SIAM J. SCI. STAT. COMPUT.
Vol. 1, No. 1, March 1980

0196-5204/80/0101-0008 $01.00/0

NUMERICAL SOLUTIONS FOR MAXIMUM SUSTAINABLE CONSUMPTION GROWTH WITH A MULTI-GRADE EXHAUSTIBLE RESOURCE*

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Abstract. An efficient method is developed for the accurate numerical solution of the nonlinear boundary value problem (BVP) governing the optimal economic growth with a finite multi-grade deposit of nonrenewable and nonaugmentable essential resource under R. M. Solow's maximum sustainable per head consumption level criterion. Unusual computational features of the BVP include: (1) the semi-infinite (time) domain of the problem is divided into a number of subintervals of unknown (and unequal) lengths with a different set of differential equations for each subinterval and with the subinterval lengths to be determined in the solution process; (2) some solution components are known to be unbounded at infinity while others decay very slowly, and (3) in a certain range of parameter values, a previously used solution method is known to be sensitive to boundary data. The new method is used to generate new accurate numerical solutions for single-grade resource problems with a high unit extraction cost and more accurate results for two-grade deposit problems previously investigated. As well, it enables us to investigate for the first time problems with more than two grades of deposits. The implications of these new results are analyzed.

Key words. exhaustible resource, boundary value problem, semi-infinite domain, switch joints, collocation

1. Introduction. One of the principal societal concerns of the nineteen seventies has been the proper use of the Earth's natural resources. This concern is reflected in the considerable amount of research activities in natural resource economics[1]. In the area of exhaustible resources, questions on the proper management and exploitation of finite nonrenewable deposits, such as fossil fuel and minerals, are usually formulated quantitatively as mathematical problems in optimal control. An appropriate formulation of the relevant optimal control problem is not always straightforward, and an exact solution of the problem in terms of elementary or special functions is not always possible. When a numerical solution of the optimal control problem is necessary, the computational procedure required is not always routine.

In this article, we consider a problem of current interest in exhaustible resource economics, and develop a new efficient method for a numerical solution of the associated optimal control problem. The economic problem is a slightly more general version of R. M. Solow's optimal growth under the maximum sustainable consumption rate criterion with a single-grade [1], [3] and a two-grade [2], [3] nonrenewable and nonaugmentable resource deposit. We consider here the same optimal growth problem, but now for a multi-grade resource deposit. The relevant boundary value problem (BVP) for the determination of the optimal growth program for this problem is substantially more complex than the corresponding BVP investigated in [1], [2], [3]. Correspondingly, the computational difficulties associated with a numerical solution for our BVP are more extensive than those experienced in [2] and [3] and are summarized as follows:

* Received by the editors June 1, 1979, and in revised form October 18, 1979.

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[1] For example, an entire 1974 volume of The Review of Economics Studies is devoted to investigations in resource economics.

(1) The domain of the solution is semi-infinite when the economic planning is for the entire future; the solution for some of the unknown quantities is known to be unbounded at infinity while other solution components decay slowly.
(2) The solution domain $[0, \infty)$ is divided up into a number of consecutive subintervals, $[0, T_1)$, $[T_1, T_2)$, $\cdots$, $[T_J, \infty)$, with a different set of differential equations for each of the subintervals; the location of the switch points $T_1, T_2, \cdots, T_J$ is not known and is to be determined as a part of the solution process.
(3) In a certain range of parameter values, a numerical solution of the BVP in its natural (reduced) form is very sensitive to boundary data [3].

Many of these features are direct consequences of economic considerations and their meaning will be clear once we describe the economic model which gives rise to the optimal control problem.

As pointed out in [3], an accurate numerical method of solution that is practical beyond the two-grade deposit case is needed for the above nonconventional nonlinear BVP. In this article, we develop such a numerical method which turns out to be also more efficient than those used in [2] and [3] when applied to problems with a single-grade or a two-grade deposit. Our approach is essentially to eliminate some of the computational difficulties by various reductions, transformations and simplifications of the BVP for the optimal program. The reduced problem is in a form suitable for the application of a BVP solver COLSYS [4], [5]. A brief description of this general purpose code is given in the Appendix of this paper.

To report the method developed and its applications to specific problems, we begin with a brief summary of the relevant optimal economic growth model in § 2; full details and justifications can be found in [1], [2], [3]. Here, we proceed to formulate the BVP for the determination of the optimal growth program for the general multi-grade deposit case. The two principal results of this section are: (1) the reduction of the complicated BVP to a sequence of simple BVP's of the same type over consecutive time intervals, each involving only one first order ODE, and (2) the demonstration of the coincidence of the optimal growth program and the program for maximum capital stock accumulation for the particular consumption rate of the optimal growth program. The second result generalizes a corresponding result in [3] and is obtained by a different, more general means.

The special case of a single-grade resource problem previously treated in [1], [3] is considered in § 3 in the framework of the reduced BVP of § 2. We use it as a vehicle to describe the way we handle the semi-infinite domain aspect of the problem (including the unboundedness and slow decay of the solution components). The same technique is also used later for the general multi-grade problem. With the efficient method of solution developed in § 3, we solve several more difficult single-grade resource problems with high extraction costs to show how close the actual solution for the consumption level is to its upper bound in some cases. These results explain, for the first time, why a larger resource deposit may not notably alter the consumption level under some circumstances.

