From: Proc. Conf. on Boundary & Interior Layer, Comp. & Asymp. Methods, (Dublin), ed. J. Miller, 1980, 415-419.

HORIZONTAL AND FLAT POINTS IN SHALLOW CAP DIMPLING

F.Y.M. Wan[1] and U. Ascher[2]

1. Introduction

For shallow caps under an axisymmetric external pressure distribution which is inward (with a representative magnitude p_o) in a neighborhood of the apex and outward away from the apex, it has been shown in [1] and [2] that a finite axisymmetric dimple state of deformation is possible if p_o is at least of the order of magnitude of the classical buckling pressure for a spherical shell, p_c, and that to a first approximation, the corresponding dimple base is located by the condition of no resultant vertical force over the dimpled region. Thus, the determination of the approximate dimensionless dimple base radius x_t for our type of loading does not require an extra condition (e.g., minimum energy) beyond the elasto-statics of the shell as it does in [3] and [4] for a point load.

In a later paper [5], the more difficult load magnitude range $p_o/p_c << 1$ is treated more systematically by the method of inner-outer asymptotic expansions.

The studies in [1], [2] and [5] are restricted to caps which are of dome shape, i.e., the undeformed shell meridian does not have a horizontal tangent except at the apex and the apex is not a flat point (of zero curvature). If the apex is a flat point, it is known to shell theorists that the shell is plate-like there, and an "edge bending" layer develops in a small neighborhood of the flat point. For a sufficiently thin cap, this layer is not expected to inhibit the occurence of the dimple state, though it may have quantitative effects on the details of the dimple. The situation for a point along the shell meridian with a horizontal tangent is less obvious. An "edge bending" layer around such a "horizontal point" may inhibit the occurence of a dimple if the location of the dimple base (as determined in [1,2,5] for the particular load distribution) lies within the layer. The main objective of this paper is to delineate the qualitative and quantitative effects of horizontal and flat points on the dimpling phenomenon by an asymptotic analysis of the relevant boundary value problem and to support the asymptotic results by accurate numerical solutions of the (original) boundary value problem. For comparison with the results obtained in [1], numerical solutions are again presented for a quadratic pressure load distribution $p(x)$, where x is the dimensionless radial distance from the axis of revolution to a point on the middle surface of the shell.

While considerable insight on the dimpling of shallow caps is gained from the asymptotic analysis, the actual asymptotic description

---

[1] Department of Mathematics, UBC. Supported in part by National Research Council of Canada Grant No. A9259, and the Applied Mathematics Group, University of Washington (Seattle, WA., U.S.A.)

[2] Department of Computer Science, UBC. Supported in part by National Research Council of Canada Grant No. A4306.

of the dimple state still involves the solution of nonlinear differential equations which in general have to be solved numerically, if detailed deformations and stress distributions of the dimple state are needed for a particular shell. As in the previous studies [1,2,5], a general boundary value problem solver COLSYS [6] is used to generate the desired accurate numerical solutions. A second objective of this paper is to point out that qualitative results deduced from the asymptotic analysis such as the occurence of layer phenomena at various locations of the shell, the order of magnitude of their layer width, etc., are invaluable for an efficient application of COLSYS (or any other numerical scheme). The simple leading term outer solution and approximate location of the dimple base alone provide us with a qualitatively correct initial guess for the iterative numerical solution scheme. Without this qualitatively correct approximation of the exact solution, convergence to a numerical solution with a prescribed error tolerance would not have been possible in many cases. Even with the help of the asymptotic results, careful parameter continuations are sometimes required for a dimple state solution. Without the asymptotic results, we could have arrived at the erroneous conclusion that a dimple state does not exist.

A third objective of the paper is to propose the dimple problem (2.1) - (2.6) as a possible test problem for numerical methods designed to solve singular perturbation problems in ODE's. The proposed dimple problem is a nonlinear system with possibly multiple solutions, a boundary layer, and an interior layer which may shift as we alter the shell thickness and load distribution [1].

