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ON STRESS STRAIN RELATIONS AND STRAIN DISPLACEMENT RELATIONS OF THE LINEAR THEORY OF SHELLS

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Summary

The stress strain relations of Flügge, Lurje and Byrne for lines of curvature coordinates are inverted to give strains in terms of stresses. It is then shown that these inverted relations may be written particularly simply in terms of strain measures which contain explicitly an angular displacement component ω turning about the normal to the middle surface, thereby furnishing an explicit example of a system of stress strain relations involving eight strain measures ϵ_{jk} and κ_{jk} and a strain potential $W(N_{11}, N_{12}, \ldots, M_{21}, M_{22})$ which had earlier been proposed by one of the authors. The strain potential $W = W^*$ for the Flügge-Lurje-Byrne relations in lines of curvature coordinates is then used to derive a simplified version of a system of FLB relations for general orthogonal coordinates.

Introduction

Using the principle of virtual stress in a form which is appropriate for a two-dimensional theory of shells we have earlier (Reissner 1962) established a system of strain displacement relations which for general orthogonal coordinates is of the form

$$\epsilon_{11} = \frac{u_{1,1}}{\alpha_1} + \frac{\alpha_{1,2}u_2}{\alpha_1\alpha_2} + \frac{w}{R_{11}}, \qquad \epsilon_{22} = \frac{u_{2,2}}{\alpha_2} + \frac{\alpha_{2,1}u_1}{\alpha_1\alpha_2} + \frac{w}{R_{22}}$$

$$\epsilon_{12} = \frac{u_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}u_1}{\alpha_1\alpha_2} + \frac{w}{R_{12}} - \omega, \qquad \epsilon_{21} = \frac{u_{1,2}}{\alpha_2} - \frac{\alpha_{2,1}u_2}{\alpha_1\alpha_2} + \frac{w}{R_{12}} + \omega,$$

$$\kappa_{11} = \frac{\beta_{1,1}}{\alpha_1} + \frac{\alpha_{1,2}\beta_2}{\alpha_1\alpha_2} + \frac{\omega}{R_{12}} , \qquad \kappa_{22} = \frac{\beta_{2,2}}{\alpha_2} + \frac{\alpha_{2,1}\beta_1}{\alpha_1\alpha_2} - \frac{\omega}{R_{12}} ,$$

$$\kappa_{12} = \frac{\beta_{2,1}}{\alpha_1} - \frac{\alpha_{1,2}\beta_1}{\alpha_1\alpha_2} - \frac{\omega}{R_{11}} , \qquad \kappa_{21} = \frac{\beta_{1,2}}{\alpha_2} - \frac{\alpha_{2,1}\beta_2}{\alpha_1\alpha_2} + \frac{\omega}{R_{22}} , \tag{1}$$

$$\gamma_1 = \beta_1 + \frac{w_{,1}}{\alpha_1} - \frac{u_1}{R_{11}} - \frac{u_2}{R_{12}} , \qquad \gamma_2 = \beta_2 + \frac{w_{,2}}{\alpha_2} - \frac{u_1}{R_{12}} - \frac{u_2}{R_{22}} .$$

With these strain displacement relations we have postulated a system of stress strain relations of the form

$$\varepsilon_{11} = \frac{\partial W}{\partial N_{11}}, \quad \varepsilon_{12} = \frac{\partial W}{\partial N_{12}}, \ \ldots\ldots , \ \gamma_1 = \frac{\partial W}{\partial Q_1}, \ \ldots\ldots ,$$

$$, \ldots\ldots , \kappa_{21} = \frac{\partial W}{\partial M_{21}}, \quad \kappa_{22} = \frac{\partial W}{\partial M_{22}}, \tag{2}$$

where the strain potential W is a suitable function of the ten arguments N_{11}, N_{12}, N_{21}, N_{22}, Q_1, Q_2, M_{11}, M_{12}, M_{21}, M_{22}.

The system of strain displacement relations (1) differs from the customary forms of such relations through the occurrence of the (sixth) displacement variable ω. The system of stress strain relations (2) differs from the customary form of such relations by way of the fact that in them the resultants N_{12}, N_{21} and the couples M_{12}, M_{21}, individually, are related to certain measures of strain.

