Studies in Applied Mathematics. Vol. LIII, No. 1, March 1974.

Lateral Bending and Twisting of Thin-Walled Curved Tubes*

By W. J. Seaman[1] *and F. Y. M. Wan*[2]

1. Introduction

A curved tube is known to be more flexible in bending than a straight tube of the same cross section. Moreover, a thin-walled curved tube may be much more flexible than what is predicted by a simple curved beam theory. An explanation of these (and other) phenomena for the case of symmetric pure bending of ring tubes with circular cross section was given by von Karman in 1911 [1]. Variations of his method of a Rayleigh–Ritz solution via a minimum principle formed the basis of several subsequent investigations along this line. A completely different approach to this class of problem was initiated in [2] in which the problem of pure bending of ring tube sections was analyzed in the framework of symmetric bending of thin elastic shells of revolution. Among the many notable results of [2] are a unified perspective of the earlier works on tubes with a circular cross section, and simple expressions for the rigidity factor for the whole range of a thickness parameter.

With the recent extension [3, 4] of Chernin's [5] two simultaneous equations formulation of lateral bending of shells of revolution to include the possibility of polarly nonperiodic displacement fields, it is now possible to undertake a shell theoretic study of the flexibility of thin-walled ring tube sections subject to lateral end forces and moments acting in the plane of the ring (Figure 1), in a manner similar to [2]. Such a study involves the solution of a two-point boundary value problem with periodic boundary conditions and is carried out in this paper for tubes with a circular cross section. If b is the cross section radius and a is the ring radius, then a numerical solution of the periodic boundary value problem valid for $0 < b/a < 1$ is in principle straightforward. But both the Gaussian elimination procedure and the shooting method for the relevant finite difference equations give erroneous solutions for very thin shells because of the effect of accumulated roundoff errors and other machine limitations. An accurate solution is obtained by the method of parallel shooting.

The effect of roundoff errors is also significant in the numerical evaluation of the overall stiffness coefficients. This difficulty is circumvented with the help of a uniqueness theorem proved in this paper.

* On the occasion of his sixtieth aniversary (January 5, 1973), the authors dedicate this paper to Professor Eric Reissner from whom both authors learned the theory of elasticity and shell theory and to whom the second author owes so much.

[1] Department of Mathematics, Worcester Polytechnic Institute, Worcester, Massachusetts 01609.
[2] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

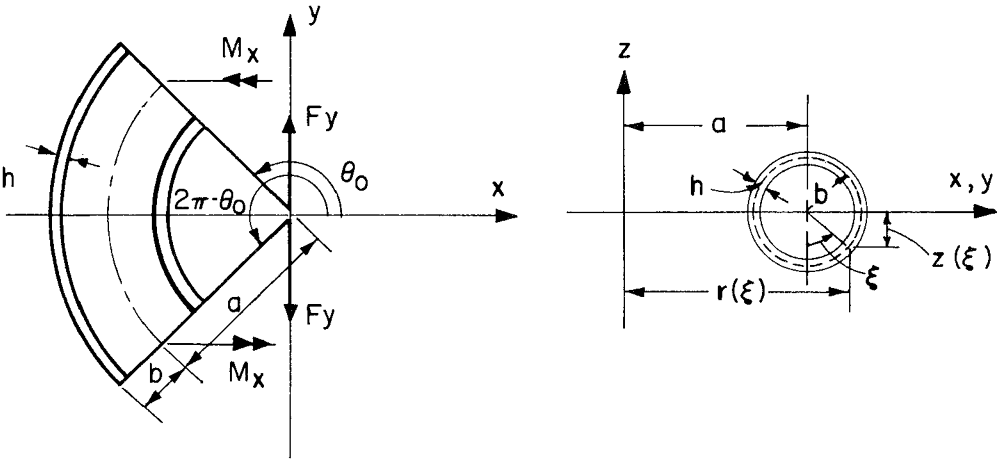

Figure 1. Curved tube with circular cross section subject to lateral end forces and twising moments.

Just as in the case of symmetric pure bending, our results for the lateral bending and twisting of curved tubes show that a thin-walled curved tube is considerably more flexible than what is predicted by a simple curved beam theory if the thickness-to-cross section radius ratio, h/b, is small compared to b/a.

To assess the reliability of our numerical scheme, two alternate methods of solution are also used for shells with $b/a \ll 1$. A truncated Fourier series solution appropriate for $(b/a)/(h/b) = O(1)$ is given in Section 4. An asymptotic solution appropriate for $(b/a)/(h/b) \gg 1$ is given in the appendix.

Finally, we will investigate how the stiffness of the tube is affected by a slit in a plane parallel to the plane of the ring. For such a slit tube, the governing differential equations are supplemented by stress free conditions along both sides of the slit. The finite difference analogue of the resulting boundary value problem can now be solved by Gaussian elimination. The results show that the overall torsional stiffness is reduced by an order of magnitude while the flexural stiffness does not change by more than 50%.

2. Formulation

In cylindrical coordinates (r, θ, z), the middle surface of a toroidal shell with a circular cross section may be described by the parametric equations

$$r = a + b\sin\xi, \qquad z = -b\cos\xi \qquad (\xi_i \le \xi \le \xi_0) \tag{1}$$

where the constants b and a are the cross section radius and the ring radius of the toroid, respectively. Correspondingly, we have $ds = b\,d\xi$ as the elemental arc-length along a meridian and

$$R_\xi = -b, \qquad R_\theta = -(a + b\sin\xi)/\sin\xi \tag{2}$$

as the two principal radii of curvature.

We consider here a section of a uniform, isotropic, homogeneous elastic shell of constant thickness, h, extending from $\theta = \theta_0$ to $\theta = 2\pi - \theta_0$. The shell is free of surface load and is subject to equal and opposite lateral end forces in the direction of the y-axis and end moments turning about axes parallel to the x-axis (see Figure 1). The elastostatics of such a shell may be described by way of a dimensionless stress function* ψ and a strain variable ϕ [3, 4]. Upon specializing the general formulation of [4] to a toroidal shell with circular cross section, the stress resultants and couples, $\{N_{\xi\xi}, N_{\theta\theta}, M_{\xi\xi}, M_{\theta\theta}\} = \{n_\xi(\xi), n_\theta(\xi), m_\xi(\xi), m_\theta(\xi)\} \cos\theta$ and $\{N_{\xi\theta}, M_{\xi\theta}\} = \{n_{\xi\theta}(\xi), m_{\xi\theta}(\xi)\} \sin\theta$, are given in terms of ϕ and ψ by the relations:

$$\sqrt{A/D}bn_\theta = \psi' + \frac{\lambda c}{1+\lambda s}\psi - \frac{\lambda}{\mu^2(1+\lambda s)}\left[\lambda s\phi' + \frac{1-\nu+2\lambda s}{1+\lambda s}\lambda c\phi + \frac{\lambda s(\lambda c\Omega - \lambda s U)}{(1+\lambda s)^2}\right]$$

$$\sqrt{A/D}bn_\xi = \frac{\lambda c}{1+\lambda s}\psi + \frac{\lambda c P - \lambda s T}{(1+\lambda s)^2} - \frac{\lambda}{\mu^2}\frac{\lambda s}{1+\lambda s}\left[\phi' + \frac{2\lambda c}{1+\lambda s}\phi + \frac{\nu\lambda c\Omega - \nu\lambda s U}{(1+\lambda s)^2}\right]$$