2. The Inextensional Bending Dimple Solution

The finite axisymmetric deformation elasto-statics of the shallow shell of revolution is governed by the following two nonlinear ODE's for a dimensionless stress function $\Psi(x)$ and a normalized meridional slope angle change $\Phi(x)$ [5,7,8]:

$$\varepsilon^2[\Psi'' + \frac{1}{x}\Psi' - \frac{1}{x^2}\Psi] - \frac{1}{x}\Phi(\Phi_o - \frac{1}{2}\Phi) = 0 \qquad (2.1)$$

$$(0 < x < 1)$$

$$\varepsilon^2[\Phi'' + \frac{1}{x}\Phi' - \frac{1}{x^2}\Phi] + \frac{1}{x}\Psi(\Phi_o - \Phi) = 2\kappa P(x) \qquad (2.2)$$

where $\xi(x) \equiv \xi_o\Phi_o(x)$ is the meridional slope angle with respect to the base plane of the middle surface of the undeformed shell (with $\Phi_o(1) = 1$), $\beta(x) \equiv \xi_o(\Phi_o(x) - \Phi(x))$ is the angle of the deformed middle surface and

$$\varepsilon^2 = \frac{ha}{r_o^2\sqrt{12(1-\nu^2)}}\ , \quad a = \frac{r_o}{\xi_o}\ , \quad \kappa = \frac{p_o}{p_c}\ , \quad p_c = \frac{2Eh^2}{a^2\sqrt{3(1-\nu^2)}} \qquad (2.3)$$

In (2.3), h is the uniform shell thickness, r_o is the radial distance from the axis of revolution to the edge of the shell, E is the Young's modulus of the shell material and ν is its Poisson's ratio. In (2.2),

$P(x)$ is the dimensionless vertical stress resultant taken in this paper to be

$$P(x) = -\frac{2}{x}\int_0^x tp(t)dt = x(1 - \gamma + \frac{\gamma}{2}x^2) \quad (2.4)$$

corresponding to the quadratic pressure over a portion of the cap with an edge radius $r_o x$ as in [1,5]. All stress, strain and displacement measures of the shells are expressed in terms of $\xi_o\Phi(x)$ and the stress function $\frac{1}{4}p_c r_o a\Psi(x)$; these expressions can be found in [1,7,8]. The boundary conditions for (2.1), (2.2) considered in this paper consist of the boundedness conditions at the apex,

$$\Phi(0) = \Psi(0) = 0 \quad (2.5)$$

and the clamped edge conditions (cf. [1,2,5]),

$$\Phi(1) = \Psi'(1) - \nu\Psi(1) = 0. \quad (2.6)$$

In all computations reported here, we take $\nu = .3$ and $\gamma = 1.2$.

The boundary value problem (2.1), (2.2), (2.5), (2.6) determines $\Psi(x)$ and $\Phi(x)$ for a given $p(x)$. If the shell is sufficiently thin so that $\varepsilon^2 << 1$ and if κ is not too small, the deformed configuration of the shell for the class of pressure load distributions considered may take the form of an axisymmetric dimple [1,2,5]. Furthermore, an approximate but qualitatively correct description of the dimple state is given by the following inextensional bending solution of (2.1) and (2.2):

$$\begin{cases} \Phi = \Phi^{(1)}(x) \equiv 2\Phi_o(x) \text{ and } \Psi = \Psi^{(1)}(x) \equiv -\dfrac{2\kappa x P(x)}{\Phi_o(x)} & (0 < x < x_T) \\ \Phi = \Phi^{(2)}(x) \equiv 0 \quad\text{ and } \Psi = \Psi^{(2)}(x) \equiv \dfrac{2\kappa x P(x)}{\Phi_o(x)} & (x_T < x < 1) \end{cases} \quad (2.7)$$

