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POLAR DIMPLING OF COMPLETE SPHERICAL SHELLS*

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Many observed phenomena may be studied as problems of axisymmetric finite deformation of thin elastic shells of revolution. Under favorable conditions, the finite deformation experienced by the shell is, except for layer phenomena, essentially inextensional bending. A typical example is the polar dimpling of a complete spherical shell under an axisymmetric pressure distribution which varies along the meridional direction. This paper develops an asymptotic solution for this problem to illustrate a technique for more general situations. The asymptotic solution exhibits the following novel features:

(1) two adjacent regions of the shell experience two different types of inexensional bending deformation,

(2) incompatibilities between the inextensional bending solutions at the boundary of these two regions, i.e., the dimple base, are removed by layer solutions,

(3) this interior boundary varies with the shell stiffness and external load and is determined in the solution process, and

(4) the leading term approximate solution for the interior boundary location is obtained very simply from the inextensional bending solutions without any reference to the more complicated layer solutions.

1. Introduction

Many observed phenomena may be studied as problems in axisymmetric finite deformation of thin elastic shells of revolution. Under favorable conditions, the finite deformation experienced by the shell is, except for layer phenomena, essentially inextensional bending. The polar dimpling of a complete spherical shell under an axisymmetric pressure distribution which varies along the meridional direction is a typical example. This type of deformation is known to be relevant to the Collapsing Spherical Bladder Problem in the study of propellant storage devices with efficient fuel expulsion characteristics [3, 11].

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The main purpose of this paper is to describe a technique for constructing a simple approximate characterization of the dimpling and related types of finite axisymmetric deformation of thin dome-like shells of revolution under suitable axisymmetric loads. We do this by way of a specific problem involving a homogeneous, isotropic, complete spherical shell of constant thickness subject to a smooth axisymmetric normal load distribution which is inward near the poles and outward near the equator. If the shell is sufficiently thin, we expect polar dimpling to be an admissable mode of deformation[1] and an asymptotic solution of the relevant boundary value problem to be appropriate. While it is not difficult to see that the leading term of the *outer* expansion solution satisfies the equations for the finite inextensional bending theory of shells, a novel feature of our method consists of using two different types of inextensional bending solution for two adjacent regions within a hemisphere of the shell. The role of the *inner* expansion solution is to connect up the two pieces of the outer solution, smoothing out the discontinuities in the composite outer solution across the boundary of these two adjacent regions, i.e. the dimple base. The procedure is the counterpart of that used in [9] where two different types of nonlinear membrane solution in two different portions of a rotating shell were connected up by a transition layer (or inner expansion) solution across the boundary of the two regions. The actual location of the dimple base depends on the external load as well as the stretching and bending stiffness of the shell and is in principle determined as a part of the matching (of the inner and outer expansion) process. However, a good first approximation of the exact dimple radius can be obtained very simply from the inextensional bending solution itself without any reference to the more complicated inner solution (which contributes to the composite asymptotic expansion only in a small neighborhood of the dimple base where there is a significant coupling of stretching and bending shell actions).

Accurate numerical solutions of the relevant boundary value problem for the axisymmetric deformation of spherical shells confirm the adequacy of the inextensional bending solution constructed by the above technique as an approximation of the exact solution in favorable ranges of load and geometrical parameters. The numerical solutions were obtained by a general computer code developed by Ascher et al. [1] for nonlinear two point boundary value problems based on a spline-collocation method. The computer code allows the users to prescribe acceptable error tolerances for

[1]Other modes of finite axisymmetric deformation and, for sufficiently large load magnitude, axisymmetric or asymmetric buckling are also possible. We are not concerned with the actual mode of deformation for the prescribed loading in this report, but confine ourselves to a simple accurate solution for the polar dimpling mode.

the unknown functions in the differential equations and their derivatives and provides error estimates for the numerical solutions obtained.

Applications of the solution procedure described in this note to the Collapsing Spherical Bladder and other problems will be reported elsewhere [11].

2. Formulation

Consider a homogeneous, isotropic, spherical shell of constant thickness h and middle surface radius a subject only to an axisymmetric normal distributed surface load $p_n(\xi)$ where ξ is the angle between the meridional tangent at a point of the midsurface of the undeformed shell and the base plane. The finite deformation elastostatics of such a shell has been shown to be governed by the following pair of coupled nonlinear second order (integro-) differential equations for the meridional angle change ϕ of the deformed middle surface (with $\beta \equiv \xi - \phi$ being the meridional angle of the deformed middle surface) and a stress function Ψ [6, 8]:

$$\frac{A}{a}\left[\Psi'' + \cot\xi\,\Psi' - (\cot^2\xi - \nu)\Psi\right] - \frac{1}{\sin\xi}(\cos\beta - \cos\xi) =$$
$$= \frac{A}{a}\left[\nu(rV)' + (1+\nu)\cot\xi(rV) - \frac{1}{\sin\xi}(r^2p_H)' - \nu\frac{\cot\xi}{\sin\xi}(r^2p_H)\right] \tag{1}$$

$$\frac{D}{a}\left[\phi'' + \cot\xi\,\phi' + \frac{\cos\beta}{\sin^2\xi}(\sin\beta - \sin\xi) - \frac{\nu}{\sin\xi}(\cos\beta - \cos\xi)\right]$$
$$+\frac{\sin\beta}{\sin\xi}\Psi = \frac{\cos\beta}{\sin\xi}(rV) \tag{2}$$

for $0 \leqslant \xi \leqslant \pi$, where primes indicate differentiation with respect to ξ and where

$$p_H = -p_n\sin\beta, \qquad rV = -\int^{\xi} a^2 p_n \cos\beta\sin\xi\,d\xi,$$
$$r = a\sin\xi, \qquad A = \frac{1}{Eh}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}. \tag{3}$$

In the above equations, E is the constant Young's modulus of the material and ν is Poisson's ratio.