$$\sqrt{A/D}an_{\xi\theta} = \frac{\psi}{1+\lambda s}, \qquad (a/D)m_{\xi\theta} = \frac{1-\nu}{1+\lambda s}\left[\phi + \frac{\lambda}{\mu^2}\frac{1+\nu}{2}\frac{\psi}{1+\lambda s}\right]$$

$$(b/D)m_\xi = -\phi' - \frac{1+\nu}{1+\lambda s}\lambda c\phi + \frac{\nu\lambda s U - \nu\lambda c\Omega}{(1+\lambda s)^2} - \frac{\lambda}{\mu^2}\frac{1+\nu}{1+\lambda s}\left[\lambda s\psi' + \frac{1+2\lambda s}{1+\lambda s}\lambda c\psi + \frac{(1-\nu)\lambda s(P\lambda c - T\lambda s)}{(1+\lambda s)^2}\right]$$

$$(b/D)m_\theta = -\nu\phi' - \frac{1+\nu}{1+\lambda s}\lambda c\phi + \frac{\lambda s U - \lambda c\Omega}{(1+\lambda s)^2} - \frac{\lambda}{\mu^2}\frac{1+\nu}{1+\lambda s}\left[\lambda s\psi' + \frac{\nu+2\lambda s}{1+\lambda s}\lambda c\psi\right] \tag{3}$$

where

$$A = \frac{1}{Eh}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \qquad s = \sin\xi, \qquad c = \cos\xi$$

$$\mu^2 = \frac{b^2}{a\sqrt{DA}} = \frac{b^2\sqrt{12(1-\nu^2)}}{ah}, \tag{4}$$

and where primes indicate differentiation with respect to ξ, ν is Poisson's ratio, E is Young's modulus, P, T, Ω and U are dimensionless constants. If $\theta_0 = 0$, then $U_y = -\pi a U$ and $\Omega_x = \pi\Omega$ are the relative displacement and rotation of the disconnected ends $\theta_0 = 0$ and 2π of the ring tube [4].† The midsurface strain (ε)

* $\sqrt{D/A}\psi$ is the stress function $rn_{\xi\theta}$ used in [3, 4].

† U_y here is the negative of that used in [3, 4] so that a positive U_y means the ends are moving away from each other.

and curvature change (κ) measures are then obtained from (3) and the stress strain relations

$$\varepsilon_{\xi\xi} = A(N_{\xi\xi} - \nu N_{\theta\theta}), \quad \varepsilon_{\theta\theta} = A(N_{\theta\theta} - \nu N_{\xi\xi}), \quad \varepsilon_{\xi\theta} = \varepsilon_{\theta\xi} = \tfrac{1}{2}(1+\nu)A(N_{\xi\theta} + N_{\theta\xi}),$$
$$M_{\xi\xi} = D(\kappa_{\xi\xi} + \nu\kappa_{\theta\theta}), \quad M_{\theta\theta} = D(\kappa_{\theta\theta} + \nu\kappa_{\xi\xi}), \quad M_{\xi\theta} = M_{\theta\xi} = \tfrac{1}{2}(1-\nu)D(\kappa_{\xi\theta} + \kappa_{\theta\xi}). \tag{5}$$

with $N_{\theta\xi} = N_{\xi\theta} - M_{\xi\theta}/[b(1+\lambda s)]$.

The two functions ϕ and ψ are the solution of the two coupled differential equations [4]

$$L[\phi] - \frac{s}{1+\lambda s}[(1+\nu)\lambda\phi + \mu^2\psi]$$
$$= \mu^2 P\frac{\lambda+s}{(1+\lambda s)^2} + \mu^2 T\frac{c}{(1+\lambda s)^2} + \nu U\frac{\lambda c}{(1+\lambda s)^3}$$
$$+\lambda\Omega\left[\frac{\nu(\lambda+s)}{(1+\lambda s)^3} + \frac{\lambda(1+\nu)}{(1+\lambda s)^2}\right]$$
$$L[\psi] - \frac{s}{1+\lambda s}[(1-\nu)\lambda\psi - \mu^2\phi]$$
$$= -\mu^2\Omega\frac{\lambda+s}{(1+\lambda s)^2} - \mu^2 U\frac{c}{(1+\lambda s)^2} - \nu T\frac{\lambda c}{(1+\lambda s)^3}$$
$$-\lambda P\left[\frac{\nu(\lambda+s)}{(1+\lambda s)^3} - \frac{\lambda(1-\nu)}{(1+\lambda s)^2}\right] \tag{6}$$

where

$$L[\chi] = (1+\lambda s)^{-1}\{[(1+\lambda s)\chi']' - 2\lambda^2(1+c^2)(1+\lambda s)^{-1}\chi\}. \tag{7}$$

Except for section (7), we consider in this paper only *closed cross section tubes* so that $0 \le \xi \le 2\pi$ and $0 < \lambda < 1$. From (3) and (5), it is evident that we must have

$$\phi(0) = \phi(2\pi), \qquad \phi'(0) = \phi'(2\pi)$$
$$\psi(0) = \psi(2\pi), \qquad \psi'(0) = \psi'(2\pi) \tag{8}$$

for a closed cross section tube in order that the stress and strain measures be single-valued in ξ. In addition, the middle surface displacement components will also be single-valued in ξ if we have

$$\int_0^{2\pi} k_\xi\, d\xi = 0, \qquad \int_0^{2\pi} (e_\xi + bk_\xi)c\, d\xi = 0 \tag{9}$$

where k_ξ and e_ξ are the ξ-dependent part of κ_ξ and ε_ξ, i.e. $(\varepsilon_\xi, \kappa_\xi) = (e_\xi, k_\xi)\cos\theta$. The two conditions (9) follow from the general expressions for the displacement components obtained in [4].

The two differential equations (6), and the periodicity conditions (8) determine ϕ and ψ in terms of the four parameters P, T, Ω and U. By linearity, we can write

$$\phi = P\phi_P + T\phi_T + \Omega\phi_\Omega + U\phi_U$$
$$\psi = P\psi_P + T\psi_T + \Omega\psi_\Omega + U\psi_U \tag{10}$$

where $\phi_P \ldots$, and ψ_U do not depend on P, T, Ω and U. The two integral conditions (9) can then be written as two linear equations relating the four parameters $P, \ldots,$ and U:

$$A_{1P}P + A_{1T}T + A_{1\Omega}\Omega + A_{1U}U = 0 \qquad A_{2P}P + A_{2T}T + A_{2\Omega}\Omega + A_{2U}U = 0 \tag{11}$$

where

$$\begin{aligned} bA_{1P} &= \int_0^{2\pi} \left\{ \phi_P' + \frac{\lambda c}{1 + \lambda s}\phi_P \right. \\ &\qquad \left. + \frac{\lambda/\mu^2}{1 + \lambda s}\left[\lambda s \psi_P' + \frac{2\lambda s + 1 + v}{1 + \lambda s}\lambda c \psi_P + \frac{\lambda^2 sc}{(1 + \lambda s)^2}\right]\right\} d\xi \\ &= \int_0^{2\pi} \left\{ \phi_P + \frac{\lambda/\mu^2}{1 + \lambda s}(2\lambda s + v)\psi_P \right\} \frac{\lambda c}{1 + \lambda s} d\xi \end{aligned} \tag{12}$$

etc. The two linear relations, (11), may be used to express P and T in terms of U and Ω.