It is one of the purposes of the present note to show that the relations (1) and (2) are compatible with more customary forms of such relations, and in particular with the stress displacement relations for lines-of-curvature coordinates established by Flügge (1932), Lurje (1940) and Byrne (1944). In the process of establishing this compatibility, earlier results (Knowles and Reissner 1960) concerning an inverted form of these lines-of-curvature stress displacement relations are significantly simplified.

Having a system of stress strain relations of the form (2) for an isotropic medium, for lines of curvature coordinates, we deduce the corresponding relations for arbitrary orthogonal coordinates upon writing the strain potential W in terms of certain invariants of the transformation laws for N_{jk}, M_{jk} and the curvature measures $1/R_{jk}$. We also invert this system and thereby confirm earlier results obtained by different procedures for stress resultants and couples in terms of strain measures. (Knowles and Reissner 1958; Wan 1965).

The reason for choosing the relations given by Flügge, Lurje and Byrne for an analysis in conjunction with equations (1) and (2) is that these relations can be thought of, relatively simply, as a completely rational consequence of a three-dimensional formulation of shell theory for a limiting-type transversely isotropic medium (Reissner 1966).

Stress strain relations for Lines-of-curvature coordinates

We designate stress resultants, stress couples and strain components referred to lines of curvature coordinates by stars. Furthermore, we designate strain components minus the terms with ω by tildes and the principal radii of curvature by R_1 and R_2. With these conventions the Flügge-Lurje-Byrne stress strain relations are

$$N^*_{11} = C\,(\tilde{\varepsilon}^*_{11} + \nu\tilde{\varepsilon}^*_{22}) + \rho D\,(\tilde{\kappa}^*_{11} - \tilde{\varepsilon}^*_{11}/R_1)$$

$$M^*_{11} = D\,(\tilde{\kappa}^*_{11} + \nu\tilde{\kappa}^*_{22} + \rho\tilde{\varepsilon}^*_{11})$$

$$N^*_{12} = \tfrac{1}{2}\,(1-\nu)\,C\,(\tilde{\varepsilon}^*_{12} + \tilde{\varepsilon}^*_{21}) + \tfrac{1}{2}\,(1-\nu)\rho D\,(\tilde{\kappa}^*_{12} - \tilde{\varepsilon}^*_{12}/R_1) \tag{3}$$

$$M^*_{12} = \tfrac{1}{2}\,(1-\nu)\,D\,(\tilde{\kappa}^*_{12} + \tilde{\kappa}^*_{21} + \rho\tilde{\varepsilon}^*_{12})$$

and

$$\gamma^*_1 = \gamma^*_2 = 0 \tag{4}$$

where

$$C = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \qquad \rho = \frac{1}{R_2} - \frac{1}{R_1} \tag{5}$$

and where N_{22}^*, M_{22}^*, N_{21}^*, M_{21}^* are defined correspondingly.

In order to attempt writing the system (3) and (4) in the form (2), it is necessary to solve (3) so as to have strains in terms of stress resultants and couples. Equations (3) and the corresponding equations for N_{22}^*, M_{22}^*, N_{21}^*, M_{21}^* consist of two separate systems of four equations each, one system for $\tilde{\varepsilon}_{11}^*$, $\tilde{\varepsilon}_{22}^*$, $\tilde{\kappa}_{11}^*$, $\tilde{\kappa}_{22}^*$ and the other for $\tilde{\varepsilon}_{12}^*$, $\tilde{\varepsilon}_{21}^*$, $\tilde{\kappa}_{12}^*$, $\tilde{\kappa}_{21}^*$. The solution of the first system is straightforward. The solution of the second system is complicated by the fact that the four equations are not independent but are such that, identically

$$N_{12}^* - N_{21}^* = \frac{M_{21}^*}{R_2} - \frac{M_{12}^*}{R_1} \tag{6a}$$

At the same time the four unknowns in them are also not independent but satisfy the compatibility equation

$$\tilde{\kappa}_{12}^* - \tilde{\kappa}_{21}^* = \frac{\tilde{\varepsilon}_{12}^*}{R_2} - \frac{\tilde{\varepsilon}_{21}^*}{R_1} \tag{6b}$$

which, on the basis of (3), implies the further relation

$$\left\{1 + \frac{(\rho h)^2}{24}\right\} (M_{12}^* - M_{21}^*) = \frac{\rho h^2}{24} (N_{12}^* + N_{21}^*) \tag{6c}$$