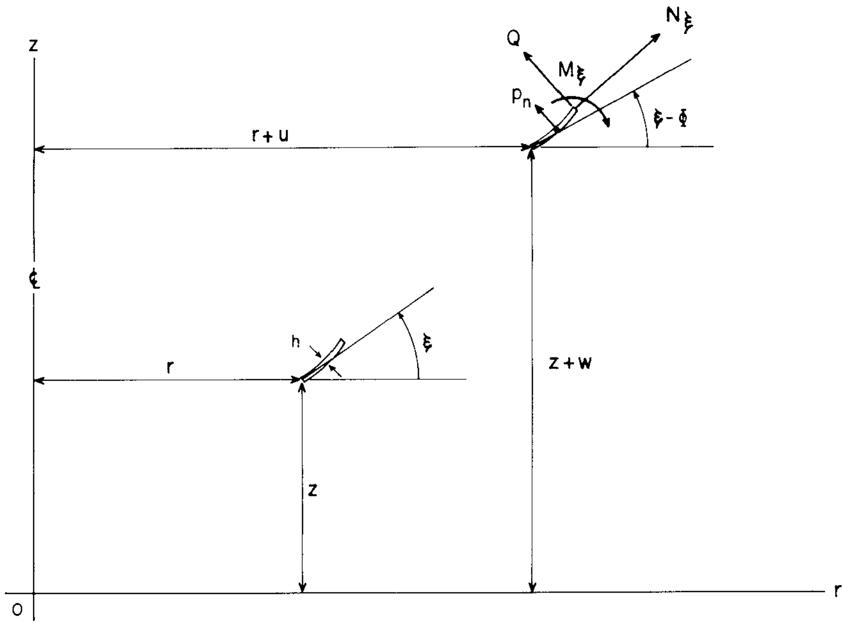

Fig. 1

In terms of ϕ (or $\beta \equiv \xi - \phi$) and Ψ, we have the following expressions for the transverse shear resultant Q, the inplane stress resultant N_ξ and N_θ, the stress couples M_ξ and M_θ, and the radial and axial midsurface displacement components u and w (see Fig. 1):

$$Q = \frac{(rV)\cos\beta - \Psi\sin\beta}{a\sin\xi}, \tag{4}$$

$$N_\xi = \frac{(rV)\sin\beta + \Psi\cos\beta}{a\sin\xi}, \qquad N_\theta = \frac{1}{a}\Psi' + a\sin\xi\, p_H, \tag{5}$$

$$M_\xi = \frac{D}{a}\left[\phi' - \frac{\nu}{\sin\xi}(\sin\beta - \sin\xi)\right],$$

$$M_\theta = \frac{D}{a}\left[\nu\phi' - \frac{1}{\sin\xi}(\sin\beta - \sin\xi)\right], \tag{6}$$

$$u = A\left[\sin\xi\,\Psi' - \nu\cos\beta\,\Psi + r^2 p_H - \nu\sin\beta(rV)\right], \tag{7}$$

$$w = \int^\xi \left\{ a(\sin\beta - \sin\xi) + A\frac{\sin\beta}{\sin\xi} \times\left[\cos\beta\,\Psi + \sin\beta(rV) - \nu\sin\xi\,\Psi' - \nu r^2 p_H\right]\right\} d\xi. \tag{8}$$

The (integro-) differential equations (1, 2) supplemented by four suitable auxiliary conditions to be specified later determine ϕ and Ψ. The stress and deformation measures of the shell as given by (4–8) will be known once we have ϕ and Ψ. For the purpose of an asymptotic analysis, we introduce appropriate dimensionless dependent and independent variables and write the governing equations in dimensionless form. Let p_i be a representative magnitude of the inward portion of $p_n(\xi)$. We write

$$p_n = p_i\rho, \qquad rV = p_i a^2 P, \qquad p_H = p_i\rho_H,$$
$$\Psi = p_i a^2\psi, \qquad \mu = p_i a A = \frac{p_i a}{Eh}, \qquad \varepsilon^4 = \frac{DA}{a^2} = \frac{h^2}{12(1-\nu^2)a^2}. \tag{9}$$

