Journal of Applied Mathematics and Physics (ZAMP)
Vol. 38, January 1987

0044-2275/87/001079-13 $ 4.10/0

Boundary layer solutions for some nonlinear elastic membrane problems

By Frederic Y. M. Wan, Dept. of Applied Mathematics FS-20, University of Washington, Seattle, WA 98195, USA and
Hubertus J. Weinitschke, Institut für Angewandte Mathematik der Universität Erlangen-Nürnberg, 8520 Erlangen, Germany

1. Introduction

In this paper, we are interested in some nonlinear elastic membrane problems which exhibit boundary layer phenomena. The first type of problems is concerned with a circular membrane subject to a polarly symmetric surface load and to a uniform radial tension at its edge. The second type of problems is concerned with an annular membrane subject also to polarly symmetric surface load and to a uniform radial tension or displacement along the outer edge, but free of tractions at the inner edge. Under favorable conditions, the stress distributions in these membranes exhibit steep gradients near at least one edge. For the circular membrane problem, a boundary layer appears in the hoop stress near the edge if the magnitude of the radial tension is sufficiently small. In the annular membrane problem, a steep gradient occurs in both radial and hoop stresses adjacent to the inner edge if the inner radius is small compared to the outer radius. In either case, a direct numerical solution of the relevant boundary value problem (BVP) is difficult and/or costly to obtain. Therefore, it is of considerable interest to seek simpler (but accurate) approximate solutions for these problems by asymptotic analyses. The existence of boundary layer phenomena in elastic membranes and their description by asymptotic methods have been known ever since the observation of Bromberg and Stoker [1]. The layer phenomena of our problems involve some features not encountered in previously treated problems of elastic membranes.

Within the nonlinear membrane theory of Föppl [2], the problems described above may be reduced to two-point BVP's for a second-order nonlinear ordinary differential equation (ODE) with a small parameter. The BVP for the circular membrane with small radial edge tension b may be rescaled and transformed into a canonical form of a singular perturbation problem. But this standard problem does not have a bounded outer solution in the limit $b = 0$. On the other hand, the BVP in its original form admits no simplification for the outer problem (except for the loss of the edge boundary condition) and no explicit (leading term) outer solution. The method of matched asymptotic expansions, when adapted to this problem must therefore cope with these difficulties.

For annular membrane problems, the BVP is in the form of a *singular boundary problem* similar to the model example for the Stokes-Oseen problem discussed in [3], except that an explicit outer solution is not possible for our problem. The interesting feature here is that the matching process yields an appropriate second boundary condition for the leading term outer solution which can then be found numerically. Moreover, the BVP for the leading term outer solution is just the circular membrane problem; its solution supplies a constant of integration for the leading term inner solution.

The Föppl theory is known to be inadequate for large deformation problems. A membrane theory which allows for arbitrarily large deflections and rotations (but small strains) can be obtained from the corresponding plate theory of E. Reissner [4] by setting the bending stiffness equal to zero. This less restrictive membrane theory may also be reduced to two-point BVP's for a second-order ODE for both the circular and annular membrane problems. Here the stress distributions in the first problem may or may not have a boundary layer even if the normalized radial edge tension is small. The existence of a layer phenomena now depends more critically on the relative magnitude of the surface and edge loads, as one would expect. In the annular membrane problem, the boundary layer near the inner edge is quite similar to that in the small rotation theory.

2. Formulation of the membrane problems

2.1. The Föppl formulation for the circular membrane

Consider a circular elastic membrane subject to a vertical surface load $p(r)$ and to a uniform radial tension σ_a (Problem S) or to a uniform horizontal displacement u_a (Problem H) at the edge $r = a$, so that $\sigma_r(a) = \sigma_a$ or $u(a) = u_a$. Within the framework of the nonlinear membrane theory of Föppl [2] (finite displacements, but moderately small rotations), the polarly symmetric stresses and displacements are determined by the solution of the following BVP

$$y'' + \frac{3}{x} y' + \frac{2}{y^2}[Q(x)]^2 = 0, \quad 0 < x < 1, \quad y'(0) = 0 \quad By(1) = b \tag{2.1}$$

$$By := \begin{cases} y \\ y' + (1-\nu)\, y \end{cases} \qquad b := \begin{cases} c_1 \sigma_a & \text{(Problem } S\text{)} \\ c_2 u_a & \text{(Problem } H\text{)} \end{cases}$$

$$x = \frac{r}{a}, \quad y = c_1 \sigma_r, \quad \bar{p} = c_3 p(r), \quad Q(x) = \frac{2}{x^2} = \int_0^x t \bar{p}(t)\, dt$$

where ν is Poisson's ratio and the constants c_i, $i = 1, 2, 3$ depend only on a, $p_m = \max |p(r)|$, Young's modulus E and the membrane thickness h.