Solution of the four equations for $\tilde{\varepsilon}_{11}^*$, $\tilde{\varepsilon}_{22}^*$, $\tilde{\kappa}_{11}^*$, $\tilde{\kappa}_{22}^*$ leads to

$$\tilde{\varepsilon}_{11}^* = \frac{1}{Eh\Delta_1}\Bigg[N_{11}^* \left\{1 - \frac{(\rho h)^2}{12(1-\nu^2)} + \frac{\rho h^2}{12R_2}\right\} - \nu N_{22}^* \left\{1 - \frac{(\rho h)^2}{12(1-\nu^2)}\right\}$$

$$- \rho M_{11}^* \left\{1 + \frac{\rho h^2}{12(1-\nu^2)R_1}\right\} + \nu\rho M_{22}^* \left\{\frac{\rho h^2}{12(1-\nu^2)R_2}\right\}\Bigg] \tag{7a}$$

$$\tilde{\kappa}^*_{11} = \frac{12}{Eh^3\Delta_1}\left[M^*_{11}\left\{1 - \frac{(\rho h)^2 h^2}{144(1-\nu^2)R_1^{\ 2}}\right\} - \nu M^*_{22}\left\{1 - \frac{(\rho h)^2 h^2}{144(1-\nu^2)R_1 R_2}\right\}\right]$$

$$- \frac{\rho}{Eh\Delta_1}\left[N^*_{11}\left\{1 + \frac{\rho h^2}{12(1-\nu^2)R_1}\right\} - \nu N^*_{22}\left\{\frac{\rho h^2}{12(1-\nu^2)R_1}\right\}\right] \tag{7b}$$

where

$$\Delta_1 = 1 - \frac{(\rho h)^2}{12(1-\nu^2)} - \frac{(\rho h)^2 h^2}{144(1-\nu^2)R_1 R_2} \tag{8}$$

with corresponding expressions for $\tilde{\varepsilon}^*_{22}$ and $\tilde{\kappa}^*_{22}$.

Solution of the four equations for $\tilde{\varepsilon}^*_{12}$, $\tilde{\varepsilon}^*_{21}$, $\tilde{\kappa}^*_{12}$, $\tilde{\kappa}^*_{21}$, subject to the observation of (6), leads to two relations of the form

$$\gamma^* \equiv \tilde{\varepsilon}^*_{12} + \tilde{\varepsilon}^*_{21} = \frac{1+\nu}{Eh\Delta_2}\{N^*_{12} + N^*_{21}\}$$

$$\tau^* \equiv \tilde{\kappa}^*_{12} + \tilde{\kappa}^*_{21} - \left(\frac{\tilde{\varepsilon}^*_{12}}{R_1} + \frac{\tilde{\varepsilon}^*_{21}}{R_2}\right) \tag{9}$$

$$= \frac{12(1+\nu)}{Eh^3}\{M^*_{12} + M^*_{21}\} - \frac{1+\nu}{2Eh\Delta_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\{N^*_{12} + N^*_{21}\}$$

where

$$\Delta_2 = 1 + \frac{(\rho h)^2}{24} \tag{10}$$

Equations (7) to (10) agree with previously given inversion formulas for the system (3) (Knowles and Reissner 1960).

Evidently, equations (7) for $\tilde{\varepsilon}^*_{11} = \varepsilon^*_{11}$ and $\tilde{\kappa}^*_{11} = \kappa^*_{11}$ as they stand are of the form (2), with a suitable function W^*. At the same time, equations (9) are not of such form. We now show that equations (9) can be transformed in such a way as to become equivalent to the appropriate portions of (2). In doing this, the displacement variable ω, which does not occur in $\tilde{\varepsilon}^*_{12}$, $\tilde{\varepsilon}^*_{21}$, $\tilde{\kappa}^*_{12}$, $\tilde{\kappa}^*_{21}$, appears as a natural consequence of the transformation procedure.