The reduced BVP's of § 2 for the optimal growth program are coupled through the unknown constant consumption rate and must be solved simultaneously, each for an unknown solution domain. In § 4, we transform this sequence of BVP's over consecutive (unknown) time intervals into a single BVP over the interval (0, 1) for a system of $2(J+1)$ first order ODE's. The resulting problem is in a form suitable for the application of COLSYS. More accurate solutions of the two-grade resource problems analyzed in [2], [3] are obtained by the new method of solution to demonstrate its efficiency. The method also enables us to study, for the first time, problems involving more than two grades of resource deposits. Sample results for a three-grade deposit problem are reported and analyzed for the effects of various input parameters on the maximum sustainable consumption level. From a computational viewpoint, we note the important fact that the computing time required for a solution with the same degree of accuracy increases only moderately with the number of resource grades. Thus, our method can be used for multi-grade resource problems with the number of different grade deposits considerably larger than three.

2. Problem formulation and simplifications. Recall the simple economic model studied in [1] and [2], in which a single nonrenewable and nonaugmentable resource is an essential input to the production of some commodity. At any instant t, let $k(t)$ and $r(t)$ be the capital stock and resource flow per head, respectively, and take the per head output of the commodity to be $q=k^a r^b$ with $0<b<a<a+b<1$. Capital accumulation in this model is governed by

$$\dot{k}=k^a r^b-\theta r-c, \tag{2.1}$$

where c is the per head consumption rate and θ is the known unit extraction cost of the resource which characterizes the quality of the deposit.

The Rawls–Solow max-min principle [1] requires that c be a constant. Our problem is to maximize a permanently sustainable consumption level c subject to equation (2.1), a prescribed initial stock of capital

$$k(0)=\bar{k}_o, \tag{2.2}$$

the nonnegativity constraints

$$r(t)\geqq 0 \quad \text{and} \quad k(t)\geqq 0 \tag{2.3}$$

(which are incorporated automatically into (2.1)) and the conditions of a finite stock of exhaustible resource in several grades of deposits specified below.

Suppose the stock of resource consists of $J+1$ different grade deposits of amount $\bar{D}_1, \bar{D}_2, \cdots, \bar{D}_{J+1}$, respectively, and with constant unit extraction costs $\theta_1, \theta_2, \cdots, \theta_{J+1}$, respectively $(0\leqq\theta_1<\theta_2<\cdots<\theta_{J+1})$. According to [2], the resource stock must be depleted in the order of increasing unit extraction cost under the optimal program. Let T_j be the time when the jth resource is exhausted, $0\equiv T_0<T_1<\cdots<T_J<T_{J+1}\equiv\infty$, and let the remaining resource stock at time t, $D(t)$ be defined by

$$\dot{D}=-r, \tag{2.4}$$

$$D(\infty)=0, \tag{2.5}$$

as the stock of resource will be eventually exhausted under the optimal program. Then, we have also

$$D(T_j)=\sum_{i=j+1}^{J+1}\bar{D}_i, \qquad (j=0,1,\cdots,J) \tag{2.6}$$

and $\theta(t)=\theta_j$ in the interval $T_{j-1}<t<T_j$, $j=1,2,\cdots,J+1$.

Thus our problem is to choose $r(t)$ to maximize the constant c subject to the equations of state (2.1) and (2.4) and the auxiliary conditions (2.2), (2.5) and (2.6). In addition, k and D must be continuous across T_j, $j=1,\cdots,J$. The switch points T_j are unspecified and are to be determined in the solution process.

The necessary conditions for an interior maximum c consist of a system of differential equations and transversality conditions for the Lagrange multipliers λ_k and

λ_D [6]:

(2.7,8) $$-\dot{\lambda}_k = ak^{a-1}r^b\lambda_k, \qquad -\dot{\lambda}_D = 0 \qquad (T_{j-1} < t < T_j),$$

(2.9) $$0 = \lambda_D - \lambda_k(bk^a r^{b-1} - \theta_j)$$

for $j = 1, 2, \cdots, J+1$, with

(2.10,11) $$\lambda_k(T_j-) = \lambda_k(T_j+), \qquad \lim_{t\to\infty} \lambda_k(t) = 0,$$

(2.12) $$[\lambda_k(k^a r^b - \theta_j r - c) - \lambda_D r]_{t=T_j-}^{T_j+} = 0$$

for $j = 1, 2, \cdots, J$. The conditions (2.7)–(2.12) can be reduced to the corresponding conditions in [3] for the two-grade deposit case but have been obtained without the plausible assumption of maximum capital stock accumulation during the cheaper-grade phase of the growth program adopted there. Note that when the planning is for the entire future, the maximum principle may not apply so that these necessary conditions are formal conditions. In this article (and in [1], [2] and [3]), we are concerned only with the process of obtaining a continuous, piecewise differentiable interior solution [6] of these formal necessary conditions. All indications are that the interior solution obtained is unique. For brevity, we will henceforth refer to it as *the* optimal program for the economic growth problem.

In theory, the BVP defined by (2.1)–(2.12) determines within each subinterval the resource extraction rate $r(t)$, the resource stock depletion $D(t)$ and the capital accumulation $k(t)$ (as well as other quantities) for the optimal program, with $D(t)$ and $k(t)$ continuous across the switch points. The task on hand is to find an efficient method for the numerical solution of this complex BVP. For this purpose, we perform some preliminary reductions of the BVP. We begin the reductions by observing that the final (semi-infinite) phase of the optimal program is merely the optimal program for a single-grade resource case with an initial capital stock $\bar{k}_J = k(T_J)$ inherited from an earlier phase of the program. Also, the relevant system of ODE's is autonomous; so the starting time T_J for the final phase is just a reference time and has no substantive effect on the solution. Thus we can use the results in [3] to conclude that for the $(J+1)$-th phase $(T_J < t < \infty)$:

(2.13) $$(1-b)k^a r^b = c.$$

For the growth program to sustain the same consumption rate for all $t > 0$ in spite of the finiteness of the resource deposit needed for the production of consumption goods, it is necessary to have $k \to \infty$ as $t \to \infty$, as is apparent from (2.15) below (e.g., $k(t)$ is linear in t for the special case $\theta_{J+1} = 0$). To avoid a numerical solution of $k(t)$ directly, we use (2.13) to eliminate $k(t)$ from the BVP. We can write (2.1) for $t > T_J$ as

(2.14) $$\dot{k} = \frac{c}{1-b} - \theta_{J+1}r - c = \frac{bc}{1-b} + \theta_{J+1}\dot{D}$$

or

(2.15) $$k(t) = \frac{bc}{1-b}(t - T_J) + \theta_{J+1}(D - \bar{D}_{J+1}) + \bar{k}_J,$$

where $\bar{k}_J \equiv k(T_J+) = k(T_J-)$. This explicit integration not only reduces the size of the resulting BVP (by one differential equation per resource grade), but also takes care of possible numerical sensitivity to small changes in $\bar{k}_0$ which was experienced in [3]. Using (2.13) again, we can write (2.4), (2.6) and (2.5) as

$$\text{(2.16, 17, 18)}\quad \dot{D} = -\left[\frac{c}{1-b}\right]^{1/b} k^{-(a/b)}, \quad D(T_J+) = \bar{D}_{J+1}, \quad D(\infty) = 0.$$

In the neighborhood of T_J, we may use (2.9), (2.10) and (2.13) to deduce from the continuity condition (2.12)

$$[(1-b)k^a r^b - c]_{t=T_J^-} = [(1-b)k^a r^b - c]_{t=T_J^-} = 0$$

or

$$\text{(2.19)}\qquad [(1-b)k^a r^b]_{t=T_J^-} = c.$$

Incidentally, the condition (2.19) implies the continuity of r across the switch point T_J (and therewith $\lambda_k(T_J) \neq 0$). We now use (2.19) in a reduction similar to that for a single grade resource case to get for (see [3]) the

J-th phase ($T_{J-1} < t < T_J$):

$$\text{(2.20)}\qquad (1-b)k^a r^b = c,$$

$$\text{(2.21)}\qquad k = \frac{bc}{1-b}(t - T_{J-1}) + \theta_J(D - \bar{D}_J - \bar{D}_{J+1}) + \bar{k}_{J-1},$$

$$\text{(2.22, 23, 24)}\qquad \dot{D} = -\left[\frac{c}{1-b}\right]^{1/b} k^{-(a/b)}; \quad D(T_{J-1}+) = \bar{D}_J + \bar{D}_{J+1}, \quad D(T_J-) = \bar{D}_{J+1},$$

where $\bar{k}_{J-1} \equiv k(T_{J+1}+) = k(T_{J-1}-)$.

Upon repeating the same reduction for each subinterval, we get for the

j-th phase ($T_{j-1} < t < T_j$):

$$\text{(2.25)}\qquad (1-b)k^a r^b = c,$$

$$\text{(2.26)}\qquad k = \frac{bc}{1-b}(t - T_{j-1}) + \theta_j\left(D - \sum_{i=j}^{J+1} \bar{D}_i\right) + \bar{k}_{j-1},$$

$$\text{(2.27, 28, 29)}\qquad \dot{D} = -\left[\frac{c}{1-b}\right]^{1/b} k^{-(a/b)}, \quad D(T_{j-1}) = \sum_{i=j}^{J+1} \bar{D}_i, \quad D(T_j) = \sum_{i=j+1}^{J+1} \bar{D}_i$$

for $j = 2, 3, \cdots, J$ with $\bar{k}_{j-1} \equiv k(T_{j-1}+) = k(T_{j-1}-)$.

Finally, the same reduction also gives for the

1-st phase ($0 < t < T_1$):

$$\text{(2.30)}\qquad (1-b)k^a r^b = c,$$

$$\text{(2.31)}\qquad k = \frac{bct}{1-b} + \theta_1(D - \bar{D}) + \bar{k}_o, \qquad \bar{D} \equiv \sum_{i=1}^{J+1} \bar{D}_i,$$

$$\text{(2.32, 33, 34)}\qquad \dot{D} = -\left[\frac{c}{1-b}\right]^{1/b} k^{-(a/b)}, \quad D(0) = \bar{D}, \quad D(T_1) = \sum_{i=2}^{J+1} \bar{D}_i.$$

We now simplify the expressions (2.15), (2.21) and (2.26) for $k(t)$ by deriving an explicit expression for $\bar{k}_j$. From (2.26), (2.28) and (2.29) we obtain the recursive relation

$$\text{(2.35)}\qquad \bar{k}_j = k(T_j) = \frac{bc}{1-b}(T_j - T_{j-1}) - \theta_j \bar{D}_j + \bar{k}_{j-1}.$$

Hence

$$\bar{k}_j = \frac{bc}{1-b} T_j - \sum_{i=1}^{j} \theta_i \bar{D}_i + \bar{k}_0 \qquad (j = 1, \cdots, J). \tag{2.36}$$

Substituting back in (2.26), we get for $T_{j-1} < t < T_j$,

$$k(t) = \frac{bct}{1-b} + \theta_j D(t) + \bar{K}_j, \tag{2.37}$$

where

$$\bar{K}_j \equiv \bar{k}_0 - \theta_j \sum_{i=j}^{J+1} \bar{D}_i - \sum_{i=1}^{j-1} \theta_i \bar{D}_i \qquad (j = 1, \cdots, J+1). \tag{2.38}$$