In terms of the above variables and parameters, we have the following two dimensionless differential equations for ϕ and ψ,

$$\mu[\psi'' + \cot\xi\,\psi' + (\nu - \cot^2\xi)\psi] - \frac{1}{\sin\xi}(\cos\beta - \cos\xi) = \\ = \mu\left[\nu P' + (1+\nu)\cot\xi\,P - \frac{1}{\sin\xi}(\sin^2\xi\,\rho_H)' - \nu\cos\xi\,\rho_H\right], \tag{10}$$

$$\frac{\varepsilon^4}{\mu}\left[\phi'' + \cot\xi\,\phi' + \frac{\cos\beta}{\sin^2\xi}(\sin\beta - \sin\xi) - \frac{\nu}{\sin\xi}(\cos\beta - \cos\xi)\right] \\ + \frac{\sin\beta}{\sin\xi}\psi = \frac{\cos\beta}{\sin\xi}P \tag{11}$$

and the auxiliary equations

$$\frac{Q}{p_i a} = \frac{P\cos\beta - \Psi\sin\beta}{\sin\xi},$$

$$\frac{N_\xi}{p_i a} = \frac{P\sin\beta + \Psi\cos\beta}{\sin\xi}, \qquad \frac{N_\theta}{p_i a} = \psi' + \sin\xi\,\rho_H,$$

$$\frac{u}{a} = \mu[\sin\xi\,\psi' - \nu\cos\beta\,\psi + \sin^2\xi\,\rho_H - \nu\sin\beta\,P], \tag{12}$$

$$\frac{w}{a} = \int^{\xi}\left\{\sin\beta - \sin\xi + \mu\frac{\sin\beta}{\sin\xi} \right. \\ \left. \times\left[\cos\beta\,\psi + \sin\beta\,P - \nu\sin\xi\,\psi' - \nu\sin^2\xi\,\rho_H\right]\right\} d\xi.$$

The expressions for M_ξ and M_θ are essentially unaffected by the introduction of (9).

For thin shells, $\varepsilon^4 = O(h^2/a^2)$ is small compared to unity. If p_i is not large (say by a factor h/a) compared to the classical buckling pressure $p_c = Eh^2/a^2$ of the linear theory, $\mu = O(p_i h/p_c a)$ is also small compared to unity. If at the same time p_i is not too small so that $\varepsilon^4/\mu = O(p_c h/p_i a)$ remains small compared to unity , e.g. $\varepsilon^3 \ll \mu \ll 1$, then the structure of the governing differential equations suggests the possibility that, except for layer phenomena, an inextensional bending solution (corresponding to the limiting case $A = 0$) dominates throughout the shell. The proper form of the inextensional bending solution depends on the external load function $\rho(\xi)$. In this paper, we consider a particularly simple normal surface load distribution which facilitates a polar dimpling mode of deformation and thereby allows us to bring out the essence of an elementary method for a simple approximate description of the dimple type deformation in terms of an appropriate inextensional bending solution.

3. Polar dimpling

Consider a normal surface load distribution of the form

$$p_n(\xi) = p_i - p_0 \sin\xi = p_i(1 - \delta \sin\xi) \tag{13}$$

where $p_0 > p_i > 0$ and $\delta = p_0/p_i > 1$. Since p_n is positive inward, this particular normal pressure distribution is directed toward the center of the sphere at the two poles $\xi = 0$ and $\xi = \pi$, gradually weakens as we move toward the equator and eventually changes into an outward pressure which has its peak value at the equator, $\xi = \pi/2$. Corresponding to (13), we have from (3) and (9)

$$P \equiv -\int_0^\xi (1 - \delta \sin\xi) \cos\beta \ \sin\xi \, d\xi,$$

$$\rho_H = -\sin\beta(1 - \delta \sin\xi) \tag{14}$$

With $p_n(\xi)$ symmetric about the equator, we only have to consider the hemisphere $0 \leqslant \xi \leqslant \pi/2$ with $Q = \phi = 0$ at $\xi = \pi/2$ assuming the dimple base to be above the equator. In view of (12), these conditions are equivalent to

$$\xi = \frac{\pi}{2}: \quad \phi = \psi = 0 \tag{15}$$

keeping in mind $\beta = \xi - \phi$. Eq. (15) provides two boundary conditions for the fourth order system (10, 11). Two more auxiliary conditions come from the symmetry requirements, $\phi = 0$ and $u = 0$, at the pole $\xi = 0$ which may be written as

$$\xi = 0: \quad \phi = \psi = 0 \tag{16}$$

since the stress measures N_ξ and N_θ are expected to be bounded.

For $\mu \ll 1$, we expect, to a good first approximation, the solution of the two point boundary value problem defined by (10), (11), (15) and (16) (see also (14) and (9)) to be effectively the same as that for the limiting case $\mu = 0$ except possibly for layer phenomena. Setting $\mu = 0$, the differential equation (10) reduces to a transcendental equation

$$\cos\ \beta_0 = \cos\ \xi \tag{17}$$

where a subscript 0 has been used to indicate the fact that β_0 is only an approximate solution for the problem. Eq. (17) is satisfied by $\beta_0^{(1)} = \xi$ or $\beta_0^{(2)} = -\xi$. (We have ruled out other solutions as they are physically unrealizable.) Rather than choosing either one of the two acceptable solutions to hold for the entire hemisphere, we follow the procedure used for a related problem in shallow shells [10] and consider the possibility of using both solutions, each for a different portion of the shell. More specifically, we consider a solution of the form