Equations (9) may be written, with two arbitrary functions Ω and Λ, in the form

$$\begin{Bmatrix} \tilde{\varepsilon}^*_{12} \\ \tilde{\varepsilon}^*_{21} \end{Bmatrix} = \frac{1+\nu}{2Eh\Delta_2} \{N^*_{12} + N^*_{21}\} \pm \Omega \tag{11}$$

$$\begin{Bmatrix} \tilde{\kappa}^*_{12} \\ \tilde{\kappa}^*_{21} \end{Bmatrix} = \frac{6(1+\nu)}{Eh^3} \{M^*_{12} + M^*_{21}\} - \frac{1+\nu}{4Eh\Delta_2} \left(\frac{1}{R_1} + \frac{1}{R_2}\right) \{N^*_{12} + N^*_{21}\}$$

$$+ \frac{1}{2} \left\{\frac{\tilde{\varepsilon}^*_{12}}{R_1} + \frac{\tilde{\varepsilon}^*_{21}}{R_2}\right\} \pm \Lambda \tag{12}$$

Use of the compatibility equations (6b,c) expresses Λ in terms of Ω,

$$\Lambda = \frac{(1+\nu)\rho}{4Eh\Delta_2} \{N^*_{12} + N^*_{21}\} + \frac{1}{2} \left(\frac{1}{R_2} + \frac{1}{R_1}\right) \Omega \tag{13}$$

Introduction of (13) into (12) and observation of the consequences of (11) leaves us with the much simplified relations

$$\begin{Bmatrix} \tilde{\kappa}^*_{12} \\ \tilde{\kappa}^*_{21} \end{Bmatrix} = \frac{12(1+\nu)}{Eh^3} \begin{Bmatrix} M^*_{12} \\ M^*_{21} \end{Bmatrix} \pm \begin{Bmatrix} \Omega/R_1 \\ \Omega/R_2 \end{Bmatrix} \tag{14}$$

Upon comparing the definitions of $\tilde{\varepsilon}^*_{12}$ and ε^*_{12}, etc., we see that the stress strain relations (11) and (14) assume the desired form if we set

$$\Omega = \omega \tag{15}$$

With (15) and (1) we may write equations (11) and (14) as

$$\begin{Bmatrix} \epsilon_{12}^* \\ \epsilon_{21}^* \end{Bmatrix} = \frac{N_{12}^* + N_{21}^*}{2(1-\nu)C\Delta_2} \tag{16}$$

and

$$\begin{Bmatrix} \kappa_{12}^* \\ \kappa_{21}^* \end{Bmatrix} = \frac{1}{(1-\nu)D} \begin{Bmatrix} M_{12}^* \\ M_{21}^* \end{Bmatrix} \tag{17}$$

The correctness of equations (16) and (17) may be verified by establishing that their substitution on the right of the last two of equations (3) reduces these relations to identities.

It may be noted that equations (16) and (17) agree with stress strain relations which have recently been stipulated in a different context (Reissner 1963), upon omitting the small additive term $(\rho h)^2/24$ in the definition of Δ_2.

It is now a simple matter to construct the function W^* which upon substitution in (2) leads to the stress strain relations (17), (16),(7) and (4). Rather than doing this, we shall first deduce a simpler system of stress strain relations from (17), (16) and (7) which is consistent, in a specific sense, with the relations of Flügge, Lurje and Byrne.