Throughout the paper we use the convention that when a sum is empty, its value is 0. Thus the original complex BVP summarizing the necessary conditions for the optimal program is reduced to the much simpler BVP

$$\dot{D} = -\left[\frac{c}{1-b}\right]^{1/b} \left[\frac{bct}{1-b} + \theta_j D + \bar{K}_j\right]^{-(a/b)} \qquad (T_{j-1} < t < T_j) \tag{2.39}$$

subject to the boundary conditions (2.6) for $j = 1, \cdots, J+1$, with $T_0 = 0$, $T_{J+1} = \infty$, $D(\infty) = 0$, and with $k(t)$ and $r(t)$ given in terms of $D(t)$ by (2.37) and (2.25) (or (2.4)), respectively. Even without solving this simpler problem, our reduction has already yielded the remarkable result that, independent of the number of grades of deposit, *the maximum sustainable per head consumption rate over an infinite horizon is achieved by steering the economy along the "optimal trajectory", $(1-b)k^a r^b = c$, in the capital-resource flow space. This trajectory was found in* [3] *(by a conceptually different argument) to be the path of maximum capital accumulation for the particular consumption rate c, and is of course the optimal growth path for the two-grade resource problem obtained there.*

For the solution of the reduced BVP, we may heuristically regard the BVP of the initial phase as one for determining $D(t)$, $k(t)$, $r(t)$ and T_1 for $0 < t < T_1$ with c as an unknown parameter. This solution gives us $T_1(c)$ to be used in the BVP of phase 2 for determining $D(t)$, $k(t)$, $r(t)$ and T_2 for $T_1 < t < T_2$, still with c as an unknown parameter. $T_2(c)$ will then be used in the BVP of phase 3, etc. Finally, the $(J+1)$th (semi-infinite) phase uses $T_J(c)$ from the Jth phase to determine $D(t)$, $k(t)$, $r(t)$ and c for $T_J(c) < t < \infty$.

3. Numerical solutions for a single-grade deposit. Before we discuss the numerical procedure for the general problem of a multi-grade resource deposit, we consider, in this section, the single-grade deposit case (with $\theta =$ constant for all $t > 0$) to illustrate how we handle the semi-infinite domain in our numerical scheme and to investigate the effects of high extraction costs on the solution.

The problem has been reduced in § 2 to

$$\dot{D} = -\left[\frac{c}{1-b}\right]^{1/b} \left[\frac{bct}{1-b} + \theta D - \theta \bar{D} + \bar{k}_0\right]^{-(a/b)} \qquad (0 < t < \infty), \tag{3.1}$$

$$D(0) = \bar{D}, \qquad D(\infty) = 0. \tag{3.2, 3}$$

It was pointed out in [3] that (3.1) (or the equivalent equation for $r(t)$) admits an exact solution in the form of a quadrature. However, the quadrature has to be evaluated numerically, and we might as well obtain a numerical solution of (3.1)–(3.3) directly. While it is possible to integrate the initial value problem (3.1) and (3.2) for a fixed c, and then to iterate on c until (3.3) is satisfied [3], we prefer a different approach which takes advantage of an available BVP solver, COLSYS. A brief description of COLSYS can be found in the Appendix.

To use COLSYS, the consumption rate will be treated as a function of time, and an auxiliary differential equation

$$\dot{c}=0 \tag{3.4}$$

is introduced to stipulate the fact that c is really independent of time. The system (3.1)–(3.4) now defines a standard two point BVP, and the BVP solver, COLSYS, will be applicable to this problem once we decide how to handle the semi-infinite interval.

Next we transform the semi-infinite domain $[0, \infty)$ into a finite interval $(0, 1]$ by making a nonlinear change of independent variable

$$x=\left(\frac{1}{1+t}\right)^{\sigma} \tag{3.5}$$

for some constant $\sigma>0$. In terms of x, the BVP takes the form

$$\frac{dD}{dx}=x^{-(\sigma+1)/\sigma}\left[\frac{c}{1-b}\right]^{1/b}\frac{1}{\sigma}\left[\frac{bc}{1-b}(x^{-(1/\sigma)}-1)+\theta(D-\bar{D})+\bar{k}_0\right]^{-(a/b)} \quad (0<x<1) \tag{3.6}$$

$$\frac{dc}{dx}=0 \tag{3.7}$$

with the boundary conditions

$$D(0)=0, \qquad D(1)=\bar{D}. \tag{3.8, 9}$$

The second-order transformed problem (3.6)–(3.9) is now in standard form for COLSYS.

It should be noted that the mapping of the semi-infinite interval $[0, \infty)$ onto $(0, 1]$ by (3.5) is at the expense of a singular point in the ODE (3.6) at $x=0$. Fortunately, such singularity poses no problem for COLSYS in this context, provided dD/dx is bounded there. For (3.6), dD/dx is bounded if $-((\sigma+1)/\sigma)+a/(b\sigma)\geqq 0$, or σ should satisfy

$$\sigma\leqq\frac{a-b}{b}. \tag{3.10}$$

In practice, σ should not be too small in order to avoid loss of significant digits in the right side of (3.6).

The frequently used procedure for handling a semi-infinite interval is simply to cut it at some finite point L, with L "large enough", and to solve the problem on $[0, L]$ with the boundary conditions, originally given at ∞, imposed at $t=L$. An adequate value for L is found experimentally and depends on the rate at which the solution components approach their asymptotic values. Here, however, the slow decay of $D(t)$ would necessitate taking extremely large values of L, particularly when $b\geqq a/2$, or when θ is relatively large. In fact, in [3] the asymptotic expression of $D(t)$ had to be used to provide a better approximation of the boundary condition at L, thus making the size of L manageable (but still very large) for one-grade resource problems. The transformation (3.5) together with COLSYS resolves this numerical difficulty in an automatic, elegant way.

