An important observation at this point is that the system of matrix equations (4.2) can be put in the form of a single matrix equation

$$AW + WA^T = F \tag{4.3}$$

where

$$A = \begin{bmatrix} 0 & I \\ -B & -D \end{bmatrix}, \qquad W = \begin{bmatrix} U & S \\ T & V \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 \\ 0 & -G \end{bmatrix}. \tag{4.4}$$

Equation (4.3) is of the form (3.2) and the solution (3.4) is therefore applicable (with $B = A^T$), provided that the eigenvalues of A all have a negative real part, can be diagonalized, etc.

The matrices A, W and F are only $2M \times 2M$ matrices. We may limit ourselves to storing a total of four such $2M \times 2M$ matrices without incurring any penalty in the programming of our solution scheme. Since A is not symmetric, we should allow for the possibility of complex eigenvalues and eigenvectors. With a $100K$ words core, we can use a 51×51 grid and stay in core, which is more than adequate for most problems.

The solution scheme described above has been applied to the case where

$$R_S(x, y) = e^{-\varepsilon|x-y|}. \tag{4.5}$$

The computer program can be easily modified to allow for a different $R_S(x, y)$. For a fixed set of ζ, γ, and ε, the eigenvalues and eigenvectors are obtained by the routine CMPXQR developed by the author's colleague S. Orszag in conjunction with [9]. The routine is based on the QR method [10] with the eigenvectors computed by back substitution and the version for the IBM 360 machines uses double precision complex arithmetics.

With $\varepsilon = 0(1)$, for which a 21×21 grid is adequate, it takes less than 0.63 minute on an IBM 360/75 to generate the solution of (4.3). For the string case ($\zeta = 0$),* the results agree up to at least four significant figures with those obtained by two other methods. One of these is to solve the initial-boundary value problem (1.8)–(1.10) (also with a 21×21 grid) which requires more than 5 minutes on a UNIVAC 1106 [11]. The other is to solve the boundary value problem, (1.12) and (1.10), with $\zeta = 0$ by expanding the unknowns in double Fourier Legendre series [12]. For $\varepsilon = 0(10)$, a 31×31 grid is needed. This larger grid increases the computing time to almost 2 minutes on an IBM 360/75. On the other hand, the initial-boundary value problem approach with the same grid would take over an hour on the UNIVAC 1106.

* Associated with the lower order operator L_z when $\zeta = 0$ is a reduction of the number of boundary conditions from four to two. The two appropriate boundary conditions are $w(0, \tau) = 0$ and $w_x(1, \tau)$ being bounded.

The results have been further confirmed by using a different eigenvalue–eigenvector routine, ALLMAT (modified slightly by the author), which uses the Wielandt inverse power method to obtain the eigenvectors [10]. The routine is written in single precision complex arithmetics. The calculations for a 41×41 grid consume 216 seconds on a CDC 6600.* The results differ from those for a 31×31 grid (using CMPXQR or ALLMAT) by less than 5% in the mean square (m.s.) velocity and by less than 0.5% in the m.s. displacement.

On a UNIVAC 1106, ALLMAT gave very good results for the Poisson's equation but, because of the effect of accumulated roundoff errors, gave erroneous results for the rotor blade problem.

5. Numerical results

Extensive numerical results for the flexible blade problem, (1.12) and (1.10), have been obtained by the method described in section (4). With the help of these results, we can study the effect of the three parameters, ζ, γ and ε, on the blade behavior. In this paper, we will confine ourselves to delimiting the applicability of the steady state solution,

$$\langle w^2(x, x, \tau)\rangle = \langle w_\tau^2(x, x, \tau)\rangle = \frac{\gamma}{16}x^2, \tag{5.1}$$

of the problem based on the assumption that the blade is rigid and the random excitation is uniform along the blade span, i.e., $\theta(x, \tau) = \theta(\tau)$. Such a simplified model was used in all previous analyses of the forced random vibration of lifting rotor blades [13–18].

Table 1

The Variation of Tip Mean Square Response with Effective Bending Stiffness ($\gamma = 4$)

	$U(1, 1)$			$V(1, 1)$		
ζ^4	$\varepsilon = 0$	$\varepsilon = 1.0$	$\varepsilon = 10$	$\varepsilon = 0$	$\varepsilon = 1.0$	$\varepsilon = 10$
0	0.2593	0.2322	0.1255	0.3003	0.3822	0.6369
10^{-4}	0.2590	0.2298	0.1139	0.2975	0.3452	0.4747
10^{-2}	0.2547	0.2199	0.0945	0.2855	0.3107	0.3763
1.0	0.2504	0.2118	0.0835	0.2796	0.3024	0.3675
10^2	0.2503	0.2116	0.0832	0.2795	0.3023	0.3674

For fixed γ and ε, we see from Table 1 that the maximum m.s. displacement and velocity, attained at the blade tip, decreases as the effective bending stiffness factor, ζ, increases from the extreme case of vanishing bending stiffness, $\zeta = 0$ (a string or sail). These m.s. quantities reach their limiting values as ζ increases beyond unity.