$$\beta_0(\xi) = \begin{cases} \beta_0^{(2)} = -\xi, & 0 \leqslant \xi < \xi_t, \\ \beta_0^{(1)} = \xi, & \xi_t < \xi \leqslant \pi/2 \end{cases} \tag{18}$$

for some transition point ξ_t in $(0, \pi/2)$ with the corresponding ψ_0 obtained from (11):

$$\psi_0(\xi) = \begin{cases} \psi_0^{(2)} \equiv -\cot\xi P_0 = \dfrac{1}{2}\cos\xi\sin\xi\left(1 - \dfrac{2\delta}{3}\sin\xi\right), & 0 \leqslant \xi < \xi_t \\ \psi_0^{(1)} \equiv \cot\xi P_0 = -\dfrac{1}{2}\cos\xi\sin\xi\left(1 - \dfrac{2\delta}{3}\sin\xi\right), & \xi_t < \xi \leqslant \dfrac{\pi}{2} \end{cases} \tag{19}$$

In other words, a dimple develops in a region centered at the apex while the rest of the hemisphere experiences no change in the meridional slope.

For $A = 0$ (so that the shell is inextensible) and therefore $\mu = 0$, the solution (18) and (19) satisfies the two differential equations (10) and (11) as well as the four auxiliary conditions (15) and (16). But both β_0 (or ϕ_0)

and ψ_0 are discontinuous at the yet unspecified transition point ξ_t resulting in unbounded hoop stress resultant and stress couples. We can eliminate the discontinuity in ψ_0 by choosing the unknown parameter ξ_t so that $\psi_0^{(2)}(\xi_t) = \psi_0^{(1)}(\xi_t)$ or

$$\cos \xi_t \sin \xi_t \left(1 - \frac{2\delta}{3} \sin \xi_t\right) = 0 \tag{20}$$

where ξ_t is physically restricted to the interval $(0, \pi/2)$. To satisfy (20) with $\xi_t = 0$ or $\pi/2$ would mean one or the other of the two inextensional bending solutions prevails throughout the shell. Therefore, we make ψ_0 continuous by taking ξ_t to be the solution of

$$\sin \xi_t = \frac{3}{2\delta}\,. \tag{21}$$

Eq. (21) determines a unique ξ_t inside the interval $(0, \pi/2)$ only if $3/2 < \delta \equiv p_0/p_i < \infty$. As p_0/p_i decreases toward $3/2$, the size of the dimple increases with the dimple base approaching the equator. For a complete sphere deforming symmetrically about the equator, too large a dimple in both hemispheres is of course physically impossible. Rather than pursuing a discussion of the actual physical constraints on the size of the dimple or the possibility of a dimple at only one of the two poles, we turn now to the discontinuity in β_0.

A discontinuity in β_0 is acceptable if the shell has no bending stiffness so that $D = 0$. For shells with a small bending stiffness factor, we anticipate that such a discontinuity may be removed by a layer type solution in the neighborhood of ξ_t. A small bending stiffness factor is also expected to give rise to a small correction to the location of the transition point ξ_t.

When $\mu \neq 0$ (however small) as it is usually the case, the solution (18–21) no longer satisfies (10) while the discontinuity in ϕ_0 at $\xi = \xi_t$ persists. The satisfaction of the two governing differential equations can be accomplished by a regular perturbation series solution in powers of μ for ϕ and ψ. In fact, it is not difficult to see that the inextensional bending solution is the leading term of this (outer asymptotic) expansion of the solution of the boundary value problem. We may therefore focus our attention on the elimination of the discontinuity in our composite inextensional bending solution (18, 19). This will be done in Section 4 by an ad hoc method similar to the one used in [10] for shallow spherical caps. The analysis of [10], limited to the $\mu = O(\varepsilon)$ range, will be extended here to cover a much wider and more realistic range of load and geometric parameter values.

4. A composite asymptotic solution

For $0 < \mu \ll 1$ and $0 < \varepsilon^4/\mu \ll 1$, we expect the inextensional bending solution of Section 3 to be a good approximation for the solution of the boundary value problem defined by (10), (11), (14), (15) and (16) outside the transition layer, a narrow region in the neighborhood of the transition point ξ_t. Inside the narrow transition layer, an inner asymptotic expansion of the solution is appropriate. As in [10][2], we limit ourselves here to a brief discussion of the leading term of a composite expansion of the solution in the form

$$\phi \sim \phi_0(\xi) + f_0(y), \qquad \psi \sim \psi_0(\xi) + \alpha g_0(y) \tag{22}$$

where the stretched variable y is defined in terms of a small parameter λ by

$$y = \frac{\xi - \xi_t}{\lambda}. \tag{23}$$

The small parameter λ and the multiplicative factor α in the expansion for ψ in (22) are to be specified below. We note that α was set equal to λ in [10] since only the $\mu = O(\varepsilon)$ case was discussed there. Here, we allow for the range $\mu \ll \varepsilon$ as well, e.g. $\mu = O(\varepsilon^2)$. With $\mu = p_i a/Eh = O(\varepsilon^2 p_i/p_c)$, the complementary range $\mu \gg \varepsilon$ implies $p_c \ll p_i\varepsilon$; in other words, p_i is orders of magnitude larger than the classical buckling load. While the pressure distribution is not uniform throughout the shell in our case, we nevertheless do not anticipate the range $\mu \gg \varepsilon$ to be of much interest from a practical viewpoint.