Accurate numerical solutions for the maximum sustainable per head consumption rates c of the single-grade resource problem for several sets of parameter values have been obtained by the method described above and are shown in Table 1 to illustrate the efficiency of our approach to the handling of the semi-infinite interval. (The corresponding $r(0)$ is obtained from (2.25).) For all runs reported in this paper, we use $n=4$ collocation points per element, a tolerance tol $=10^{-5}$ on all solution components and a uniform initial mesh of $N=5$ elements. For all cases, unless otherwise stated, the initial guess for the nonlinear iteration consists of $c=1.2$ and a linear interpolant of the boundary conditions for D. Also we fix $\sigma=0.3$ if $b=.05$; $\sigma=1/3$ otherwise[2].

TABLE 1
Single-grade resource ($a=0.2$, $\bar{k}_0=2.4$, $\bar{D}=50$).

Case	b	θ	c	E	Eest	N	CPU
(1)	.05	0	1.21287	.16–10	.15–9	20	0.7
(2)	.05	.03	1.16540		.40–9	20	0.8
(3)	.05	.09	1.10742		.51–10	40	1.3
(4)	.10	0	1.18618	.30–9	.28–8	12	0.4
(5)	.10	.03	1.15741		.27–10	20	0.4
(6)	.10	.09	1.09629		.34–9	20	0.8
(7)	.15	0	1.05198	.16–9	.15–8	10	0.3
(8)	.15	.03	1.04384		.67–9	10	0.3
(9)	.15	.09	1.02466		.80–8	10	0.3

For all cases in Table 1, we fix $a=0.2$, $\bar{k}_0=2.4$ and $\bar{D}=50$. Values of the maximum per head consumption rate c for various values of b and θ are tabulated, together with the estimated error in c (under "Eest"), the final mesh size ("N") and the computer run time in seconds ("CPU"). For the special case $\theta=0$, the exact solution is known [1], [3] and the exact error is listed under "E". The notation $.\alpha-\beta$ means $.\alpha\times10^{-\beta}$. All results reported here were run on an Amdahl 470 V/6-II computer using double precision (14 hexadecimal digits).

From Table 1, we see that the values of c are determined very accurately and efficiently by COLSYS. In fact these values are much more accurate than the 5 digits sought. This is because the solution component c is more stable than D. Also for c the estimate (A.4) applies. Compared to other solution methods for handling the semi-infinite interval, our experience with the cases reported here and others indicates that the transformation (3.5) together with COLSYS is superior, particularly for $b\geqq a/2$. For example, while the results for the same cases obtained in [3] agree with those given in Table 1 to at least four significant figures (and six significant figures for $b=0.05$ cases), the efficiency of the method used in [3] varies with b. To achieve the same accuracy for cases with relatively large b in [3] required more iterations and a larger terminal time L (where we prescribed the expected asymptotic behavior). While it is possible to reduce L by using more terms in the asymptotic boundary value, the method developed here is more attractive in that even without any asymptotic analysis, the CPU time required for all cases considered does not vary by an order of magnitude.

For a given set of parameter values for a, b, θ and $\bar{k}_0$, it follows from a result of [3] that the sustainable per head consumption rate c is bounded by

$$c_{\max}=(1-b)\bar{k}_0^{a/(1-b)}\left(\frac{b}{\theta}\right)^{b/(1-b)}. \tag{3.11}$$

(It would not be possible to maintain the consumption with any finite amount of resources otherwise.) The actual solution for c is of course determined by the size of the

[2] This choice of σ makes $-(\sigma+1)/\sigma+a/(b\sigma)$ a nonnegative integer.

resource deposit; the larger the deposit, the closer c is to c_{max}. However, the value c_{max} can not be attained for it corresponds to a constant resource extraction rate which would exhaust the finite resource deposit in finite time [3]. Table 2 shows how close the solution c is to c_{max} for cases with a high resource extraction cost and a relatively large amount of resource. More importantly, the results indicate that a larger deposit has very little effect on c in this range of parameter values since c_{max} is independent of $\bar{D}$ (see last two cases in Table 2). For larger values of b (or larger b/θ ratios) however, c is considerably smaller than c_{max} (e.g., in the last two cases of Table 1 as well as the corresponding results in [3]) and is expected to increase with $\bar{D}$.

The limitation on the parameter values for the purpose of optimal growth with a sustainable per head consumption rate may be viewed from still another perspective. For a given set of values of a, b and $\bar{k}_0$ and a desired level of per head consumption rate c, to sustain c for the whole future imposes an upper bound on θ,

$$\theta < \theta_{max} \equiv b\bar{k}_0^{a/b}\left(\frac{1-b}{c}\right)^{(1-b)/b}. \tag{3.12}$$

For the cases in Table 2, the actual values for θ are extremely close to this bound.

TABLE 2

*Effect of high extraction costs in single-grade resource growth** ($a = 0.2$, $b = 0.05$, $\bar{k}_0 = 2.4$).

Case	$\bar{D}$	θ	c_{max}	c	θ_{max}	CPU	N
(1)	50	.09	1.1075	1.1074	.09008	0.9	14
(2)	50	.10	1.1013	1.1013	.10004	0.8	16
(3)	50	.11	1.0958	1.0958	.11002	1.0	16
(4)	50	.12	1.0908	1.0908	.12001	0.8	16
(5)	50	.15	1.0781	1.0781	.15000	1.3	28
(6)	50	.20	1.0619	1.0619	.20000	2.1	18
(7)	50	.25	1.0495	1.0495	.25000	2.3	18
(8)	100	.25	1.0495	1.0495	.25000	3.1	20

* The initial guess $D \equiv \bar{D}$ was used for the nonlinear iterations in these cases. Also for $\theta \geqq .12$ a larger initial value for c is used.