Modification of stress strain relations (17), (16) and (7)

Considering that the stress strain relations (3) may be taken as a consequence of a three-dimensional analysis, in which terms involving powers of h higher than the third are disregarded, we ask for a simplified version of (17), (16) and (7) such that upon inversion of this simplified system there follows a set of relations for stress resultants and couples in terms of strain components which coincides with the system (3) insofar as all terms linear and cubic in h are concerned. We obtain such a simplified system upon expanding

$$\frac{1}{\Delta_1} = 1 + \frac{(\rho h)^2}{12(1-\nu^2)} + \ldots\ldots , \qquad \frac{1}{\Delta_2} = 1 - \frac{(\rho h)^2}{24} + \ldots\ldots \tag{18}$$

and upon retaining terms in the expanded equations (7) and (17) as follows

$$\varepsilon^*_{11} = \frac{1}{Eh} \{N^*_{11} - \nu N^*_{22} - \rho M^*_{11} + \frac{\rho h^2}{12R_2} N^*_{11}\}$$

$$\kappa^*_{11} = \frac{12}{Eh^3} \{M^*_{11} - \nu M^*_{22}\} - \frac{\rho}{Eh} \{N^*_{11}\} \tag{19}$$

$$\varepsilon^*_{12} = \varepsilon^*_{21} = \frac{1+\nu}{2Eh} \left[1 - \frac{(\rho h)^2}{24} \right] \{N^*_{12} + N^*_{21}\}$$

Corresponding equations hold for ε^*_{22} and κ^*_{22}, while (17) is written in the equivalent form

$$\kappa^*_{12} = \frac{12(1+\nu)}{Eh^3} M^*_{12} , \qquad \kappa^*_{21} = \frac{12(1+\nu)}{Eh^3} M^*_{21} \tag{20}$$

The function W^* implied by (19) and (20) is

$$\begin{aligned} W^* = \frac{1}{2Eh} \{ & (1 + \frac{\rho h^2}{12R_2}) N^{*2}_{11} + (1 - \frac{\rho h^2}{12R_1}) N^{*2}_{22} - 2\nu N^*_{11} N^*_{22} \\ & + \frac{1+\nu}{2} (1 - \frac{\rho^2 h^2}{24})(N^*_{12} + N^*_{21})^2\} \\ & + \frac{\rho}{Eh} \{N^*_{22} M^*_{22} - N^*_{11} M^*_{11}\} \\ & + \frac{6}{Eh^3} \{M^{*2}_{11} + M^{*2}_{22} - 2\nu M^*_{11} M^*_{22} + (1+\nu) (M^{*2}_{12} + M^{*2}_{21})\} \end{aligned} \tag{21}$$

Equation (21) is meaningful as it stands provided the shell possesses the limiting kind of transverse isotropy associated with the derivation of the stress strain relations (3) and (4). In contemplating simplifications of (21) the most natural procedure would be to omit all terms of order $(h/R^2)N^2$, or incorporate additional such terms. In this way we may, through use of equation (6c), in particular modify the term $M_{12}^{*2} + M_{21}^{*2}$ in (21) to $(M_{12}^* + M_{21}^*)^2$ or to $2M_{12}^* M_{21}^*$.

In considering other possible simplifications one notes that the terms of order NM/hR in (21) are the same terms which are the difference in W^* between a first and second approximation as formulated by Trefftz (1935) for an isotropic medium. This difference, for a transversely isotropic medium, was subsequently shown (Reissner 1952) to contain additional terms of the same order, depending on transverse normal strain deformations, except in the limiting case of transverse isotropy which forms the basis of the stress strain relations (3).

Strain potential for general orthogonal coordinates

Given that N_{11}^*, M_{11}^* etc. are resultants and couples with respect to the special orthogonal coordinate system of lines of curvature, and N_{11}, M_{11}, etc., the corresponding resultants and couples with reference to an orthogonal system intersecting the lines of curvature at an angle ϕ, we have the transformation relations

$$
\begin{aligned}
N_{11}^* &= N_{11}\cos^2\phi + N_{22}\sin^2\phi + (N_{12} + N_{21})\cos\phi\sin\phi \\
N_{22}^* &= N_{11}\sin^2\phi + N_{22}\cos^2\phi - (N_{12} + N_{21})\cos\phi\sin\phi \\
N_{12}^* &= N_{12}\cos^2\phi - N_{21}\sin^2\phi + (N_{22} - N_{11})\cos\phi\sin\phi \\
N_{21}^* &= N_{21}\cos^2\phi - N_{12}\sin^2\phi + (N_{22} - N_{11})\cos\phi\sin\phi
\end{aligned}
\tag{22}
$$

and analogous relations for M_{11}^*, etc.