Finally, the dimensionless end displacement parameters, U and Ω, are related to the edge loads, F_y and M_x, by the two overall equilibrium conditions [3]

$$\int_0^{2\pi} n_\theta \, d\xi = -F_y/b, \qquad \int_0^{2\pi} (m_\theta - bn_\theta)c \, d\xi = -M_x/b \tag{13}$$

which, in view of (10) and (3), are two linear equations of the form

$$\begin{aligned} A_{3P}P + A_{3T}T + A_{3\Omega}\Omega + A_{3U}U &= -F_y/b \\ A_{4P}P + A_{4T}T + A_{4\Omega}\Omega + A_{4U}U &= -M_x/b. \end{aligned} \tag{14}$$

If we now use (11) to eliminate P and T from (14), we get two stiffness relations of the form

$$F_y = C_{FU}U_y + C_{F\Omega}\Omega_x, \qquad M_x = C_{MU}U_y + C_{M\Omega}\Omega_x \tag{15}$$

where $\Omega_x = \pi\Omega$ and $U_y = -\pi a U$. It will be shown later that, for closed cross section tubes, we have $C_{MU} = C_{F\Omega} = 0$ so that the stiffness relations (15) become uncoupled.

3. A single complex equation and a uniqueness theorem

One conspicuous difference between Chernin's equations and the new set obtained in [4] is that the two coupled equations for toroidal shells, (6), obtained from [4] can be combined into a single second order complex equation

$$L[\chi] - \frac{s}{1 + \lambda s}[\lambda + i\mu^2\sqrt{1 - (v\lambda/\mu^2)^2}]\chi = f(\xi) \tag{16}$$

where f is a known function of ξ and

$$\chi = \phi + \left[\frac{v\lambda}{\mu^2} - i\sqrt{1 - \left(\frac{v\lambda}{\mu^2}\right)^2}\right]\psi. \tag{17}$$

We note parenthetically that $\lambda/\mu^2 = O(h/b)$ is negligibly small compared to unity. In terms of χ, the periodic boundary conditions (8) become simply $\chi(0) = \chi(2\pi)$ and $\chi'(0) = \chi'(2\pi)$. The single complex equation formulation will now be used to prove a uniqueness theorem for the periodic boundary value problem defined by (6) and (8).

Let (ϕ_h, ψ_h) be the difference of any two solutions of (6) and (8). Then the corresponding χ_h is a solution of (16) with $f(\xi) = 0$ and with $\chi_h(0) = \chi_h(2\pi)$ and $\chi_h'(0) = \chi_h'(2\pi)$. Upon multiplying the homogeneous complex ODE by $(1 + \lambda s)\bar{\chi}_h$, where $\bar{\chi}_h$ is the complex conjugate of χ_h, and integrating the result from 0 to 2π, we get

$$[(1 + \lambda s)\chi_h'\bar{\chi}_h]_0^{2\pi} - \int_0^{2\pi} \left[(1 + \lambda s)|\chi_h'|^2 + \frac{2\lambda^2(1 + c^2)}{1 + \lambda s}|\chi_h|^2 + s(\lambda + i\mu^2\sqrt{1 - (v\lambda/\mu^2)^2})|\chi_h|^2\right] d\xi = 0. \tag{18}$$

The first term on the left vanishes because of the periodic conditions. What remains is equivalent to the two conditions

$$\int_0^{2\pi} |\chi_h|^2 s\, d\xi = 0, \qquad \int_0^{2\pi} \left[\frac{2\lambda^2(1 + c^2)}{1 + \lambda s}|\chi_h|^2 + (1 + \lambda s)|x_h'|^2\right] d\xi = 0. \tag{19}$$

Note that the first condition of (19) has been used to get the second. Since $0 < \lambda < 1$, the second condition of (19) implies $\chi_h = 0$. The solution of the periodic boundary value problem (6) and (8) is therefore unique. As we shall see later, this uniqueness theorem (which is not at all obvious in the two simultaneous ODE formulation for ϕ and ψ) is very useful in the computational aspects of our problem.

4. Fourier series solution and stiffness relations

The form of the differential equations (6) and the periodicity conditions (8) suggests that the four fundamental pairs of solutions, $\{\phi_P, \psi_P\}, \ldots$ and $\{\phi_U, \psi_U\}$, introduced in equation (10) may be written in the form of Fourier series in ξ. In fact, it has been shown in [6] that the periodic boundary value problem is solved by series for $\{\phi_P, \psi_P\}$ and $\{\phi_\Omega, \psi_\Omega\}$ which contain only *even cosine* and *odd sine* terms and series for $\{\phi_U, \psi_U\}$ and $\{\phi_T, \psi_T\}$ which contain only *odd cosine* and *even sine* terms (see equation (20) below).* It should be noted that several other families of series also solve the (homogeneous) boundary value problem formally. While we expect these other series solution to be divergent, to prove that they are so is a very tedious task. On the other hand, the uniqueness theorem proved in the last section makes such a proof unnecessary.

The particular form of the Fourier series solutions implies that we have in (11) and (14)

$$A_{1P} = A_{1\Omega} = A_{2T} = A_{2U} = A_{3P} = A_{3\Omega} = A_{4U} = A_{4T} = 0 \tag{20}$$

for all $0 < \lambda < 1$ and $\mu > 0$. For example, with

$$\begin{pmatrix}\phi_P\\ \psi_P\end{pmatrix} = \sum_{n=0}^{\infty} \left\{\begin{pmatrix}A_n\\ C_n\end{pmatrix} \cos 2n\xi + \begin{pmatrix}B_n\\ D_n\end{pmatrix} \sin(2n + 1)\xi\right\} \tag{21}$$

* Formal Fourier series solutions have also been obtained for similar boundary value problems in a Chernin type formulation with no discussion of their convergence and uniqueness [7].

and with

$$\int_0^{2\pi} \frac{\cos \xi}{(1+\lambda \sin \xi)^k} \begin{Bmatrix} \cos 2n\xi \\ \sin(2n+1)\xi \end{Bmatrix} d\xi = 0,$$
$$\int_0^{2\pi} \frac{\sin 2\xi}{(1+\lambda \sin \xi)^2} \begin{Bmatrix} \cos 2n\xi \\ \sin(2n+1)\xi \end{Bmatrix} d\xi = 0 \tag{22}$$

where $k = 1$ or 2, we have $A_{1P} = 0$ from (12). The vanishing of the other coefficients in (20) is proved in a similar way.