where a first approximation, x_t, of x_T is determined by $P(x_t) = 0$ when $\kappa^{-1} = 0(1)$ (see [5] for the determination of x_t when $\kappa << 1$). The expressions (2.7) correspond to the leading term of an outer (asymptotic expansion) solution of the boundary value problem. This outer solution is to be supplemented by layer (inner) solutions in the neighborhood of the dimple base x_T (to smooth out the discontinuities in the inextensional bending solution) and adjacent to the shell edge (to satisfy the boundary conditions). The condition $P(x_t) = 0$ follows from a continuity requirement on Ψ across the dimple base, which is a consequence of a distinguished limit consideration in the analysis of the inner solution, though it may also be imposed on physical grounds.

3. Horizontal Points and Flat Points

From the expressions in (2.7), it is evident that the approximate solution $\Psi^{(1)}(x)$ is not well defined at $x = 0$ if the shell is sufficiently flat at the apex. For the quadratic pressure load corresponding to (2.4), we need a bending layer solution around $x = 0$ to ensure the boundedness of the stress, strain and displacement measures of the shell at the apex if $\Phi_o(x) = 0(x^m)$ with $m > 1$. For thin shells, this additional layer solution is confined to a neighborhood of the apex. Therefore, it is not expected to inhibit the occurence of a finite axisymmetric dimple since $0 << x_t \leq 1$. However, the interior transition layer around the dimple base is wider and the dimple is not as pronounced for shells of similar thickness with a relatively flat apex. This indirect effect of a flat apex may be seen from an inner (or transition layer) solution of (2.1) - (2.2) in the neighborhood of the approximate dimple base x_t in the form

$$\{\Phi_o,\Phi\} = \Phi_t\{\phi_o(y),\phi(y)\}, \quad \Psi = \Psi_t\psi(y), \quad x = x_t(1+\lambda y) \tag{3.1}$$

for an appropriate choice of λ and Ψ_t. For simplicity, we limit discussion here to $1/\kappa = O(1)$, so that $\bar{P}(x_t) = 0$. In this case, a suitable distinguished limit of (2.1) - (2.2) gives $\lambda = \varepsilon/\sqrt{x_t\Phi_t}$ and $\Psi_t = \Phi_t$. For spherical caps, we have $\Phi_o(x) = x$ and $\Phi_t = x_t$ so that $\lambda = \varepsilon/x_t = O(\varepsilon)$ whenever $0 < x_t \leq 1$; but for shells with a flat apex such as $\Phi_o(x) = x^m$ with $m > 1$, we have $\lambda = \varepsilon/\sqrt{x_t^{m+1}}$ so that λ may be an order of magnitude larger than ε even if $0 << x_t < 1$. Therefore, the transition layer width is considerably wider for these cases and the dimple not as pronounced. Accurate numerical solutions obtained by COLSYS, a general purpose computer code for boundary value ODE's based on spline-collocation at Gaussian points [6], provide quantitative confirmation of these observations. Some sample solutions in Figure (1) show that a much smaller value of ε (10^{-4}) than that for a spherical cap (10^{-2}) is necessary for the exact (numerical) solution with $\Phi_o = x^3$ to approach the inextensional bending solution (2.7). We see from the graph of $\Phi(x)$ for $\Phi_o(x) = x^3$ and $\varepsilon = 10^{-4}$ that a (bending) layer persists in the neighborhood of the apex.