At the same time the curvature radii R_{11}, R_{12} and R_{22} are related to R_1 and R_2 through

$$\frac{1}{R_1} = \frac{\cos^2\phi}{R_{11}} + \frac{\sin^2\phi}{R_{22}} + \frac{2\cos\phi\,\sin\phi}{R_{12}}$$

$$\frac{1}{R_2} = \frac{\cos^2\phi}{R_{22}} + \frac{\sin^2\phi}{R_{11}} - \frac{2\cos\phi\,\sin\phi}{R_{12}} \tag{23}$$

$$0 = \frac{\cos^2\phi - \sin^2\phi}{R_{12}} + \frac{\cos\phi\,\sin\phi}{R_{22}} - \frac{\cos\phi\,\sin\phi}{R_{11}}$$

Introduction of (22), of the corresponding equations for M_{11}, etc., and of the first two of equations (23), with ϕ determined by means of the third of equations (23), will transform W^* as given by (21) into the corresponding function W which is to be used in (2) in conjunction with the components of strain of equation (1). In carrying out the transformation of W^* into W we take advantage of various invariants associated with the transformation laws (22) and (23). The invariants are

$$N_{11} + N_{22}\,, \quad N_{12} - N_{21}\,, \quad N_{11}N_{22} - N_{12}N_{21}\,,$$

$$\frac{1}{R_{11}} + \frac{1}{R_{22}}\,, \quad \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)^2 + \frac{4}{R_{12}^2}\,,$$

$$\frac{N_{11}}{R_{11}} + \frac{N_{22}}{R_{22}} + \frac{N_{12}+N_{21}}{R_{12}}\,, \quad \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(N_{12} + N_{21}) + \frac{2}{R_{12}}(N_{11} - N_{22})$$

and

$$\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)(N_{22}M_{22} - N_{11}M_{11}) +$$

$$+ (N_{12} + N_{21})\frac{M_{11}+M_{22}}{R_{12}} + (N_{11} + N_{22})\frac{M_{12}+M_{21}}{R_{12}}\,.$$

Therewith

$$W = \frac{1}{2Eh}\left\{N_{11}^2 + N_{22}^2 - 2\nu N_{11}N_{22} + \frac{1+\nu}{2}(N_{12} + N_{21})^2\right\}$$

$$+ \frac{6}{Eh^3}\left\{M_{11}^2 + M_{22}^2 - 2\nu M_{11}M_{22} + (1+\nu)(M_{12}^2 + M_{21}^2)\right\}$$

$$+\frac{1}{Eh}\{(\frac{1}{R_{22}}-\frac{1}{R_{11}})(N_{22}M_{22}-N_{11}M_{11})+\frac{M_{11}+M_{22}}{R_{12}}(N_{12}+N_{21})$$

$$+\frac{M_{12}+M_{21}}{R_{12}}(N_{11}+N_{22})\}$$

$$+\frac{h}{24E}\Big[\{(\frac{1}{R_{22}}+\frac{1}{R_{11}})(N_{11}+N_{22})-(\frac{N_{11}}{R_{11}}+\frac{N_{22}}{R_{22}}+\frac{N_{12}+N_{21}}{R_{12}})\}^{2}$$

$$-(\frac{1}{R_{11}R_{22}}-\frac{1}{R_{12}^{2}})(N_{11}+N_{22})^{2} \qquad (24)$$

$$-(1+\nu)\{\frac{N_{11}-N_{22}}{R_{12}}+(\frac{1}{R_{22}}-\frac{1}{R_{11}})\frac{N_{12}+N_{21}}{2}\}^{2}\Big]$$