With (20), the relations (11) and (14) become two uncoupled sets

$$A_{1T}T + A_{1U}U = 0, \qquad A_{3T}T + A_{3U}U = -F_y/b \tag{23}$$

and

$$A_{2P}P + A_{2\Omega}\Omega = 0, \qquad A_{4P}P + A_{4\Omega}\Omega = -M_x/b. \tag{24}$$

From (23), we get

$$T = -\frac{A_{1U}}{\Delta_F}\frac{F_y}{b}, \qquad U = \frac{A_{1T}}{\Delta_F}\frac{F_y}{b} \tag{25a, b}$$

and from (24), we get

$$P = -\frac{A_{2\Omega}}{\Delta_M}\frac{M_x}{b}, \qquad \Omega = \frac{A_{2P}}{\Delta_M}\frac{M_x}{b} \tag{26a, b}$$

where

$$\Delta_F = A_{3T}A_{1U} - A_{3U}A_{1T}, \qquad \Delta_M = A_{4P}A_{2\Omega} - A_{4\Omega}A_{2P}. \tag{27}$$

Equations (25b) and (26b) show that there is no coupling between lateral bending and twisting for tubes with a closed circular cross section, i.e. $C_{MU} = C_{F\Omega} = 0$ in the stiffness relation (15). As we shall see later, the results given by (20) also play an important role for an accurate evaluation of the influence coefficients.

The Fourier series solutions for the four fundamental pairs $\{\phi_P, \psi_P\}$, etc. converge rapidly only for $\lambda \ll 1$ and $\mu^2 \leq 1$. In this range, we have to a good approximation

$$\begin{aligned}
\phi \simeq & -\Omega\left\{1 + \lambda \sin \xi + \frac{\mu^4}{72}\lambda \sin 3\xi\right\} - \frac{2\lambda P}{\mu^2}\left\{\left(1 + \frac{\mu^4}{16}\right) + \frac{\mu^4}{8}\cos 2\xi\right\} \\
& - \frac{72\mu^4 U}{576 + 41\mu^4}\left\{\sin 2\xi + \frac{\mu^4}{576}\sin 4\xi\right\} \\
& - \mu^2 T\left\{\left[1 - \frac{36\mu^4}{576 + 41\mu^4}\right]\cos \xi + \frac{4\mu^4}{576 + 41\mu^4}\cos 3\xi\right\} \\
\psi \simeq & -P\left\{1 + \lambda \sin \xi + \frac{\mu^4}{72}\lambda \sin 3\xi\right\} + \frac{2\lambda\Omega}{\mu^2}\left\{\left[1 + \frac{\mu^4}{16}\right] + \frac{\mu^4}{8}\cos 2\xi\right\} \\
& - \frac{72\mu^4 T}{576 + 41\mu^4}\left\{\sin 2\xi + \frac{\mu^4}{576}\sin 4\xi\right\} \\
& + \mu^2 U\left\{\left[1 - \frac{36\mu^4}{576 + 41\mu^4}\right]\cos \xi + \frac{4\mu^4}{576 + 41\mu^4}\cos 3\xi\right\}.
\end{aligned} \tag{28}$$

The calculations for these truncated Fourier series solutions have been omitted since they are similar to those for the pure bending problem described in [2].

With (28), the following two uncoupled approximate overall load-deformation relations follow from (25b) and (26b):

$$U_y \approx \frac{a^3 F_y}{Ehb^3} \frac{1}{1 - \dfrac{36\mu^4}{576 + 41\mu^4}}, \qquad \Omega_x \approx \frac{aM_x}{Ehb^3}[2 + \nu + \tfrac{3}{32}\mu^4]. \tag{29}$$

For $\mu^2 \leq 1$, the flexibility of the tube is nearly that predicted by simple curved beam theory (see Appendix B). But for thinner tubes for which $\mu^4 > 1$, equations (29) suggest that the rigidity of the tube may be significantly smaller than that predicted by curved beam theory (see Appendix A). It is interesting that the flexibility coefficient for U_y is the same as the rigidity factor given by (4.14b) of [2] keeping in mind that μ^4 here is the same as μ^2 in [2].

5. A finite difference solution with parallel shooting

To have a solution usable for all $\mu > 0$ and $0 < \lambda < 1$, we seek a finite difference solution of the differential equation (16) with the periodic conditions $\chi(0) = \chi(2\pi)$ and $\chi'(0) = \chi'(2\pi)$. To do this, we choose a set of $M + 1$ equally spaced mesh points ξ_j in the interval $(0, 2\pi)$ with spacing t and $\xi_1 = 0$ and $\xi_{M+1} = 2\pi$. With a second order central difference approximation for χ' and χ'', we get from (16) a complex difference equation at each mesh point ξ_j of the form

$$A_j\chi_{j-1} + B_j\chi_j + C_j\chi_{j+1} = f_j t^2, \quad j = 1, 2, \ldots, M + 1 \tag{30}$$

where $\chi_j = \chi(\xi_j)$, $f_j = f(\xi_j)$, $\xi_0 = -t$ and $\xi_{M+2} = 2\pi + t$. The coefficients A_j, B_j and C_j which involve the values of the coefficients of the differential equation at ξ_j will not be listed here.

Now, (30) is a set of $M + 1$ equations for $M + 3$ unknowns. The two additional equations needed to complete the system come from the periodic boundary conditions written in finite difference form

$$\chi_1 = \chi_{M+1}, \quad \text{and} \quad \chi_{M+2} - \chi_M = \chi_2 - \chi_0. \tag{31}$$

Equations (30) and (31) may be solved by forward elimination and backward substitution. Unfortunately, this procedure is very sensitive to machine roundoff errors for large μ.

Alternately, we may use the shooting method in conjunction with (30). That is, we seek a fundamental set of solutions $\chi^{(k)}(\xi)$, $k = 1$ and 2, for the ODE (16) with $f(\xi) = 0$ such that

$$\chi^{(k)}(0) = \delta_{k1}, \qquad \chi^{(k)\prime}(0) = \delta_{k2} \tag{32}$$

where δ_{kj} is the Kronecker delta symbol, and also a particular solution $\chi^{(3)}(\xi)$ of the inhomogeneous ODE (16) such that

$$\chi^{(3)}(0) = \chi^{(3)\prime}(0) = 0. \tag{33}$$

Evidently, the general solution of (16) is

$$\chi = c_1\chi^{(1)}(\xi) + C_2\chi^{(2)}(\xi) + \chi^{(3)}(\xi). \tag{34}$$

The solution of the periodic boundary value problem for χ is obtained by choosing the two *complex* constants C_1 and C_2 in such a way that $\chi(0) = \chi(2\pi)$ and $\chi'(0) = \chi'(2\pi)$. These two conditions require that C_1 and C_2 satisfy the two complex linear equations

$$C_1 = C_1\chi^{(1)}(2\pi) + C_2\chi^{(2)}(2\pi) + \chi^{(3)}(2\pi)$$
$$C_2 = C_1\chi^{(1)\prime}(2\pi) + C_2\chi^{(2)\prime}(2\pi) + \chi^{(3)\prime}(2\pi). \tag{35}$$

The determinant of the coefficient matrix of the system (35) does not vanish because of the uniqueness theorem in Section 3. The fundamental set $\chi^{(1)}$ and $\chi^{(2)}$ and the particular solution $\chi^{(3)}$ may be generated approximately by (30), (32) and (33).

The above so-called shooting method was also found to be impractical for large μ because the step-by-step integration with the prescribed initial conditions (32) and (33) gave very large values of $\chi^{(k)}$ at some later steps, so large that they often exceeded the magnitude allowed by the computer. Even when they were within the machine limit at $\xi = 2\pi$, the coefficient matrix of the system (35) was ill-conditioned, so that the results were still not reliable.