To determine α and λ, we substitute (22) into (10) and (11) and omit terms of order μ to get

$$\frac{\mu\alpha}{\lambda^2}\left[\ddot{g_0} + \lambda \cot\xi\, \dot{g_0} + (\nu - \cot^2\xi)\lambda^2 g_0\right] - \frac{1}{\sin\xi}(\cos\beta - \cos\xi) = 0, \tag{24}$$

$$\frac{\varepsilon^4}{\alpha\mu\lambda^2}\left[\ddot{f_0} + \lambda\cot\xi\, \dot{f_0} + \frac{\lambda^2\cos\beta}{\sin^2\xi}(\sin\beta - \sin\xi) - \frac{\nu\lambda^2}{\sin\xi}(\cos\beta - \cos\xi)\right] + \frac{\sin\beta}{\sin\xi}g_0 = \frac{1}{\alpha\sin\xi}\left[P\cos\beta - \psi_0\sin\beta\right] \tag{25}$$

[2]A systematic treatment of the inner expansion and the matching process for different ranges of parameter values will be reported elsewhere by D. F. Parker and the present author [4].

where $\beta \sim \xi - \phi_0(\xi) - f_0(y) \equiv \beta_0(\xi) - f_0(y)$. With ξ_t specified by (21), the net contribution from the terms inside the brackets on the right-side of (25) is $O(\lambda)$:

$$P \cos \beta - \psi_0 \sin \beta = c_t P_0 \sin f_0 [1 + O(\lambda)]$$

$$= \frac{3\lambda c_t y}{4\delta} \cos \xi_t \sin f_0 [1 + O(\lambda)] \tag{26}$$

where P_0 is P with $\beta = \beta_0$ and where

$$c_t = \begin{cases} -1, & y < 0, \\ 1, & y > 0. \end{cases} \tag{27}$$

To retain the highest derivative term in both (24) and (25) we take $\mu\alpha/\lambda^2 = \varepsilon^4/\mu\alpha\lambda^2 = 1$ or

$$\lambda = \varepsilon, \qquad \alpha = \varepsilon^2/\mu. \tag{28}$$

Since $\varepsilon^4 = O(h^2/a^2)$, we omit all $O(\varepsilon)$ terms in (24) and (25) to get

$$\ddot{g_0} - [\cot \xi_t (\cos f_0 - 1) + c_t \sin f_0] = 0, \tag{29}$$

$$\ddot{f_0} + \frac{\sin(c_t \xi_t - f_0)}{\sin \xi_t} g_0 = \frac{\mu}{2\varepsilon} c_t y \cos \xi_t \sin f_0 \tag{30}$$

for the determination of f_0 and g_0. When $\xi_t^2 \ll 1$ and $f_0^2 \ll 1$, the above pair of equations reduces to the transition layer equations for a spherical cap obtained in [10] for the $\mu = O(\varepsilon)$ case.

For the stress distributions of the shell to be continuous across the transition point ξ_t, we must have

$$\begin{aligned} \phi(\xi_t+) &= \phi(\xi_t-), & \psi(\xi_t+) &= \psi(\xi_t-), \\ \phi'(\xi_t+) &= \phi'(\xi_t-), & \psi'(\xi_t+) &= \psi'(\xi_t-). \end{aligned} \tag{31}$$

From the structure of the differential equations (29) and (30) we have

$$g_0(-y) = g_0(y), \qquad f_0(-y) = -f_0(y) \tag{32}$$

so that the continuity of ψ at ξ_t is satisfied automatically (since ψ_0 is continuous at ξ_t) while the continuity of ϕ requires

$$f_0(0+) = \xi_t. \tag{33}$$

Correspondingly, the continuity of ψ' at ξ_t requires

$$g_0'(0+) = -\frac{\mu}{\varepsilon}\left[\frac{\delta}{3}\cos^2\xi_t \sin\xi_t\right] = -\frac{\mu}{\varepsilon}\left[\frac{3}{2\delta}\left(1-\frac{9}{4\delta^2}\right)\right] \tag{34}$$

while the continuity of ϕ' is satisfied identically to terms of order ε. A more thorough discussion of the continuity conditions can be found in [10].

Outside the transition layer, we expect the contribution from the f_0 and g_0 to be insignificant. Therefore, we stipulate

$$\lim_{y\to\infty} f_0(y) = \lim_{y\to\infty} g_0(y) = 0 \tag{35}$$

which, along with (32), also imply f_0 and g_0 vanishing at $-\infty$.

The two differential equations (29) and (30) and the four auxiliary conditions (33), (34) and (35) determine f_0 and g_0 for $0 < y < \infty$ provided that μ is of order ε at most, which is the range of interest in practice as pointed out earlier in this section. (The $\mu \gg \varepsilon$ case requires a different treatment of (25) leading to a different set of λ and α [4]. The right-side of (30) and (34) should be omitted if $\mu \ll \varepsilon$.) The conditions (32) give f_0 and g_0 for $-\infty < y < 0$. Thus, a solution of the boundary value problem for $0 < y < \infty$ makes the polar dimpling a possible type of deformation under the prescribed normal load distribution.