From (24) follows that the stress strain relations which generalize equations (19) and (20) to arbitrary orthogonal surface coordinates are:

$$\varepsilon_{11}=\frac{1}{Eh}\{N_{11}-N_{22}-(\frac{1}{R_{22}}-\frac{1}{R_{11}})M_{11}+\frac{M_{12}+M_{21}}{R_{12}}\}$$

$$+\frac{h}{12E}\{(\frac{1}{R_{22}^{2}}-\frac{1}{R_{11}R_{22}}-\frac{\nu}{R_{12}^{2}})N_{11}+\frac{2+\nu}{R_{12}^{2}}N_{22}-(\frac{3+\nu}{2R_{22}}-\frac{1+\nu}{2R_{11}})\frac{N_{12}+N_{21}}{R_{12}}\}$$

$$\kappa_{11}=\frac{12}{Eh^{3}}\{M_{11}-M_{22}\}-\frac{1}{Eh}\{(\frac{1}{R_{22}}-\frac{1}{R_{11}})N_{11}-\frac{N_{12}+N_{21}}{R_{12}}\} \qquad (25)$$

$$\varepsilon_{12}=\varepsilon_{21}=\frac{1}{Eh}\{(1+\nu)\frac{N_{12}+N_{21}}{2}+\frac{M_{11}+M_{22}}{R_{12}}\}$$

$$+\frac{h}{12E}\Big[\{\frac{1+\nu}{4}(\frac{1}{R_{22}}-\frac{1}{R_{11}})^{2}-\frac{1}{R_{12}^{2}}\}(N_{12}+N_{21})$$

$$+(\frac{3+\nu}{2R_{22}}-\frac{1+\nu}{2R_{11}})\frac{N_{11}}{R_{12}}+(\frac{3+\nu}{2R_{11}}-\frac{1+\nu}{2R_{22}})\frac{N_{22}}{R_{12}}\Big]$$

$$\kappa_{12}=\frac{12}{Eh^{3}}\{(1+\nu)M_{12}\}+\frac{1}{Eh}\{\frac{N_{11}+N_{22}}{R_{12}}\}$$

with corresponding formulas for ε_{22}, κ_{22} and κ_{21}.

We may invert (25) so as to obtain formulas giving stress resultants and couples in terms of strain, up to terms of order h^3, by first substituting approximations for N_{11}, N_{12}, N_{21}, N_{22} following from $Eh\varepsilon_{11} = N_{11} - \nu N_{22}$, etc., in κ_{11} and κ_{12}. This gives

$$\frac{M_{11}}{D} = \kappa_{11} + \nu\kappa_{22} + \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\varepsilon_{11} - \frac{\varepsilon_{12}+\varepsilon_{21}}{R_{12}} ,$$

$$\frac{M_{12}}{D} = (1-\nu)\kappa_{12} - \frac{\varepsilon_{11}+\varepsilon_{22}}{R_{12}} , \qquad (26a)$$

with corresponding expressions for M_{21} and M_{22}. Introduction of (26a) into equations (25) for ε_{11} and ε_{12} then gives for N_{11}, etc.,

$$\frac{N_{11}}{C} = \varepsilon_{11} + \nu\varepsilon_{22} + \frac{D}{C}\left\{\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\left(\kappa_{11} - \frac{\varepsilon_{11}}{R_{11}}\right) - \frac{1}{R_{12}}\left[\frac{1+\nu}{2}\left(\kappa_{12} - \frac{\varepsilon_{22}}{R_{12}}\right)\right.\right.$$

$$\left.\left. + \frac{3-\nu}{2}\left(\kappa_{21} - \frac{\varepsilon_{11}}{R_{12}}\right) - \frac{1+\nu}{2R_{11}}\varepsilon_{12} - \left(\frac{1}{R_{11}} + \frac{1-\nu}{2R_{22}}\right)\varepsilon_{21}\right]\right\}$$

$$\frac{N_{12}}{C} = \frac{1-\nu}{2}(\varepsilon_{12}+\varepsilon_{21}) + \frac{D}{C}\left\{\frac{1-\nu}{2}\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\left(\kappa_{12} - \frac{\varepsilon_{12}}{R_{11}}\right)\right. \qquad (26b)$$

$$- \frac{1}{R_{12}}\left[\frac{1+\nu}{2}\left(\kappa_{11} - \frac{\varepsilon_{21}}{R_{12}}\right) + \frac{3-\nu}{2}\left(\kappa_{22} - \frac{\varepsilon_{12}}{R_{12}}\right)\right.$$

$$\left.\left. - \frac{1+\nu}{2R_{11}}\varepsilon_{11} - \left(\frac{1}{R_{22}} + \frac{1-\nu}{2R_{11}}\right)\varepsilon_{22}\right]\right\}$$

with corresponding expressions for N_{21} and N_{22}.