* Single precision arithmetics on a CDC 6600 is nearly equivalent to double precision on either the UNIVAC 1106 or the IBM 360 machines.

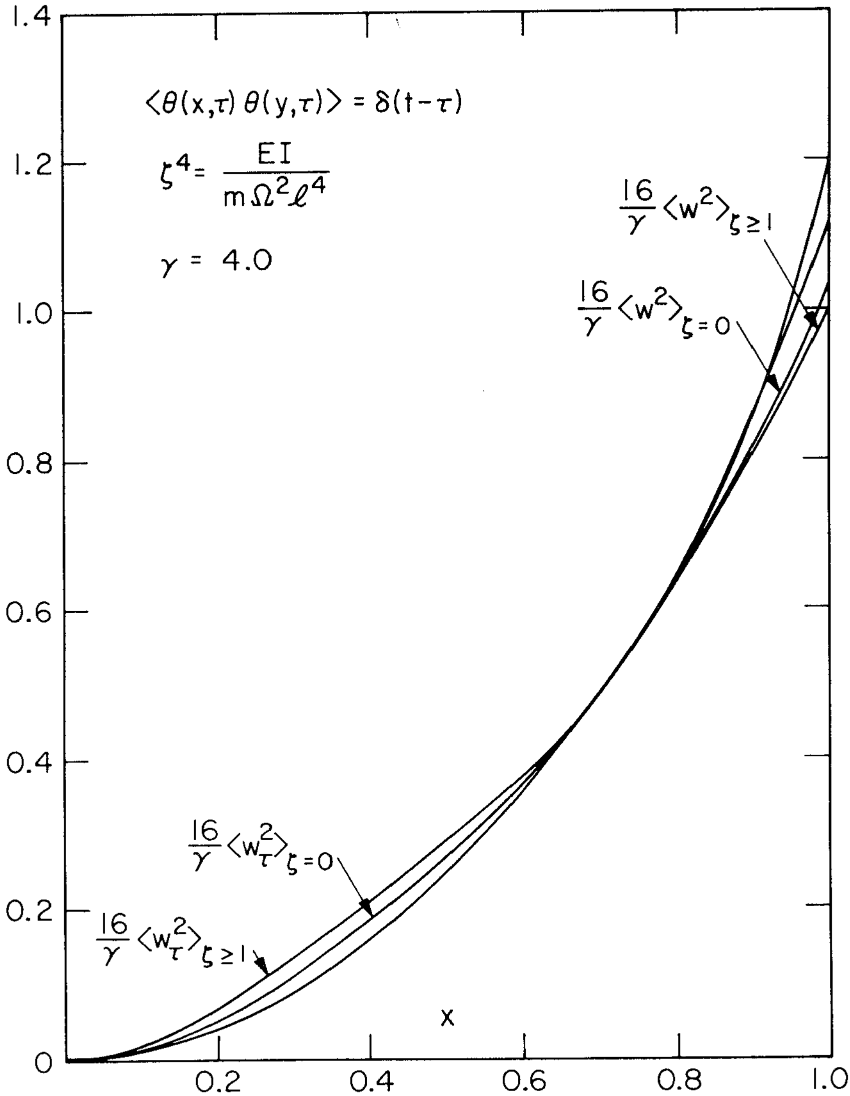

Figure 1.

However, only the m.s. displacement of the blade approaches the rigid beam solution (5.1) for $\zeta \geq 1$, while higher bending modes contribute significantly to the m.s. velocity even for $\zeta \geq 1$ (see Figures 1 and 2). The participation of the higher modes in $\langle w_\tau^2\rangle$ is still noticeable even when $\varepsilon = 0$. The difference between the m.s. response for $\zeta = 0$ and $\zeta \geq 1$ increases with ε to more than 50% of the latter when $\varepsilon = 10$.

For actual blades, ζ^4 ranges from 0.01 to 0.6 with $\zeta^4 = 0.06$ for most blades. Therefore, a solution by the simplified model of a rigid blade and spatially uniform random excitation may well be off by 25% or more in the m.s. velocity when the spanwise load correlation length is of the order of the blade length. The discrepancy increases as the load correlation length decreases.