Just as (22) gives only the leading term of an asymptotic solution for ϕ and ψ, the location of the transition point ξ_t between the two distinctly different types of inextensional bending deformation as determined by (21) is also only the leading term of an asymptotic expansion of the actual location. For sufficiently thin shells, the dependence of the transition location on the shell stiffness is expected to be a higher order effect. A discussion of the higher order correction terms for ξ_t will appear in [4]; the special case $\mu = O(\varepsilon)$ has been considered briefly in [10].

5. Numerical solutions

While the analysis of Section 4 suggests the possibility of obtaining a uniformly valid asymptotic solution of our shell problem for small values of μ and ε^4/μ, we do not actually solve the boundary value problem for f_0 and g_0 in $0 < y < \infty$ (or the corresponding problem for the leading term inner solution with the attendant matching). With a simple geometrical interpretation for its deformation pattern and an elementary description of its stress distributions, the inextensional bending solution of Section 3

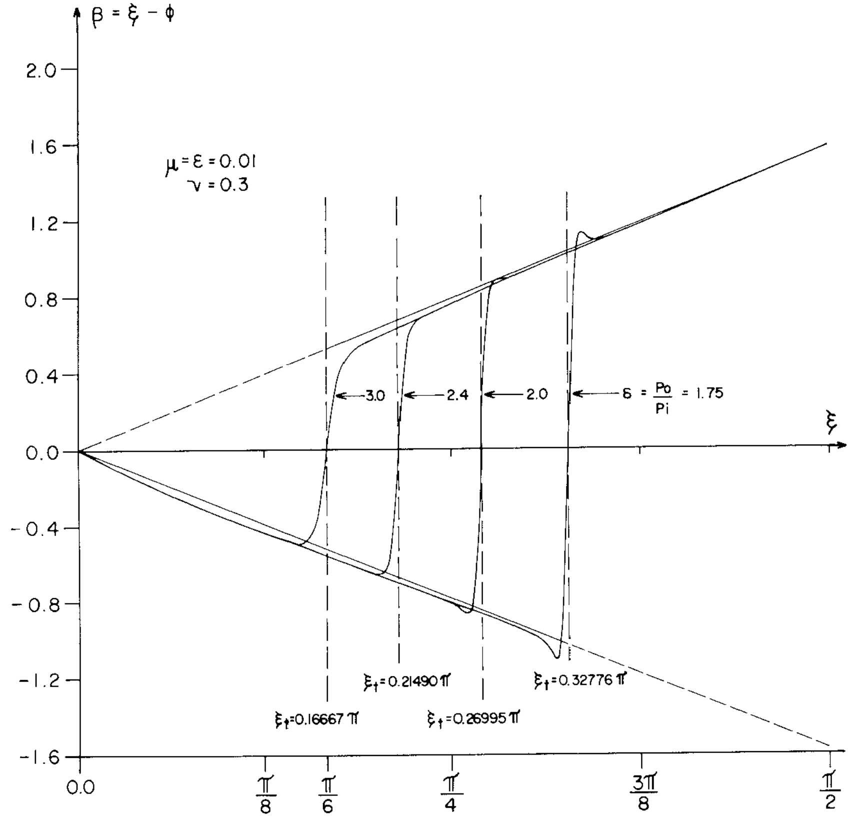

Fig. 2

often by itself provides an accurate and informative approximation of the exact solution outside a narrow transition region. In Fig. 2, we see how well β_0 approximates β for $\mu = \varepsilon = 0.01$ and for several values of $\delta \equiv p_0/p_i$. Fig. 3 shows distributions of β for several combinations of μ and ε when they are not of the same order of magnitude. All numerical solutions in this report were obtained by a general computer code developed by Ascher et al. [1] for general two point boundary value problems with an estimated error no greater than 10^{-5}. Even without the numerical solution, it is not difficult to infer from β_0 the general features of the solution within the narrow transition layer which smoothly connects up the two portions of β_0. Also, an exact solution for the layer equations in terms of elementary or special functions is not possible except for special cases. If numerical methods are contemplated, it would be just as easy (or easier) to obtain a numerical solution of the original problem.