To circumvent these difficulties, we use here a modified version of the shooting method, known as parallel-shooting [8]. The method consists of breaking up the interval $(0, 2\pi)$ into N subintervals and using the shooting method on each segment. Aside from the periodic conditions specified by the boundary value problem, the additional constraints required for a unique solution are simply continuity conditions on χ and χ' at the junctions of the adjacent segments. For $\mu^2 \leq 100$, 16 sub-intervals, with 25 mesh points in each, were found to be adequate in all cases considered. This parallel shooting method seems to have been first applied to more standard shell problems in [9] and used later for a different class of problems of toroidal shells [10] to obtain a finite difference solution with respect to θ.

6. Influence coefficients and peak stresses

The finite difference solution for the four solution pairs $\{\phi_P, \psi_P\}, \ldots, \{\phi_U, \psi_U\}$ determined by the method of parallel shooting in the last section may be used to calculate the quantities A_{1T}, A_{1U}, etc. which appear in the overall load-deformation relations (15). Though we have not listed these quantities explicitly, it is clear from (9)–(11), (13) and (14) that they are definite integrals of ϕ_P, etc. For example, we have from (3), (5), (9), (10) and (11) that

$$bA_{1T} = -\int_0^{2\pi}\left\{(1+\lambda s)^2\lambda c\phi_T + \frac{\lambda}{\mu^2}[(\nu + 2\lambda s)(1+\lambda s)\lambda c\psi_T - \lambda^2 s^2]\right\}(1+\lambda s)^{-3}\,d\xi. \tag{36}$$

Since $\lambda < 1$, the right hand side may be evaluated by Simpson's rule. Having the eight quantities A_{1T}, etc., the stiffness coefficients (see (15), (25), (26)),

$$\frac{F_y}{U_y} = C_{FU} = -\frac{b\Delta_F}{\pi a A_{1T}}, \qquad \frac{M_x}{\Omega_x} = C_{M\Omega} = \frac{b\Delta_M}{\pi A_{2P}} \tag{37}$$

where Δ_F and Δ_M are given in terms of A_{1T}, etc. in (27), can then be calculated. (We have already shown in Section 4 that $C_{MU} = C_{F\Omega} = 0$.)

We emphasize that an accurate determination of C_{FU} and $C_{M\Omega}$ depends crucially on the results (20). The approximate finite difference solution of the boundary value problem for χ in general gives very small but nevertheless non-zero values for $A_{1P}, \ldots, A_{4T}$, if the latter are calculated directly from ϕ_P, etc. These seemingly negligible errors in A_{1P}, etc. in fact lead to erroneous results for C_{FU} and $C_{M\Omega}$. (See [6] for a more detailed discussion on this point.)

To bring out the effect of the thickness parameter μ, we write the stiffness coefficients C_{FU} and $C_{M\Omega}$ in terms of the corresponding results based on a simple curved beam theory (see Appendix B):

$$\frac{F_y}{U_y} = C_{FU} = \frac{Ehb^3}{a^3} K_{FU}, \qquad \frac{M_x}{\Omega_x} = C_{M\Omega} = \frac{Ehb^3}{a(2+\nu)} K_{M\Omega}. \tag{38}$$

The lateral flexural and torsional rigidity factor, K_{FU} and $K_{M\Omega}$, are plotted in Figure 2 as functions of μ^2 for some typical values of λ. It should be noted that we must have $\mu^2/\lambda = O(b/h) \gg 1$ for shell theory to be applicable. Thus, for a fixed λ, the results for $\mu^2 = O(\lambda)$ are not meaningful and are therefore not given on the graph. For small μ, the stiffness of the tube coincides with that predicted by (29) if $\lambda \ll 1$. For a fixed λ, the tube becomes more and more flexible as the shell gets thinner. Even for $\lambda = 0.01$, the stiffness coefficients are only about half of the values predicted by simple curved beam theory if $\mu^2 \cong 5$. For sufficiently large values of μ^2, K_{FU} and $K_{M\Omega}$ both decrease linearly with μ^2 with unit slope on a log–log scale. This means the rigidity factors themselves are $O(1/\mu^2)$ for large μ^2,

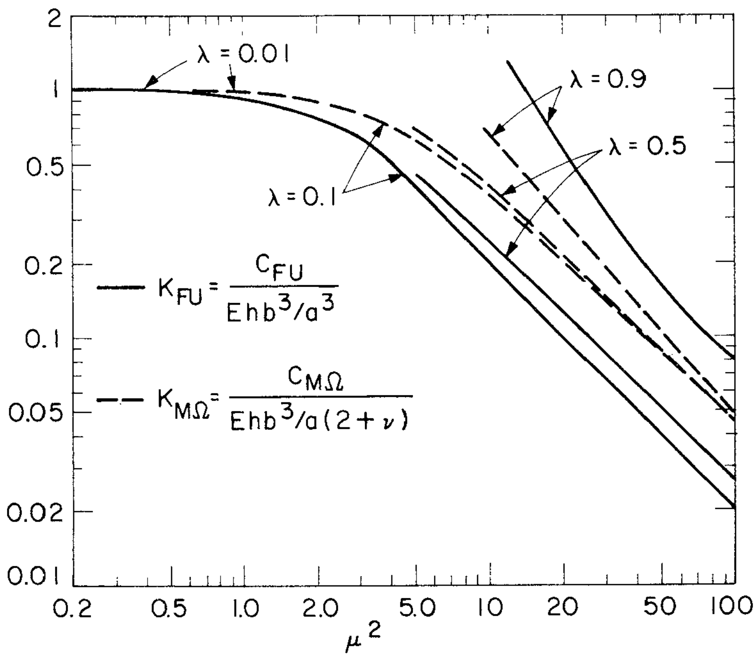

Figure 2. Variation of rigidity factors, K_{FU} and $K_{M\Omega}$, with thickness parameter μ^2 for typical values of cross section-ring-to-ring radius λ.

similar to what was found in [2] for the symmetric pure bending problem. This dependence of K_{FU} and $K_{M\Omega}$ on μ^2 is confirmed by the asymptotic solution for $\mu^2 \gg 1$ and $\lambda \ll 1$ given in Appendix A.

For both the $M_x = 0$ case and the $F_y = 0$ case, the peak direct hoop stress $|\sigma_\theta^D|_{\max} = |n_\theta/h|_{\max}$ and bending stress $|\sigma_\xi^B|_{\max} = |6m_\xi/h^2|_{\max}$ are about the same order of magnitude for moderate values of μ^2. While the maximum bending stress increases monotonically with μ^2, the maximum direct hoop stress first decreases and then increases with increasing μ^2. These peak stresses are plotted as functions of μ^2 for some typical values of λ in Figures 3 and 4. For small values of λ, they tend to the values predicted by simple beam theory as μ tends to zero.