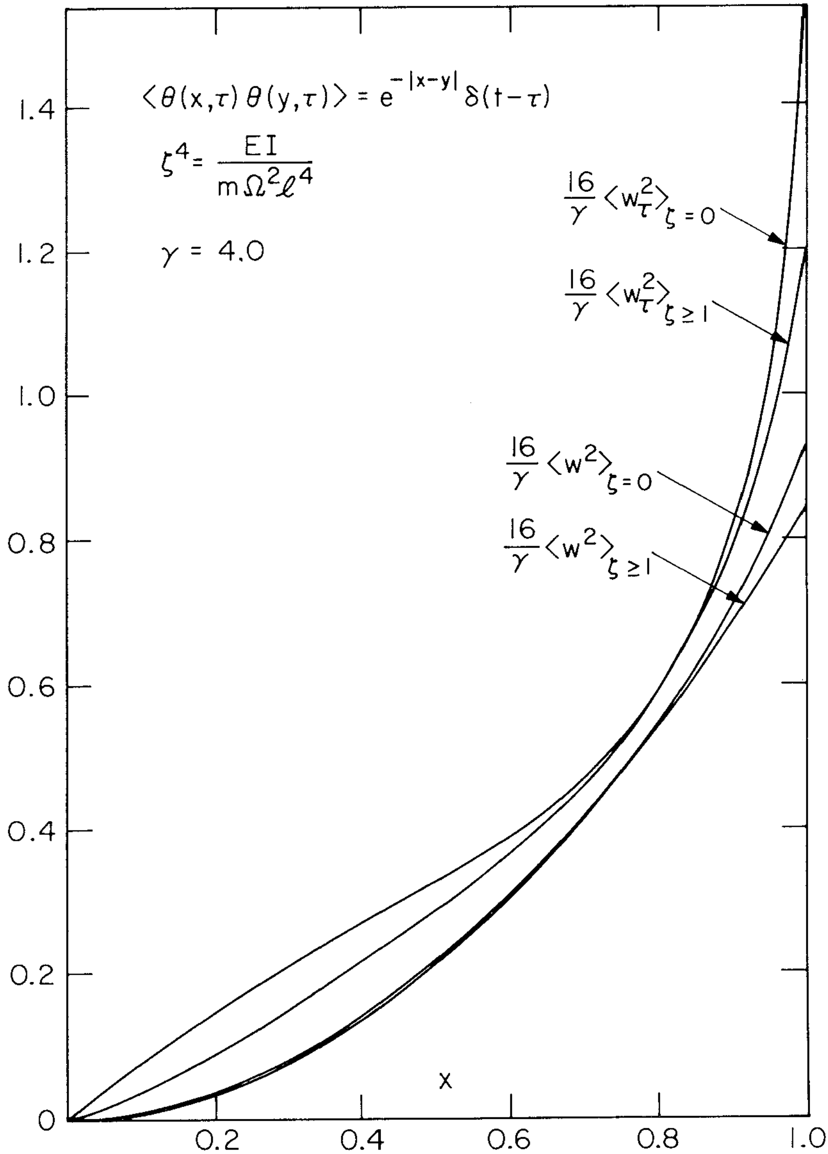

Figure 2.

The effect of decreasing spatial correlation length of the random loading is better seen from Table 2. It shows that, for fixed γ and ζ, the m.s. displacement decreases while the m.s. velocity increases with decreasing correlation length, i.e., increasing ε. For very short correlation length, the reduction in m.s. displacement from that of the simplified model may be by a factor of two or larger.

Finally, our results (not presented herein) show also that for fixed ζ and ε, the m.s. response is linear in γ, though the forcing term in (1.12) is proportional to γ^2. This feature is in agreement with the result (5.1) by the simplified model.

Table 2

The Variation of Tip Mean Square Response with Spatial Correlation Length of the Loading ($\gamma = 4$)

ε	$U(1,1)$		$V(1,1)$	
	$\zeta^4 = 0$	$\zeta^4 = 1$	$\zeta^4 = 0$	$\zeta^4 = 1$
0	0.2593	0.2504	0.3003	0.2796
0.5	0.2449	0.2297	0.3437	0.2921
1.0	0.2322	0.2118	0.3822	0.3024
2.0	0.2106	0.1826	0.4463	0.3182
5.0	0.1661	0.1272	0.5661	0.3555
10.0	0.1255	0.0835	0.6369	0.3675