The BVP (2.1) has been treated repeatedly in the literature [5–10]. It is known from [7] that (2.1) has a unique C^2 [0, 1] solution for $\sigma_a > 0$ (Problem S) and a unique tensile C^2 [0, 1] solution for any real u_a (Problem H), provided $\bar{p}(x)$ is piecewise continuous.

In what follows, we shall be interested in Problem S for small values of b. It is seen that (2.1, S) cannot have a C^2 [0, 1] solution if $b = 0$, $Q(1) \neq 0$, as y'' or even y' must become infinite at $x = 1$. In fact, a C^1 [0, 1] solution for $b = 0$ exists if and only if $Q(1) = 0$, and there is a unique C^1 [0, 1) solution for $b = 0$, $Q(1) \neq 0$ with $y'(1) = -\infty$, as shown in [7]. Therefore, a numerical solution of (2.1, S) with $Q(1) \neq 0$ becomes increasingly difficult (and costly) as b decreases because of the rapid increase of $|y'(x)|$ near the edge $x = 1$. For smoothly varying $Q(x)$, we expect $y(x)$ to be smoothly varying away from the edge, and therefore the sharp gradient in the stresses will be confined to a narrow layer adjacent to the edge. This boundary layer behavior for small b (Problem S) does not seem to have been noted in previous work on this problem.

2.2. *The Föppl formulation for the annular membrane*

Next we consider an annular elastic membrane under the same loading conditions as the circular membrane problem, with the inner circular edge $r = r_i$ assumed to be free of traction, so that $\sigma_r(r_i) = 0$. The Föppl theory then leads to the following BVP

$$z'' + \frac{3}{x} z' + \frac{2}{z^2} [R(x, \varepsilon)]^2 = 0, \quad \varepsilon < x < 1, \quad z(\varepsilon) = 0, \quad B z(1) = b$$
$$\varepsilon = \frac{r_i}{a}, \quad z = c_1 \sigma_r, \quad R(x, \varepsilon) := \frac{2}{x^2} \int_\varepsilon^x t \bar{p}(t)\, dt \tag{2.2}$$

with x, $Bz(1)$, b and $\bar{p}$ as defined before. We are interested in solutions of BVP (2.2) for small values of ε, particularly in the case of uniform load $\bar{p}(t) = 1$, for which $R(x, \varepsilon) = 1 - (\varepsilon/x)^2$.

The annular membrane problem (2.2, H) for fixed edge ($b = 0$) and $\bar{p} = 1$ was first investigated by Schwerin [11], who obtained formal power series solutions for some values of $\varepsilon \geqq 0.1$. The problem was recently analyzed by one of the authors in [12], where existence and uniqueness of solutions of the two BVP's (2.2, S) and (2.2, H) was proved for a restricted range of the parameters ε and b.

If ε is small and $R(x, \varepsilon)$ is bounded and smoothly varying in the interval [ε, 1], numerical solutions of the BVP's (2.2) show that $z(x)$ rises from $z(\varepsilon) = 0$ to values $0(1)$ within a layer of order $0(\varepsilon)$, with a steep gradient, as sketched in Fig. 1. The dotted line indicates the solution for the circular membrane subject to the same boundary conditions at $x = 1$. Again, numerical solutions of (2.2)

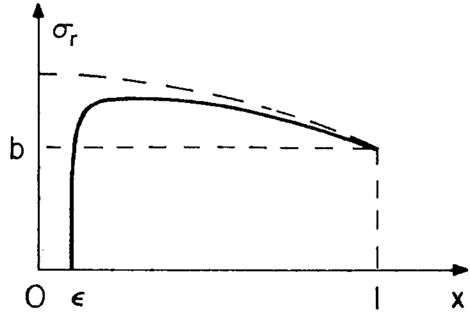

Figure 1
Boundary layer effect in annular membrane

become increasingly difficult (and costly) for decreasing values of ε because of large values of $z'(x)$ in a narrow region adjacent to $x = \varepsilon$. This boundary layer phenomenon and the possibility of an asymptotic analysis of (2.2) for $\varepsilon \ll 1$ have apparently not been noted in previous work.