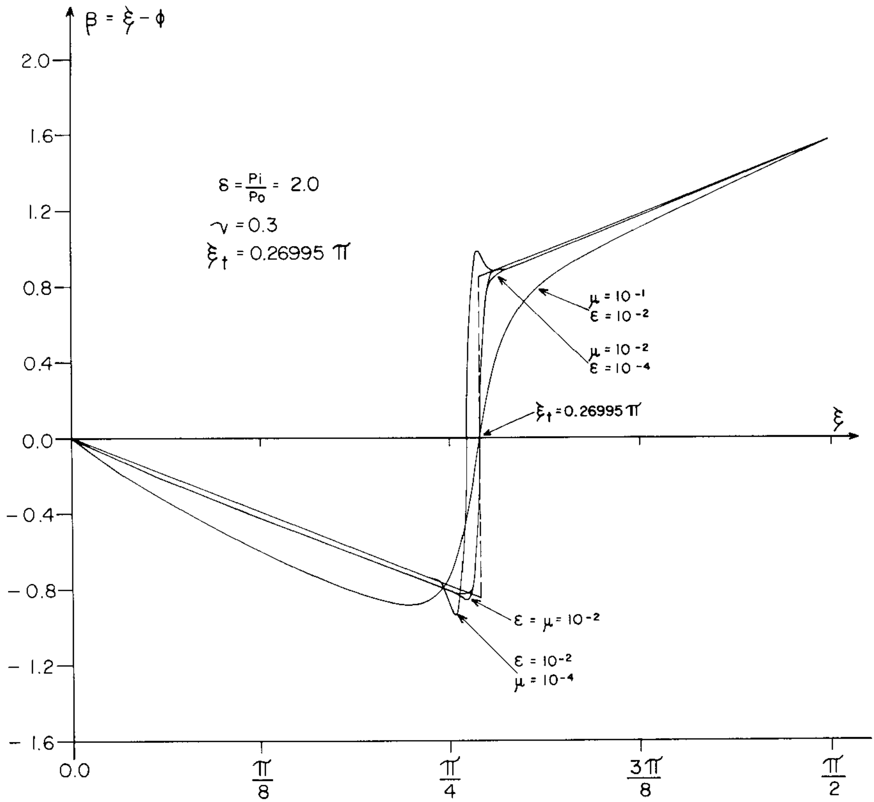

Fig. 3

In addition to being an informative approximate solution of the problem, the simple inextensional bending solution also serves another important function as an initial guess for an exact solution by iterative methods such as the one used in [1]. Without such an educated initial guess, the numerical solution often converges extemely slowly or not at all in sensitive ranges of parameter values; it may also converge to a different solution not appropriate for the dimpling phenomenon.

How well does the inextensional bending solution approximate the dimpling type behaviour depends on the values of μ and ε. An example illustrating how β approaches β_0 as both parameters decrease is shown in Fig. 4.

Numerical results for direct and bending stresses have also been obtained for different ranges of parameter values but will not be presented here as the physical problem itself was constructed only to illustrate our

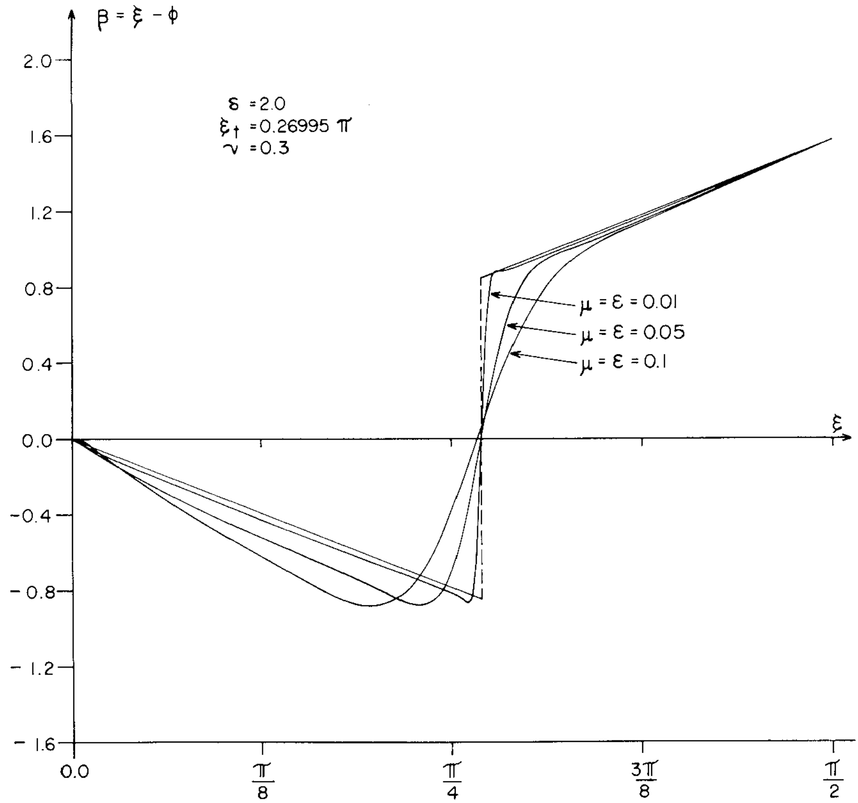

Fig. 4

method. However, a discussion of the relative magnitude of the direct and bending stresses within and outside of the transition layer will be of interest.

6. Direct and bending stresses

For a typical magnitude of the direct and bending stresses of the shell, $\sigma_D = N/h$ and $\sigma_B = \pm 6M/h^2$, consider

$$\sigma_{\theta D} = \frac{N_\theta}{h} = O\left(\frac{p_i a}{h}\psi'\right) = O\left(\frac{p_i a}{h}\left[\psi'_0 + \frac{\varepsilon}{\mu}\dot{g_0}\right]\right), \tag{36}$$

$$\sigma_{rB} = \pm\frac{6M_r}{h^2} = O\left(\frac{Eh}{a}\phi'\right) = O\left(\frac{Eh}{a}\left[\phi'_0 + \frac{1}{\varepsilon}\dot{f_0}\right]\right). \tag{37}$$

Outside the transition layer, $g_0^{\cdot}$ and $f_0^{\cdot}$ are negligible so that

$$\frac{\sigma_{\theta D}}{\sigma_{rB}} = \mathrm{O}\left(\frac{p_{\mathrm{i}}}{Eh^2/a^2}\right) = \mathrm{O}\left(\frac{\mu}{\varepsilon^2}\right). \tag{38}$$

For p_{i} not large compared to the classical buckling load p_{c} of a spherical shell under uniform external pressure, the bending stresses are always at least as important as the direct stresses and dominate the latter whenever $p_{\mathrm{i}} \ll p_{\mathrm{c}}$.