7. A slit tube

For a tube slit along some circular arc $\xi = \xi_s$, $\theta_0 \leq \theta \leq 2\pi - \theta_0$, the single-valued conditions (8) are replaced by the Kirchhoff–Bassett free edge conditions

$$N_{\xi\xi} = N_{\xi\theta} + \frac{M_{\xi\theta}}{R_\theta} = M_{\xi\xi} = Q_\xi + \frac{1}{r}\frac{\partial M_{\xi\theta}}{\partial\theta} = 0 \tag{39}$$

on both sides of the slit, $\xi = \xi_{s^+}$ and $\xi = 2\xi + \xi_{s^-}$. For a shell free of distributed surface loads, the conditions (39) imply $P = T = 0$ [3]. Moreover, these conditions are equivalent to only two independent conditions [3], say the second and third

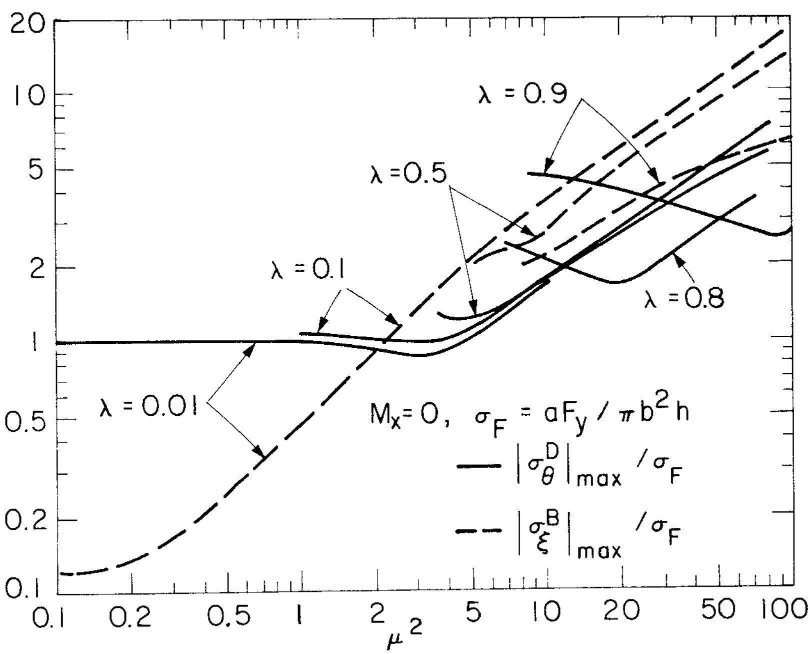

Figure 3. Variation of maximum direct and bending stress with μ^2 for the case $M_x = 0$ and for typical values of λ.

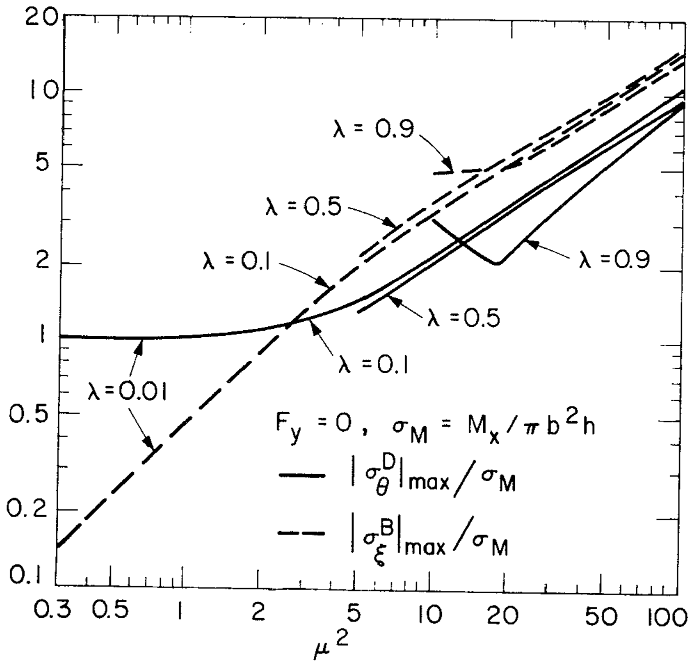

Figure 4. Variation of maximum direct and bending stress with μ^2 for the case $F_y = 0$ and for typical values of λ.

of (39). In terms of ψ and ϕ, they take the form (see (3))

$$\xi = \xi_{s^+}, 2\pi + \xi_{s^-}: \qquad \psi - \frac{\lambda}{\mu^2}\frac{1-v}{1+\lambda s}\lambda s\phi = 0$$

$$\phi' + \frac{1+v}{1+\lambda s}\lambda c\phi + \frac{v\Omega\lambda c - vU\lambda s}{(1+\lambda s)^2} + \frac{\lambda}{\mu^2}\frac{1+v}{1+\lambda s}\left(\lambda s\psi' + \frac{1+2\lambda s}{1+\lambda s}\lambda c\psi\right) = 0. \quad (40)$$

The two ODE (6) (with $P = T = 0$) and the boundary conditions (40) define a two point boundary value problem. For this problem, a finite difference solution may be obtained by Gaussian elimination applied directly to the difference analogue of the boundary value problem (6) and (40). The solution of the boundary value problem may be written in the form

$$\phi = \Omega\Phi_\Omega + U\Phi_U, \qquad \psi = \Omega\Psi_\Omega + U\Psi_U. \quad (41)$$

Finally, the end displacement parameters U and Ω are related to the applied end force and moment by two overall equilibrium conditions

$$\int_{\xi_s^+}^{2\pi+\xi_s^-} n_\theta \, d\xi = -F_y/b, \qquad \int_{\xi_s^+}^{2\pi+\xi_s^-} (m_\theta - bn_\theta)\cos\xi \, d\xi = -M_x/b. \quad (42)$$

With (41), the two conditions (42) are effectively two stiffness relations of the form

$$B_{F\Omega}\Omega_x + B_{FU}U_y = F_y, \qquad B_{M\Omega}\Omega_x + B_{MU}U_y = M_x. \quad (43)$$

The stiffness coefficients are definite integrals of the functions Φ_Ω, Φ_U, Ψ_Ω and Ψ_U. These integrals can be evaluated numerically once we have the four functions $\Phi_\Omega, \ldots, \Psi_U$.

To show how the stiffness of the tube is weakened by the slit, we give in Tables 1 and 2 the ratios B_{FU}/C_{FU} and $B_{M\Omega}/C_{M\Omega}$ for $\lambda = 0.1$ and for some typical values

Table 1

Flexural Stiffness Factor, B_{FU}/C_{FU}, for Slit Tubes ($\lambda = 0.1$)

μ^2 \ ξ_s	0	$-\frac{\pi}{8}$	$\frac{\pi}{8}$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
1.0	0.65	0.65	0.69	0.88	0.96
2.0	0.59	0.60	0.63	0.79	0.89
5.0	0.58	0.62	0.62	0.81	0.85
10.0	0.58	0.66	0.67	0.93	0.92
20.0	0.58	0.73	0.73	0.99	0.99
50.0	0.57	0.86	0.86	1.00	1.00
100.0	0.57	0.95	0.95	1.00	1.00

Table 2

Torsional Rigidity Factor, $B_{M\Omega}/C_{M\Omega}$, for Slit Tubes ($\lambda = 0.1$)