4. Numerical solutions for the multi-grade resource problem. To facilitate an efficient solution of the (reduced) BVP for optimal growth with a multi-grade resource, we further transform (2.39) and (2.6) into a form suitable for the application of COLSYS. To avoid working with unknown subintervals (T_{j-1}, T_j), $j = 1, 2, \cdots, J+1$ (with $T_0 = 0$ and $T_{J+1} = \infty$), we map each of these subintervals onto $(0, 1)$. Evidently, the switch points T_j, $j = 1, \cdots, J$, are mapped into boundary points and the continuity conditions across them become boundary conditions. In order to have only separated boundary conditions, we let

$$x = \begin{cases} \dfrac{t - T_{j-1}}{T_j - T_{j-1}} & \text{(for odd } j \leqq J\text{)}, \\[2ex] \dfrac{T_j - t}{T_j - T_{j-1}} & \text{(for even } j \leqq J\text{)}, \\[2ex] 1 - \left(\dfrac{T_J}{t}\right)^{\sigma} & \text{(for odd } j = J+1\text{)}, \\[2ex] \left(\dfrac{T_J}{t}\right)^{\sigma} & \text{(for even } j = J+1\text{)} \end{cases} \tag{4.1}$$

for $T_{j-1}<t<T_j$ with $0<\sigma\leqq(a-b)/b$. The change of variable in the last subinterval (T_J, ∞) is again a transformation of the type (3.5) for some constant σ. With $D_j(x)\equiv D(t)$ in the subinterval $T_{j-1}<t<T_j$ and $(\ )'\equiv d(\ )/dx$, the differential equations (2.39) become

$$(4.2)\qquad D_j'=\begin{cases}-\left(\dfrac{c}{1-b}\right)^{1/b}(T_j-T_{j-1})\left[\dfrac{bc}{1-b}\{T_{j-1}+x(T_j-T_{j-1})\}+\theta_jD_j+\bar{K}_j\right]^{-(a/b)} & (j \text{ odd}),\\ \left(\dfrac{c}{1-b}\right)^{1/b}(T_j-T_{j-1})\left[\dfrac{bc}{1-b}\{T_j-x(T_j-T_{j-1})\}+\theta_jD_j+\bar{K}_j\right]^{-(a/b)} & (j \text{ even})\end{cases}$$

for $1\leqq j\leqq J$ and

$$(4.3)\qquad D_{J+1}'=\begin{cases}-\left(\dfrac{c}{1-b}\right)^{1/b}\dfrac{T_J}{\sigma}(1-x)^{-(1+\sigma)/\sigma}\left[\dfrac{bc}{1-b}T_J(1-x)^{-(1/\sigma)}+\theta_{J+1}D_{J+1}+\bar{K}_{J+1}\right]^{-(a/b)} & (J+1 \text{ odd}),\\ \left(\dfrac{c}{1-b}\right)^{1/b}\dfrac{T_J}{\sigma}x^{-(1+\sigma)/\sigma}\left[\dfrac{bc}{1-b}T_Jx^{-(1/\sigma)}+\theta_{J+1}D_{J+1}+\bar{K}_{J+1}\right]^{-(a/b)} & (J+1 \text{ even}),\end{cases}$$

where $\bar{K}_j$, $j=1,\cdots,J+1$ are given by (2.38). The switch points T_j, $j=1,\cdots,J$ now appear as unknown parameters in the system of ODE's (4.2) and (4.3). To apply COLSYS, we add one ODE for each T_j and one ODE for the unknown constant c as in § 3:

$$(4.4)\qquad T_j'=0\qquad(1\leqq j\leqq J),$$

$$(4.5)\qquad c'=0.$$

Equations (4.2)–(4.5) are $2(J+1)$ first-order ODE's for the $2(J+1)$ unknowns D_j $(1\leqq j\leqq J+1)$, T_j $(1\leqq j\leqq J)$ and c. For them, we have the following $2(J+1)$ boundary conditions from (2.28) and (2.29):

$$(4.6)\qquad D_j(0)=\begin{cases}\sum_{i=j}^{J+1}\bar{D}_i & (j \text{ odd}),\\ \sum_{i=j+1}^{J+1}\bar{D}_i & (j \text{ even}),\end{cases}\qquad(1\leqq j\leqq J+1),$$

$$(4.7)\qquad D_j(1)=\begin{cases}\sum_{i=j+1}^{J+1}\bar{D}_i & (j \text{ odd}),\\ \sum_{i=j}^{J+1}\bar{D}_i & (j \text{ even}),\end{cases}\qquad(1\leqq j\leqq J+1).$$

The BVP defined by (4.2)–(4.7) on the known interval (0, 1) is in a form suitable for the application of COLSYS.