Equations (26) may finally be written in terms of ω-free strain components $\tilde{\varepsilon}_{11}$, $\tilde{\kappa}_{11}$, etc., by suitably rearranging terms in (26) and using the relation $2\omega = \tilde{\varepsilon}_{12} - \tilde{\varepsilon}_{21}$ as

$$\frac{M_{11}}{D} = \tilde{\kappa}_{11} + \nu\tilde{\kappa}_{22} + \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\tilde{\varepsilon}_{11} - \frac{1+\nu}{2R_{12}}\tilde{\varepsilon}_{12} - \frac{3-\nu}{2R_{12}}\tilde{\varepsilon}_{21}$$

(27a)

$$\frac{M_{12}}{D} = \frac{1-\nu}{2}\left\{\tilde{\kappa}_{12} + \tilde{\kappa}_{21} + \left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\tilde{\varepsilon}_{12}\right\} - \frac{1+\nu}{2R_{12}}\tilde{\varepsilon}_{11} - \frac{3-\nu}{2R_{12}}\tilde{\varepsilon}_{22}$$

$$\frac{N_{11}}{C} = \tilde{\varepsilon}_{11} + \nu\tilde{\varepsilon}_{22} + \frac{D}{C}\left\{\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\left(\tilde{\kappa}_{11} - \frac{\tilde{\varepsilon}_{11}}{R_{11}}\right) - \frac{1}{R_{12}}\left[\frac{1+\nu}{2}\left(\tilde{\kappa}_{12} - \frac{\tilde{\varepsilon}_{22}}{R_{12}}\right)\right.\right.$$

$$\left.\left. + \frac{3-\nu}{2}\left(\tilde{\kappa}_{21} - \frac{\tilde{\varepsilon}_{11}}{R_{12}}\right) - \frac{1+\nu}{2R_{11}}\tilde{\varepsilon}_{12} - \left(\frac{1}{R_{11}} + \frac{1-\nu}{2R_{22}}\right)\tilde{\varepsilon}_{21}\right]\right\}$$

(27b)

$$\frac{N_{12}}{C} = \frac{1-\nu}{2}\left(\tilde{\varepsilon}_{12} + \tilde{\varepsilon}_{21}\right) + \frac{D}{C}\left\{\frac{1-\nu}{2}\left(\frac{1}{R_{22}} - \frac{1}{R_{11}}\right)\left(\tilde{\kappa}_{12} - \frac{\tilde{\varepsilon}_{12}}{R_{11}}\right) - \right.$$

$$- \frac{1}{R_{12}}\left[\frac{1+\nu}{2}\left(\tilde{\kappa}_{11} - \frac{\tilde{\varepsilon}_{21}}{R_{12}}\right)\right.$$

$$\left.\left. + \frac{3-\nu}{2}\left(\tilde{\kappa}_{22} - \frac{\tilde{\varepsilon}_{12}}{R_{12}}\right) - \frac{1+\nu}{2R_{11}}\tilde{\varepsilon}_{11} - \left(\frac{1}{R_{22}} + \frac{1-\nu}{2R_{11}}\right)\tilde{\varepsilon}_{22}\right]\right\}$$

Equations (27) agree, upon suitable change of notation, with analogous formulas obtained previously (Knowles and Reissner, 1958, Wan 1965) by different procedures. We observe that while the formulas for M_{ij} change when ε_{ij} and κ_{ij} are replaced by $\tilde{\varepsilon}_{ij}$ and $\tilde{\kappa}_{ij}$, this is not the case for equations giving N_{ij}.

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