2.3. Reissner's membrane theory

In the geometrically fully nonlinear membrane theory due to E. Reissner [4], the basic equations for an annular membrane under a vertical load $p(r)$ and zero traction at the inner edge can be reduced to the following BVP [12, 13]

$$z'' + \frac{3}{x} z' = \frac{4}{k^2 x^2} \left\{ \frac{z}{[z^2 + k^2 x^2 R^2(x, \varepsilon)]^{1/2}} - 1 \right\} \qquad \varepsilon < x < 1$$

$$z(\varepsilon) = 0 \quad B^* z(1) = b, \quad B^* z := \begin{cases} z & \text{Problem } S \\ z' + z - \nu\,[z^2 + k^2 R^2(1, \varepsilon)]^{1/2} & \text{Problem } H \end{cases} \tag{2.3}$$

where $k = (2 p_m a / E h)^{1/3}$, $\varepsilon > 0$, and with R, z and p as defined in (2.2). Problem (2.3, S) with $R = 1 - (\varepsilon/x)^2$, but with $z(\varepsilon) = z_i > 0$, was analyzed in [12], where existence results were proved for a restricted range of k, ε, z_i and b. For small holes and smooth $R(x, \varepsilon)$, the same type of boundary layer behavior as described for solutions of (2.2) can be expected here, suggesting an approximate solution by an asymptotic analysis for $\varepsilon \ll 1$.

The corresponding equations for the circular Reissner membrane can be obtained from (2.3) by formally setting $\varepsilon = 0$ and replacing $z(\varepsilon) = 0$ by the regularity condition $z'(0) = 0$. An existence and uniqueness theory for tensile solutions of Problem S is given in [10].

3. The circular membrane

We consider the BVP (2.1) with $y(1) = b$ and $Q(x) = 1$ (uniform load). For $0 < b \ll 1$, we expect a boundary layer in the solution $y(x)$ near $x = 1$ and obtain in this section an asymptotic solution for this problem by the method of matched asymptotic expansions.

3.1. The leading term inner solution

We introduce the stretching variables $s = (1 - x)/\lambda$ and $y = b\,Y(s)$, where $\lambda(b) \to 0$ as $b \to 0$. The ODE in (2.1) and $y(1) = b$ may then be written as

$$\frac{b^3}{\lambda^2}\left(\ddot{Y} - \frac{3\lambda}{1-\lambda s}\dot{Y}\right) + \frac{2}{Y^2} = 0, \quad Y(0) = 1 \tag{3.1}$$

where a dot denotes differentiation with respect to s. The "distinguished limit" [3] of (3.1) is obtained when $\lambda^2 = b^3$. Hence we have

$$\ddot{Y} - \frac{3\,b^{3/2}}{1 - s\,b^{3/2}}\dot{Y} + \frac{2}{Y^2} = 0, \quad Y(0) = 1. \tag{3.2}$$

The leading term inner solution $Y_0(s)$ is therefore determined by the autonomous equation $\ddot{Y}_0 = -2/Y_0^2$, which has a first integral

$$\frac{1}{2}\dot{Y}_0^2 - \frac{2}{Y_0} = C. \tag{3.3}$$

This equation was solved by Schwerin in connection with an annular membrane under edge loads [11], but no surface loads. A parametric form of the solution of (3.3) is $Y_0 = (4/C)\sin^2\varphi/2$ and $s = d_0 + d_1(\varphi - \sin\varphi)$, with the constants d_i chosen such that $Y_0(0) = 1$. We here prefer an alternate form of the solution. If $C = 0$ in (3.3), then

$$Y_0(s) = t^2, \quad t := [1 + 3(1-x)\,b^{-3/2}]^{1/3} = (1 + 3s)^{1/3}, \tag{3.4}$$

is the leading term solution in the boundary layer. For $C \neq 0$, we rewrite (3.3) in terms of the variable t as

$$\left(\frac{dY_0}{dt}\right)^2 - \frac{4t^4}{Y_0} = 2Ct^4, \quad Y_0 = 1 \quad \text{at} \quad t = 1\,(s = 0). \tag{3.5}$$

It follows from classical theorems that the initial value problem (3.5) has a solution which is analytic in some neighborhood of $t = 1$. We expand $Y(t)$ at $t = 0\,(s = -1/3)$, anticipating the solution to be analytic at $t = 0$ (which corresponds to $\varphi = 0$ in the parametric representation of $Y_0(s)$ mentioned above). Hence, we obtain from (3.5).

$$Y_0(t) = t^2 + \frac{1}{10}Ct^4 - \frac{3}{700}C^2t^6 + \cdots, \tag{3.6}$$

where C is to be determined by matching Y_0 with the outer solution.

An inner asymptotic expansion of $Y(s; b)$ will be assumed in the form

$$Y(s; b) = Y_0(s) + b^{1/2}\,Y_1(s) + b\,Y_2(s) + b^{3/2}\,Y_3(s) + \cdots. \tag{3.7}$$

This expansion is required for matching the inner and outer solution, as will be seen from the description leading to (3.7) (vide infra).