For shells with a point of horizontal tangent along the meridian $\Phi(x_o) = 0$, which may or may not be a flat point, the inextensional bending solution (2.7) is in general unbounded at the turning point x_o, and a bending layer develops around the turning point. The effect of this layer solution on shell dimpling can be analyzed in a way similar to that for a flat apex. For a sufficiently thin shell, this layer solution is localized and is not expected to interfere with polar dimpling. Numerical solutions for a shell with $\Phi_o = x(x-x_o)^2/(1-x_o)^2$ confirm these observations. In Figure (2), plots of $\Phi(x)$ for $\varepsilon = 10^{-2}$, 10^{-3} and 10^{-4} are shown for $x_o = 0.3$ ($< x_t$). Evidently, a dimple shape begins to emerge for $\varepsilon \leq 10^{-3}$ in contrast to a spherical cap which dimples conspicuously at $\varepsilon = 10^{-2}$ [2]. Numerical results not included in this article show that a shell with $x_o = 0.75$ ($> x_t$) exhibits similar behavior. For the special case $x_o = x_t$, the transition layer around the dimple base lies inside the bending layer associated with the turning point x_o since the latter is known to be wider. In contrast to shells with $x_o \neq x_t$, the coincidence of the two layers reduces the portion of the shell where the exact solution deviates from the inextensional bending solution. Therefore, a shell with $\Phi_o = x(x-x_o)^2/(1-x_o)^2$ is more conducive to polar dimpling when $x_o = x_t$. This observation is supported by the plots of $\Phi(x)$ in Figure (3) for $x_o = x_t$ which, in contrast to Figure (2), show a pronounced dimple for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$.

All COLSYS solutions in this article meet a prescribed relative error tolerance of 10^{-4}. The code solves a given problem on a sequence of meshes, automatically increasing the number of mesh points in regions of abrupt changes. The inextensional bending solution (2.7) has been used as the initial approximation for the nonlinear iteration scheme on the first mesh for a fairly small ε (usually 10^{-2}), allowing convergence to the desired dimple solution (cf. [9]). Simple continuation in ε and κ has been subsequently used for tougher cases with smaller ε or κ. For Figure (2), in particular, the solution for $\varepsilon = 10^{-2}$ was used as the initial approximation for the case with $\varepsilon = 10^{-3}$ and 8 additional such continuation steps were needed to get down to the case with $\varepsilon = 10^{-4}$.

References

1. F. Y. M. Wan, "The Dimpling of Spherical Caps". Mech. Today 5 (The Eric Reissner Anniversary Volume), 1980, 495-508.

2. F. Y. M. Wan, "Polar Dimpling of Complete Spherical Shells", Theory of Shells, Proc. of Third IUTAM Shell Symp. (Tbilisi, USSR; August 1978), Ed. W. T. Koiter & G. K. Mikhailov, North-Holland (1979), 191-207.

3. D. G. Ashwell, "On the Large Deflexion of a Spherical Shell with an Inward Point Load", Proc. of First IUTAM Symp. on Thin Elastic Shells, (Delft, The Netherlands; August 1959), Ed. W. T. Koiter, North-Holland, Amsterdam (1960), 43-63.

4. C. V. Ranjan & C. R. Steele, "Large Deflection of Deep Spherical Shells under Concentrated Load", Tech. Paper No. 77-411, AIAA/ASME 18th Structures, Structural Dynamic & Materials Conf. (San Diego, Calif.; March 1977).

5. D. F. Parker & F. Y. M. Wan, "Finite Polar Dimpling of Shallow Caps under Sub-Buckling Pressure Loading", I.A.M.S. Tech. Report No. 80-15, Univ. of British Columbia, in preparation. (Also, to be presented at the ICTAM, Toronto, (August), 1980).

6. U. Ascher, J. Christiansen & R. D. Russell, "A Collocation Solver for Mixed Order Systems of Boundary Value Problem", Math. Comp. 33, 1979, 659-679.

7. E. Reissner, "Symmetric Bending of Shallow Shells of Revolution", J. Math. Mech. 7, 1958, 121-140.

8. E. Reissner, "The Edge Effect in Symmetric Bending of Shallow Shells of Revolution", Comm. Pure & Appl. Math. 12, 1959, 385-398.

9. U. Ascher, J. Christiansen & R. D. Russell, "Collocation Software for Boundary Value ODE's", Trans. of Math. Software, to appear.