μ^2 \ ξ_s	0	$-\frac{\pi}{8}$	$\frac{\pi}{8}$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
1.0	0.90×10^{-1}	0.96×10^{-1}	0.86×10^{-1}	1.24×10^{-1}	0.81×10^{-1}
2.0	0.30×10^{-1}	0.32×10^{-1}	0.30×10^{-1}	0.46×10^{-1}	0.39×10^{-1}
5.0	0.11×10^{-1}	0.12×10^{-1}	0.12×10^{-1}	0.17×10^{-1}	0.17×10^{-1}
10.0	0.83×10^{-2}	0.95×10^{-2}	0.93×10^{-2}	0.14×10^{-1}	0.14×10^{-1}
20.0	0.69×10^{-2}	0.88×10^{-2}	0.86×10^{-2}	0.12×10^{-1}	0.12×10^{-1}
50.0	0.60×10^{-2}	0.93×10^{-2}	0.91×10^{-2}	0.11×10^{-1}	0.11×10^{-1}
100.0	0.58×10^{-2}	0.98×10^{-2}	0.98×10^{-2}	0.10×10^{-1}	0.10×10^{-1}

of the thickness parameter μ^2 and the slit location ξ_s. These results show that the lateral flexural stiffness of a tube is of the same order of magnitude with or without a slit. The torsional stiffness, on the other hand, is drastically reduced by a slit, to about 10% of a closed section tube when $\mu^2 = O(1)$ and to only about 1% when $\mu^2 \gg 1$. It appears also from these data that the tube is weakest if it is slit at one of its crowns ($\xi_s = 0$ or 2π). For a slit at a given distance away from the crown, the tube is weaker in its resistance to lateral forces and stronger in its resistance to lateral moments if the slit is at the side of negative Gaussian curvature.

There is also a coupling between the flexural and torsional action of a slit tube except when $\xi_s = \pm\pi/2$. The coupling is stronger when the slit is close to the crowns. Data for the coupling coefficient $B_{F\Omega} = B_{MU}$ as well as other information concerning the stress state of slit tubes have been obtained in [6].

Appendix A—Asymptotic solution for very thin tubes

For $\mu^2 \gg 1$, an asymptotic solution of the periodic boundary value problem (6)–(8) can be obtained by a procedure similar to that used in [2, 11, 12]. We are interested in such an asymptotic solution here mainly as a check for our numerical scheme which is applicable for all $\mu^2 > 0$ and $0 < \lambda < 1$. For this purpose, we will confine ourselves here to an asymptotic solution for shells with $\lambda^2 \ll 1$ subject only to lateral end forces, F_y. In that case, the complex ODE (16) may be simplified to read:

$$\chi'' - i\mu^2 \sin\xi\chi = (T + iU)\mu^2 \cos\xi \tag{44}$$

where now $\chi = \phi - i\psi$. For $\mu^2 \gg 1$, an asymptotic solution of (44) is [2, 11, 12]:

$$\chi \sim (T + iU)\mu^{2/3} \cos\xi Y(x) \tag{45}$$

where $x = \mu^{2/3} \sin\xi$ and $Y(x) = Y_r(x) + iY_i(x)$ is the solution of

$$\frac{d^2Y}{dx^2} - ixY = 1 \tag{46}$$

with

$$\lim_{x\to\infty} xY_r(x) = 0, \qquad \lim_{x\to\infty} xY_i(x) = 1. \tag{47}$$

A solution of (46) and (47) was found to be in terms of a suitable Lommel function [2, 11, 12], and has the properties

$$Y_r(-x) = Y_r(x), \qquad Y_i(-x) = -Y_i(x) \tag{48}$$

and

$$Y_r(x) \sim \frac{2}{x^4}\left[1 + O\left(\frac{1}{x^6}\right)\right], \qquad Y_i(x) \sim \frac{1}{x}\left[1 + O\left(\frac{1}{x^6}\right)\right], \quad (|x| \gg 1). \tag{49}$$

Correspondingly, we have

$$\phi \sim [TY_r(x) - UY_i(x)]\mu^{2/3}\cos\xi, \qquad \psi \sim -[TY_i(x) + UY_r(x)]\mu^{2/3}\cos\xi. \tag{50}$$

To determine the asymptotic behavior of the rigidity factor K_{FU} for large μ^2 (and small λ), we start with the simplified expressions

$$\begin{aligned} \{A_{1T}, A_{1U}\} &= -\frac{1}{a}\int_0^{2\pi} \{\phi_T, \phi_U\} c\, d\xi \sim \frac{\mu^{2/3}}{a}\int_0^{2\pi} \{-Y_r(x), Y_i(x)\} c^2\, d\xi \\ \{A_{3T}, A_{3U}\} &= \frac{1}{a}\sqrt{\frac{D}{A}}\int_0^{2\pi} \{\psi_T, \psi_U\} c\, d\xi \sim -\frac{\mu^{2/3}}{a}\sqrt{\frac{D}{A}}\int_0^{2\pi} \{Y_i(x), Y_r(x)\} c^2\, d\xi \end{aligned} \tag{51}$$

which appear in the relations (23) (or (25)). Evidently, we have

$$\int_0^{2\pi} Y_i(x)\cos^2\xi\, d\xi = 0 \tag{52}$$

and

$$\mu^{2/3} \int_0^{2\pi} Y_r(x) \cos^2 \xi \, d\xi = 4 \int_0^{\mu^{2/3}} Y_r(x) \sqrt{1 - \left(\frac{x}{\mu^{2/3}}\right)^2} \, dx \tag{53}$$

since Y_i and Y_r are odd and even in x, respectively. Therefore, the ratios A_{1U}/A_{1T} and A_{3T}/A_{3U} are both small (of order $\mu^{-2/3}$) compared to unity and we get from (23)

$$\frac{a^3}{Ehb^3} \frac{F_y}{U_y} = K_{FU} \sim -\frac{4}{\pi\mu^2} \int_0^{\mu^{2/3}} Y_r(x) \sqrt{1 - \left(\frac{x}{\mu^{2/3}}\right)^2} \, dx \tag{54}$$

where $U_y = -\pi a U$. It was found in [2] that

$$\int_0^{\mu^{2/3}} Y_r(x) \sqrt{1 - \left(\frac{x}{\mu^{2/3}}\right)^2} \, dx \sim -\frac{\pi}{2}. \tag{55}$$

Therefore, the rigidity factor K_{FU} for $\mu^2 \gg 1$ (and $\lambda \ll 1$) is the same as that for the pure bending of curved tube sections found in [2], namely,

$$K_{FU} \sim \frac{2}{\mu^2}. \tag{56}$$

Furthermore, it is also the same as that obtained by our numerical scheme for the case $\lambda = 0.01$ and $\lambda = 0.1$ as shown in Figure 2.

An asymptotic solution for $\mu^2 \gg 1$ and $\lambda \ll 1$ has also been obtained for the case $F_y = 0$ and $M_x \neq 0$. We can also obtain similar results for all $0 < \lambda < 1$ by the more elaborate asymptotic analysis discussed in [11, 12]. However, we will not continue our discussions along this line since a finite difference solution can be generated for all $\mu^2 > 0$ and $0 < \lambda < 1$.