3.2. The leading term outer solution

The outer solution must satisfy the ODE in (2.1) and $y'(0) = 0$, it must drop from $O(1)$ to $o(1)$ as $x \to 1$. If it is to match with the inner solution in an intermediate region, it should behave roughly like $(1 - x)^{2/3}$. This suggests the new variable $\xi = (1 - x^2)^{1/3}$. We note that $y'(0) = 0$ is satisfied automatically by $\bar{y}(\xi) = y(x)$, provided $\mathrm{d}\bar{y}/\mathrm{d}\xi$ is finite at $\xi = 1$. A short calculation shows that the Eq. (2.1) for y (with $Q = 1$) may be written in terms of the variable $H(\xi) = \xi\,\bar{y}(\xi)$ as

$$(1 - \xi^3)\,H''(\xi) - \frac{1}{\xi}(4 + 2\,\xi^3)\,H'(\xi) + \frac{1}{\xi^2}(4 + 2\,\xi^3)\,H(\xi) + \frac{9}{2}\,\xi^7\,H(\xi)^{-2} = 0. \tag{3.8}$$

The leading term outer solution $H_0(\xi)$ must satisfy the same equation, since $H(\xi)$ is of order unity away from $x = 1$. We seek a series solution for $H_0(\xi)$ in the form

$$H_0(\xi) = \sum_{n=0}^{\infty} h_n\,\xi^{n+\alpha}. \tag{3.9}$$

In order that H_0 satisfies (3.8) and matches properly with the inner solution, we must have $\alpha = 3$ and

$$h_0 = \left(\frac{3}{2}\right)^{2/3}, \quad h_1 = 0, \quad h_2 \quad \text{arbitrary}, \quad h_3 = \frac{5}{3}\,h_0, \quad h_4 = -\frac{27}{28}\left(\frac{h_2}{h_0}\right)^2, \ldots .$$

It is straightforward to write down a recurrence relation for the h_n, $n \geqq 4$. The process of matching will determine the only unknown constant h_2.

3.3. Matching of inner and outer solutions to leading order

The inner and outer solutions should be identical to any order of b in a common region of validity. We will match the two solutions through an intermediate variable $\eta = (1 - x)/\delta(b)$ where $\eta = 0(1)$, $\delta(b) \to 0$ with $b \to 0$ such that $b/\delta(b) \to 0$ as $b \to 0$ (see [3] for a detailed discussion of matching by intermediate variables). Correspondingly, we have $\xi = (2\,\delta\,\eta)^{1/3}\,(1 - \delta\,\eta/2)^{1/3}$. Upon writing the outer solution in terms of η, we get

$$\bar{y}_0(\xi) = (3\,\delta\,\eta)^{2/3} + h_2\,(2\,\delta\,\eta)^{4/3} + (3\,\delta\,\eta)^{5/3} + \cdots . \tag{3.10}$$

Expanding the leading term inner solution in terms of η, we find from (3.6)

$$y(x) = (3\,\delta\,\eta)^{2/3}\left\{(1 + b^{3/2}/3\,\delta\,\eta)^{2/3} + \frac{1}{b}\left[\frac{C}{10}\,(3\,\delta\,\eta)^{2/3}\,(1 + b^{3/2}/3\,\delta\,\eta)^{4/3} + \cdots\right] + \cdots\right\}. \tag{3.11}$$

It is seen that the term in the brackets must vanish because $\delta^{2/3}/b \to \infty$ as $b \to 0$. We conclude that $C = 0$, and as all terms of the series (3.6) except t^2 have a factor C, the leading term inner solution is simply $Y_0 = t^2$, as in (3.4). A comparison of (3.10) and (3.11) shows that the leading terms $(3\,\delta\,\eta)^{2/3}$ in both expansions agree. The terms $h_2\,(2\,\delta\,\eta)^{4/3}, \ldots$ can be properly matched only after we calculate higher order terms of the inner solution.