Inside the transition layer, the layer solutions f_0 and g_0 are important (since we are only concerned with the range $\mu \leqslant \mathrm{O}(\varepsilon)$ here) so that

$$\sigma_{rB} = \mathrm{O}\left(\frac{Eh}{a\varepsilon}\right), \qquad \sigma_{\theta D} = \mathrm{O}\left(\frac{p_{\mathrm{i}} a\varepsilon}{h\mu}\right). \tag{39}$$

Recall from (9) that $\mu = p_{\mathrm{i}}a/Eh$ and $\varepsilon^2 = \mathrm{O}(h/a)$, we get from (39)

$$\frac{\sigma_{\theta D}}{\sigma_{rB}} = \mathrm{O}\left(\frac{p_{\mathrm{i}}a^2\varepsilon^2}{Eh^2\mu}\right) = \mathrm{O}(1). \tag{40}$$

Therefore, direct and bending stresses are always of comparable magnitude independent of the relative size of μ and ε (as long as μ is not large compared to ε).

From the above analysis, we see that the maximum stress experienced by the shell in a dimple mode occurs inside the transition layer when p_{i} is not large compared to the classical buckling pressure Eh^2/a^2 (or when μ is not large compared to ε^2) and is $\mathrm{O}(Ehf_0/a\varepsilon)$ in this range of p_{i}. Since f_0 is $\mathrm{O}(\xi_t)$, the peak stress level in the dimpled shell is $\mathrm{O}(E\sqrt{(h/a)}\,)$. We have thus determined the order of magnitude of the maximum stress level without solving for the transition layer solution explicitly. The stress level outside the transition layer is of course $\mathrm{O}(Eh/a)$ which is greater than or equal to $p_{\mathrm{i}}a/h$ for $\mu \leqslant \mathrm{O}(\varepsilon^2)$ but smaller than the peak stress level by an order ε.

7. Concluding remarks

In the preceding pages, we have sketched an elementary method for constructing a simple approximate characterization of the dimple mode deformation for dome type thin elastic shells of revolution. Our results have extended those obtained in [10] in two directions, to nonshallow shells on the one hand and to a wider range of load and geometric parameter values on the other hand. Whenever the dimple base is sharply defined (i.e., the transition layer is narrow), this inextensional bending description gives an accurate and informative solution of the elasto-static shell problem with the particularly desirable feature that the dimple radius is determined by the inextensional bending solution alone. This attractive feature of the method, made possible by the stiffening effect of the outward portion of the applied load distribution near the equator, is to be contrasted with the procedures used in [2, 5] for spherical shells with a point load at the apex which require a knowledge of one or more complex layer solutions. Without the stiffening effect of the outward pressure as in the case $\delta \leqslant 1$ (see (13)), the dimple base location would depend on a proper balance between the bending and stretching shell actions in response to the inward loading similar to the situation in [2] and [5].

To the extent that our main concern here is an approximate inextensional bending description of the dimple mode deformation, we have been able to avoid a thorough investigation of the transition layer problem. While our brief discussion of the gross features of the composite asymptotic solution in the spirit of [7] offers some assurance of a correct leading term outer solution and an approximate dimple base location in the range $O(\varepsilon^2) \leqslant \mu \leqslant O(\varepsilon) \ll 1$, some of the more subtle aspects of the dimpling phenomenon cannot be delineated without a detailed analysis of the inner solution and the attendant matching process. What is the nature of the correct inner solution in the range $\varepsilon \ll \mu \ll 1$ for which a corresponding situation in [7] suggests the presence of a secondary layer? How does the first order correction for the dimple radius depend on ε and μ? (It should be evident from Fig. 3, particularly the location of ξ_t for the $\mu = 10^{-4}$ and $\varepsilon = 10^{-2}$ case, that this dependence may be rather complicated.) While we expect the shell to undergo only infinitesimally small deformation for a very small inward load magnitude, e.g., $\mu \leqslant O(\varepsilon^3) \ll 1$, how does the effect of μ on the dimple radius, which is negligible for $\varepsilon^2 \leqslant \mu \ll 1$, become more and more significant as μ decreases below ε^3, moving ξ_t toward the apex and thereby reducing the dimple size? These and other important questions will be the subjects of a thorough transition layer analysis of the dimpling phenomenon to be reported in [4].

Finally, we would like to point out that the useful inextensional bending description of the dimple mode deformation can be easily obtained by the method of Section 3 for other axisymmetric load distributions and other dome type shells of revolution encountered in engineering problems. Results for the case of a spherical shell stiffened by a uniform internal pressure and subject to a localized uniform external pressure distributed axisymmetrically over a region centered at a pole and to a point force at a pole will be reported in [11].

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