The above method for the solution of the constant consumption rate growth problem with a multi-grade exhaustible resource has been used to generate accurate solutions (to five significant figures) for all the two-grade resource problems studied in [3] and two others not reported there. It is known that the results for c obtained in [3] are also good to five significant figures, but the same can not be said about the results for T_1 and $\bar{k}_1$, the switchover time and the capital stock accumulated at that point. (Using only the leading term asymptotic behavior for the boundary condition at the finite terminal time L, only two significant figure accuracy for T_1 was assured in some cases, mostly those with $b = 0.15$, by the maximum value of L allowed before the onset of error accumulation.) The results in Table 3 of this report show that the values for $\bar{k}_1$

TABLE 3

*Maximum per head consumption rate and switchover time for a two-grade resource deposit**

$(a = .2,\ \theta_2 = .09)$

Case	b	$\bar{k}_0$	$\bar{D}_1$	$\bar{D}_2$	θ_1	c	T_1	$\bar{k}_1 = k(T_1)$	CPU	N
(1)	.05	2.4	10	50	0	1.1587	9.4760	2.9779	1.8	32
(2)	.05	2.4	10	50	.03	1.1406	11.030	2.7621	1.3	18
(3)	.10	2.4	10	50	0	1.1560	6.3030	3.2096	0.5	10
(4)	.10	2.4	10	50	.03	1.1386	6.6762	2.9446	1.0	20
(5)	.15	2.4	10	50	0	1.0743	10.096	4.3140	0.5	10
(6)	.15	2.4	10	50	.03	1.0674	9.8405	3.9535	0.5	10
(7)	.05	2.4	10	25	0	1.1566	10.015	3.0097	0.9	20
(8)	.05	2.4	10	25	.03	1.1392	11.489	2.7889	1.0	20
(9)	.10	2.4	10	25	0	1.1209	9.6156	3.5976	0.5	10
(10)	.10	2.4	10	25	.03	1.1080	9.8154	3.3084	0.6	10
(11)	.15	2.4	10	25	0	.98496	24.841	6.7177	0.6	10
(12)	.15	2.4	10	25	.03	.98053	24.014	6.2552	0.6	10
(13)	.05	2.4	50	50	0	1.2291	24.029	3.9545	1.0	20
(14)	.05	2.4	50	50	.03	1.1693	35.914	3.1102	1.9	28
(15)	.05	4.8	10	50	0	1.3154	10.389	5.5193	1.8	20
(16)	.05	4.8	10	50	.03	1.3036	11.325	5.2771	1.8	20
(17)	.10	4.8	10	50	0	1.2764	8.8278	6.0519	0.6	10
(18)	.10	4.8	10	50	.03	1.2675	8.9836	5.7652	0.5	10
(19)	.15	4.8	10	50	0	1.1254	18.391	8.4525	0.5	10
(20)	.15	4.8	10	50	.03	1.1222	18.106	8.0854	0.5	10

* The initial guess $T_1 \equiv 10$ was used for the nonlinear iterations here, with $c \equiv 1.2$ and D_j linear interpolants of their boundary conditions, $j = 1, 2$.

obtained in [3] are accurate to at least four significant figures except for one case with a 0.4% error. In contrast, the values for T_1 obtained in [3] are not as accurate though the percentage error is still less than 0.5% in all cases. Evidently, the method of solution developed here has enabled us to achieve a degree of accuracy in the numerical solution for the two-grade resource not practical hitherto. More importantly, the high accuracy is attained with no asymptotic analysis (which is needed in [3]), minimal programming and a relatively small amount of computing time for all cases investigated as shown under the CPU time column of Table 3. The effects of the various input parameters as suggested by Table 3 have already been analyzed in [3] and will not be repeated here.

With the same method, we can, for the first time, generate accurate numerical solutions for maximum constant consumption rate problems with a resource deposit of more than two grades. Again, the solution process requires minimal programming and a relatively small amount of computing time to achieve the desired accuracy (and of course no asymptotic analyses). To illustrate, we report in Table 4 some sample calculations for three-grade resource problems. In all fifteen cases, we have kept $a = 0.2$, $\bar{k}_0 = 2.4$, $\bar{D}_1 = 10$ and $\bar{D}_2 = 25$. The CPU seconds required for a single case (to achieve a five significant figure accuracy) range from 1.0 to 10.7, and increase with more low grade deposit $\bar{D}_3$ or higher unit extraction costs θ_j, $j = 1, 2, 3$. The CPU time is lower when $b = 0.1$ or $b = 0.15$ and is higher for $b = 0.05$.

TABLE 4

*Maximum per head consumption rate and switchover times for a three-grade resource deposit**

($a = .2$, $\bar{k}_0 = 2.4$, $\bar{D}_1 = 10$, $\bar{D}_2 = 25$)

Case	b	$\bar{D}_3$	θ_1	θ_2	θ_3	c	T_1	T_2	$k(T_1)$	$k(T_2)$	CPU	N
(1)	.05	50	0	.03	.09	1.1864	4.9364	26.981	2.7082	3.3348	2.3	18
(2)	.10	50	0	.03	.09	1.2090	3.6195	19.070	2.8862	4.2118	1.1	10
(3)	.15	50	0	.03	.09	1.1451	5.7698	46.515	3.5660	11.050	1.3	10
(4)	.05	100	0	.03	.09	1.1865	4.9307	26.892	2.7079	3.3293	4.7	36
(5)	.10	100	0	.03	.09	1.2202	3.2491	15.804	2.8405	3.7926	2.0	20
(6)	.15	100	0	.03	.09	1.2226	3.4120	18.347	3.1362	5.6086	1.0	10
(7)	.05	50	.03	.06	.09	1.1498	8.5208	37.518	2.6157	2.8705	4.3	36
(8)	.10	50	.03	.06	.09	1.1668	4.8616	21.508	2.7303	3.3884	1.1	10
(9)	.15	50	.03	.06	.09	1.1289	5.9716	41.451	3.2896	8.8578	1.3	10
(10)	.05	50	.03	.09	.25	1.1400	11.237	61.743	2.7742	3.5545	6.3	20
(11)	.10	50	.03	.09	.25	1.1356	6.9199	37.145	2.9731	4.5369	2.3	20
(12)	.15	50	.03	.09	.25	1.1139	6.6984	45.423	3.4167	8.7789	1.3	10
(13)	.05	100	.03	.09	.25	1.1400	11.237	61.743	2.7732	3.5545	10.7	48
(14)	.10	100	.03	.09	.25	1.1358	6.9055	36.888	2.9715	4.5051	4.4	20
(15)	.15	100	.03	.09	.25	1.1440	5.3403	27.623	3.1781	5.4265	1.4	10