6. Concluding remarks

For one dimensional linear dynamical problems with random loading governed by the vector equation

$$w_\tau = L_x[w] + f(x, \tau) \qquad (0 < x < 1, \tau > 0) \tag{6.1}$$

with the auxiliary conditions

$$w(x, 0) = B_{0x}[w(0, \tau)] = B_{1x}[w(1, \tau)] = 0 \tag{6.2}$$

where the matrix linear operators L_x, B_{0x} and B_{1x} involve only spatial derivatives, the spatial correlation method developed in [2] and [3] requires the solution of the following initial-boundary value problem for the spatial correlation matrix, $u(x, y, \tau) \equiv \langle w(x, \tau) w^T(y, \tau) \rangle$,

$$u_\tau = L_x[u] + \{L_y[u^T]\}^T + g(x, y, \tau) \qquad (0 < x, y < 1, \tau > 0) \tag{6.3}$$

$$u(x, y, 0) = 0 \tag{6.4}$$

$$B_{0x}[u(0, y, \tau)] = B_{1x}[u(1, y, \tau)] = 0 \tag{6.5}$$

$$B_{0y}[u^T(x, 0, \tau)] = B_{1y}[u^T(x, 1, \tau)] = 0. \tag{6.6}$$

If the coefficients of L_x, B_{0x} and B_{1x} do not depend on τ, the problem is separable. If, in addition, $f(x, \tau)$ is weakly stationary, the forcing term g is independent of τ [2, 3]. For a damped system, the steady state solution for u in that case is also independent of τ and can be obtained from the separable boundary value problem

$$L_x[u] + \{L_y[u^T]\}^T + g(x, y) = 0 \qquad (0 < x, y < 1) \tag{6.7}$$

with the boundary conditions (6.5) and (6.6). The numerical scheme for a finite difference solution described in sections (3) and (4) of this paper is therefore applicable to this class of problems.

Acknowledgment

The research was supported in part by the Army Research Office—Durham and the Army Materials and Mechanics Research Center (AMMRC) in Watertown, Massachusetts. The author is grateful to his colleagues, Steven Orszag and Gilbert Strang for their helpful comments during various stages of his work. He is also indebted to Steven Orszag and Ralph Metcalf of MIT, and Don Oplinger of AMMRC for helpful discussions on the use of CMPXQR and ALLMAT, respectively. The preparation of the manuscript itself was carried out while he was a Visiting Associate in the Division of Engineering and Applied Science of the California Institute of Technology, the use of whose facilities he gratefully acknowledges.

References

1. G. J. Sissingh, "Dynamics of rotors operating at high advance ratios", *J. of Helicopter Society*, **13**, 1968, 56–63.
2. F. Y. M. Wan, "Linear partial differential equations with random forcing", *Studies in Applied Mathematics* **51**, 1972, 163–178.
3. F. Y. M. Wan, "A new method for linear dynamical problems in continuum mechanics with random loads", presented at the XIIIth International Congr. of Theor. and Appl. Mech., Moscow, 1972.
4. Y. K. Lin, *Probabilistic theory of structural dynamics*, McGraw-Hill, 1967.
5. V. V. Bolotin, *Statistical Methods in Structural Mechanics*, Holden-Day, 1969.
6. V. V. Bolotin, "Applications of methods of the theory of probability and the theory of reliability in calculations of structures", Publishing Co. of Literatures in Structures, Moscow, 1971.
7. R. E. Lynch, J. R. Rice and D. H. Thomas, "Tensor product analysis of partial difference equations", *Bull. Amer. Math. Soc.*, **70**, 1964, 378–384.
8. R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, 1960.
9. S. A. Orszag, "Accurate solution of the Orr-Sommerfeld stability equation", *J. Fluid Mech.*, **50**, 1971, 689–703.
10. J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford Press, 1965.
11. F. Y. M. Wan and C. Lakshmikantham, "The spatial correlation method and a time-varying flexible structure", to be presented at the AIAA-ASME Structural Dynamics and Materials Conference, March 1973 (Williamsburg); AIAA Paper No. 73-406.
12. F. Y. M. Wan and C. Lakshmikantham, "The spatial correlation method and time-invariant flexible structures", to appear.
13. G. H. Gaonkar and K. H. Hohenemser, "Flapping response of lifting rotor blades to atmospheric turbulence", *J. of Aircraft*, **6**, 1969, 496–503.
14. G. H. Gaonkar and K. H. Hohenemser, "Stochastic properties of turbulence excited rotor blade vibration", *AIAA Journal*, **9**, 1971, 419–424.
15. G. H. Gaonkar and K. H. Hohenemser, "Comparison of two stochastic models for threshold crossing studies of rotor blade flapping vibration", AIAA paper 71-389, presented at Structural Dynamics and Materials Conf., April 1971 (Anaheim, California).
16. G. H. Gaonkar, "Interpolation of aerodynamic damping of lifting rotors in forward flight from measured response variance", *J. of Sound and Vibration*, **18**, 1971, 381–389.
17. F. Y. M. Wan and C. Lakshmikantham, "Rotor blade response to random loads: a direct time-domain approach", *AIAA Journal* **11**, 1973, 24–28.
18. F. Y. M. Wan, "Nonstationary response of linear time-varying dynamical systems to random excitation", *J. Appl. Mech.* **40**, 1973.

Massachusetts Institute of Technology

(Received December 11, 1972)