Appendix B—Lateral bending and twisting of circular ring sector

Consider a section of a circular ring in the x, y-plane, centered at the origin, with ring radius a and extending circumferentially from $\theta = \theta_0$ to $\theta = 2\pi - \theta_0$. The ring is subject to equal and opposite end forces, F_y, and moments, M_x, in a manner similar to the shell case shown in Figure 1. In the absence of distributed load, the force vector, **F**, and moment vector, **M**, are constant throughout the ring sector with

$$\mathbf{F} = -F_y \mathbf{i}_y, \qquad \mathbf{M} + \mathbf{r} \times \mathbf{F} = M_x \mathbf{i}_x \tag{57}$$

where $\mathbf{r} = a\mathbf{i}_r$. In terms of components in the directions of cylindrical coordinates,

$$\mathbf{F} = F_\theta \mathbf{i}_\theta - F_r \mathbf{i}_r + F_z \mathbf{i}_z, \qquad \mathbf{M} = M_\theta \mathbf{i}_\theta + \mathbf{i}_\theta \times (-M_r \mathbf{i}_r + M_z \mathbf{i}_z) \tag{58}$$

we have

$$F_\theta = -F_y \cos\theta, \qquad F_r = F_y \sin\theta, \qquad F_z = 0$$
$$M_\theta = -M_x \sin\theta, \qquad M_r = aF_y \cos\theta, \qquad M_z = M_x \cos\theta. \tag{59}$$

The components of the beam displacement vector **u** and rotation vector $\boldsymbol{\phi}$:

$$\mathbf{u} = u_\theta \mathbf{i}_\theta - u_r \mathbf{i}_r + u_z \mathbf{i}_z, \qquad \boldsymbol{\phi} = \phi_\theta \mathbf{i}_\theta + \mathbf{i}_\theta \times (-\phi_r \mathbf{i}_r + \phi_z \mathbf{i}_z) \tag{60}$$

are related to the components of force and moment vectors by the stress–strain relations

$$\frac{aF_\theta}{C_\theta} = \dot{u}_\theta - u_r, \qquad \dot{u}_r + u_\theta - a\phi_r = 0, \qquad \frac{aM_r}{B_r} = \dot{\phi}_r \tag{61}$$

$$\dot{u}_z - a\phi_z = 0, \qquad \frac{aM_z}{B_z} = \dot{\phi}_z - \phi_\theta, \qquad \frac{aM_\theta}{B_\theta} = \dot{\phi}_\theta + \phi_z \tag{62}$$

where dots indicate differentiation with respect to θ. The relations (61) and (62) were obtained by specializing the general results of [13] to a circular ring with negligible transverse shear deformation. For a circular cross section ring tube with a small thickness-to-tube radius ratio h/b (and a small tube radius-to-ring radius ratio b/a), we have

$$B_r = B_z = EI \cong \pi b^3 hE, \qquad B_\theta = \frac{\pi b^3 hE}{1 + \nu}$$

$$C_\theta = EA \cong 2\pi bhE. \tag{63}$$

Note that the set of equations (61) is not coupled with the set (62).

The moment–rotation relation

With M_θ and M_z known from (59), the last two equations of (62) may be solved to give ϕ_z and ϕ_θ in terms of M_x. Except for a rigid body rotation, we have

$$\phi_z = \frac{aM_x}{2}\left(\frac{1}{B_z} + \frac{1}{B_\theta}\right)\theta\cos\theta$$

$$\phi_\theta = \frac{aM_x}{2}\left[\frac{1}{B_\theta}\cos\theta - \left(\frac{1}{B_z} + \frac{1}{B_\theta}\right)\theta\sin\theta\right]. \tag{64}$$

If we wish, we can also get u_z from the first equation of (62).

The $\mathbf{i}_x$-component of $\boldsymbol{\phi}$ is then

$$\begin{aligned}\phi_x &= -\phi_\theta\sin\theta + \phi_z\cos\theta \\ &= \frac{aM_x}{2}\left[\left(\frac{1}{B_z} + \frac{1}{B_\theta}\right)\theta - \frac{1}{2B_\theta}\sin 2\theta\right]\end{aligned} \tag{65}$$

with

$$\Delta\phi_x = [\phi_x]_0^{2\pi} = \pi a M_x\left(\frac{1}{B_\theta} + \frac{1}{B_z}\right) = \frac{aM_x(2 + \nu)}{Eb^3 h}. \tag{66}$$

The force–displacement relation

With F_θ and M_r given in (59), we may write (61) as

$$\phi_r = \frac{\dot{u}_r + u_\theta}{a}, \qquad \dot{u}_\theta = u_r - \frac{aF_y}{C_\theta}\cos\theta$$

$$\frac{aF_y}{B_r}\cos\theta = \frac{\dot{\phi}_r}{a} = \frac{\ddot{u}_r + u_r}{a^2} - \frac{F_y}{aC_\theta}\cos\theta. \tag{67}$$

Except for a rigid body displacement, we have

$$u_r = \frac{a^3 F_y}{2B_r^*}\theta \sin\theta, \qquad u_\theta = -\frac{a^3 F_y}{2B_r^*}(\theta\cos\theta - \sin\theta) \tag{68}$$

where

$$B_r^* = \frac{B_r}{1 + B_r/a^2 C_\theta} \cong \pi b^3 h E.$$

The $\mathbf{i}_y$-component of the displacement vector is then

$$u_y = u_\theta \cos\theta - u_r \sin\theta = -\frac{a^3 F_y}{2B_r^*}(\theta - \tfrac{1}{2}\sin 2\theta) \tag{69}$$

with

$$\Delta u_y \equiv [-u_y]_0^{2\pi} = \frac{a^3}{Eb^3 h} F_y. \tag{70}$$

References

1. T. von Kármán, "Uber die Formämderung dünnwandiger Rohre, insbesondere federnder Ausgleichsrohre," *Z. Ver. deut. Ing* **55**, 1911, 1889–1895.
2. R. A. Clark and E. Reissner, "Bending of curved tubes," *Adv. in Appl. Mech.* **II** (Academic Press) 1951, 93–122.
3. F. Y. M. Wan, "Circumferentially sinusoidal stress and strain in shells of revolution," *IJSS* **6**, 1970, 959–973.
4. F. Y. M. Wan, "Laterally loaded shells of revolution," *Ing. Arch.* **42**, 1973, 245–258.
5. V. S. Chernin, "On the system of differential equations of equilibrium of shells of revolution under bending loads," *P.M.M.* **23**, 1959, 258–265.
6. W. J. Seaman, "Laterally loaded toroidal shells," *Ph.D. Dissertation*, Dept. of Math., M.I.T., September, 1972.
7. K. F. Chernykh, "Linear theory of shells, parts I and II," *NASA TTF*-**11**, 562, February, 1968.
8. H. B. Keller, "Numerical methods for two-point boundary value problems," *Blaisdell*, 1968.
9. A. Kalnins, "Analysis of shells of revolution subjected to symmetrical and nonsymmetrical loads," *J. Appl. Mech.* **31**, 1964, 467–476.
10. A. Kalnins, "Analysis of curved thin-walled shells of revolution," *AIAA J.* **6**, 1968, 584–588.
11. R. A. Clark, "On the theory of thin elastic toroidal shells," *J. Math. and Phys.* **24**, 1950, 146–178.
12. R. A. Clark, "Asymptotic solutions of elastic shell problems," *Proc. of a Symp. on Asymptotic Solutions of Differential Equations and Their Applications* (Ed. C. H. Wilcox), John Wiley, 1964, 185–209.
13. E. Reissner, "Variational consideration for elastic beams and shells," *Proc. ASCE* **88** (EM), 1962, 23–57.

Massachusetts Institute of Technology

(Received August 3, 1973)