* The initial guess, $T_1 \equiv 8$, $T_2 \equiv 16$, $c \equiv 1.2$ and linear interpolants of boundary conditions for D_j, $j = 1, 2, 3$, was used for all cases but No. 10, 13, 14. For the latter 3 cases, the initial guess was changed to $T_2 \equiv 50$, $c \equiv 2.0$, $D_3 \equiv \bar{D}_3$.

By comparing cases (1)–(3) with cases (4)–(6), respectively, we see that doubling the low grade resource deposit $\bar{D}_3$ hardly affects the $b = 0.05$ case, changes the consumption rate level by only 1% in the $b = 0.1$ case but changes c by about 7% in the $b = 0.15$ case. These results conform with our observations in § 3 for single-grade resource problems; evidently, c is still considerably less than c_{max} for the $b = 0.15$ case so that it can be increased by a larger low grade resource deposit $\bar{D}_3$. To achieve the 7% change in c by doubling $\bar{D}_3$ in this case involves a substantial change in the growth program. It allows the switch-over times T_1 and T_2 to go from 5.7698 and 46.515 down to 3.4120 and 18.347, respectively. Correspondingly, it allows $\bar{k}_1$ and $\bar{k}_2$ to go from 3.5660 and 11.050 down to 3.1362 and 5.6086. Evidently, with more low-grade resource, the economy requires less initial capital for the last phase of the program covering the semi-infinite interval (T_2, ∞) so that the higher grade deposits can be used up sooner to achieve a higher per head consumption rate c in all three phases of the program. In contrast, c is very nearly c_{max} in cases (10), (11), (13) and (14) with very high extraction costs; we can not increase c much by doubling $\bar{D}_3$.

Cases (1)–(3) in conjunction with cases (7)–(9) demonstrate the effects of an increase of the unit extraction costs θ_1 and θ_2 for the two high grade deposits. Higher values of θ_1 and θ_2 use up more output at each instant. To keep down the extraction cost per unit time, the new optimal programs of cases (7)–(9) reduce c to slow down the resource extraction and to build up a capital stock at the switch points comparable to those for cases (1)–(3), respectively. A good size capital stock in turn moderates the reduction in consumption level.

Finally, cases (10)–(12) in conjunction with cases (7)–(9) show that an increase in θ_2 and θ_3 gives rise to analogous effects.

By comparing the CPU time for related single-grade, two-grade and three-grade problems in Tables 1–4, e.g., cases (6), (4) and (2) in Tables 1, 3 and 4, respectively, we see that CPU time increases only moderately with the number of resource grades. This suggests that our solution procedure is still practical even when J is larger.

Appendix. The general purpose BVP solver COLSYS [4], [5] is based on spline collocation at Gaussian points and is capable of handling mixed order systems of nonlinear multi-point BVP's. Here it is sufficient to consider a two-point BVP for a first order system,

(A.1) $$\mathbf{z}' = \mathbf{f}(x, \mathbf{z}), \qquad x \in (a, b),$$

(A.2) $$\mathbf{g}^1(\mathbf{z}(a)) = 0, \qquad \mathbf{g}^2(\mathbf{z}(b)) = 0,$$

where $\mathbf{z}, \mathbf{f}$ are vector functions of order m, $\mathbf{g}^1$ is of order m_1 and $\mathbf{g}^2$ is of order $m_2 = m - m_1$. The functions $\mathbf{f}$, $\mathbf{g}^1$ and $\mathbf{g}^2$ may be nonlinear.

In COLSYS, the problem (A.1)–(A.2) is solved on a sequence of meshes, until user-specified error tolerances are satisfied. For a specific mesh $a = x_0 < x_1 < \cdots < x_N = b$, with $h_i = x_i - x_{i-1}$, $h = \max_{1 \leq i \leq N} h_i$, and an integer $n > 1$, the collocation solution $\mathbf{v}(x) = (v_1, \cdots, v_m)$ is a piecewise polynomial vector function: for each j, $1 \leq j \leq m$, $v_j \in C[a, b]$ is a polynomial of degree $\leq n$ on each element (x_{i-1}, x_i), $i = 1, \cdots, N$. The piecewise polynomial solutions are represented in terms of a B-spline basis. The approximate solution is determined by requiring that it satisfy (A.2) and the differential equation (A.1) at the images of the n zeros of the appropriate Legendre polynomial in each element. Under sufficient smoothness conditions, the error in $\mathbf{v}$ for $x \in [x_i, x_{i+1})$ is given by

(A.3) $$z_j(x) - v_j(x) = K z_j^{(n+1)}(x_i) h_i^{n+1} + O(h^{n+2}),$$

where K is a known bounded function of x, and at the mesh point x_i,

(A.4) $$z_j(x_i) - v_j(x_i) = O(h^{2n}), \qquad (j = 1, \cdots, m) \quad (i = 0, 1, \cdots, N).$$

Expression (A.3) is used both for estimating the error accurately via mesh halving and for automatic new mesh selection. For nonlinear problems, the damped Newton's method is used for the first mesh to find the collocation solution, and modified Newton iterations with a fixed Jacobian are performed for subsequent refined meshes. Full details of the code can be found in [4], [5] and references therein.

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