3.4. *Higher order terms*

We now insert (3.7) into (3.2) to get terms of order $b^{1/2}$ and b, respectively,

$$\ddot{Y}_1 = \frac{4}{Y_0^3}\,Y_1, \quad \ddot{Y}_2 = \frac{4}{Y_0^3}\,Y_2 - \frac{6}{Y_0^4}\,Y_1^2, \quad Y_i(0) = 0 \quad i = 1, 2. \tag{3.12}$$

With $Y_0^3 = (1 + 3\,s)^2$ from (3.4), the solution $Y_1\,(s)$ is given by

$$Y_1\,(s) = C_1\,[(1 + 3\,s)^{4/3} - (1 + 3\,s)^{-1/3}]. \tag{3.13}$$

Expressing Y_1 in terms of η and expanding the result for small b, we obtain

$$y\,(x) = (3\,\delta\,\eta)^{2/3}\,[1 + 0\,(b^{3/2}/\delta\,\eta)] + C_1\,b^{-1/2}\,[(3\,\delta\,\eta)^{4/3} - b^{5/2}\,(3\,\delta\,\eta)^{-1/3} + \cdots]. \tag{3.14}$$

Since there is no term in the outer solution to match the term $C_1\,b^{-1/2}\,(3\,\delta\,\eta)^{4/3}$, we must set $C_1 = 0$ and conclude $Y_1\,(s) \equiv 0$.

According to (3.12) the solution $Y_2\,(s)$ also has the form given by (3.13), with C_1 replaced by C_2. In terms of the variable η we find

$$y\,(x) = (3\,\delta\,\eta)^{2/3} + \tfrac{2}{3}\,b^{3/2}\,(3\,\delta\,\eta)^{-1/3} + \cdots + C_2\,[(3\,\delta\,\eta)^{4/3} + \tfrac{4}{3}\,(3\,\delta\,\eta)^{1/3}\,b^{3/2} + \cdots - b^{5/2}\,(3\,\delta\,\eta)^{-1/3}\,(1 - \tfrac{1}{3}\,b^{3/2}\,(3\,\delta\,\eta)^{-1} + \cdots)] + \cdots. \tag{3.12}$$

Matching this with the outer expansion (3.10), we find $h_2 = (3/2)^{4/3}\,C_2$, where C_2 can be determined by calculating $Y_3\,(s)$ and matching it with the outer solution. In this way we get $C_2 = -1$ and $h_2 = -(3/2)^{4/3}$. In order to match all terms of $Y_3\,(s)$, we must calculate the second term $H_1\,(\xi)$ in an outer expansion

$$H\,(\xi) = H_0\,(\xi) + b^{1/2}\,H_1\,(\xi) + b\,H_2\,(\xi) + \cdots. \tag{3.16}$$

The resulting linear ODE for H_1, whose first three terms are identical with those of (3.8) can be solved by the Frobenius method ($\xi = 0$ being a regular singular point).

We summarize the results up to the order of approximation carried out in this section. In the boundary layer we have, with $s = (1 - x)/b^{3/2}$

$$\begin{aligned} y\,(x) &= b\,Y_0\,(s) + b^2\,Y_2\,(s) + 0\,(b^{5/2}) \\ &= b\,(1 + 3\,s)^{2/3} - b^2\,[(1 + 3\,s)^{4/3} - (1 + 3\,s)^{-1/3}] + 0\,(b^{5/2}). \end{aligned} \tag{3.17}$$

Away from the layer we have, with $\xi = (1 - x^2)^{1/3}$

$$y(x) = y_0(x) + 0(b^{1/2}) = [(\tfrac{3}{2})^{2/3}\xi^2 - (\tfrac{3}{2})^{4/3}\xi^4 + \tfrac{5}{3}(\tfrac{3}{2})^{2/3}\xi^5 + \cdots] + 0(b^{1/2}). \tag{3.18}$$

It should be noted that the (three term) inner solution for $y(x)$ given in (3.17) has only an algebraic (not exponential) growth as it leaves the boundary layer of dimensionless thickness of order $b^{3/2}$. Thus the exact solution will exhibit a characteristic boundary layer behavior only if b is sufficiently small. At the same time, the local series expansion of the outer solution (3.18) converges rapidly only if $\xi \ll 1$, and should not be used for $x < 0.95$, say. However, using $y'(0) = 0$ and $y(x_0)$, with $x_0 \in [0.95, 1)$, from the series expansion as boundary conditions, a numerical solution of the ODE (2.1) can easily be found in the range $0 \leqq x \leqq x_0$ by standard methods.

The limited range of computational efficiency notwithstanding, the outer solution in the form (3.18) allows us to determine the inner solution (3.17) completely. We see clearly from this inner solution the behavior of the solution in the layer adjacent to the edge of the membrane. In particular, (3.17) may be used to obtain the behavior of $\sigma_\theta = c_1(xy)'$ in the boundary layer.

3.5. The circular membrane according to Reissner's theory

We now consider briefly the finite rotation problem for the circular membrane subject to a uniform axial load $\bar{p} = 1$ and to small radial tension prescribed at the edge. In the range $0 < k^2 \leqq b^2 \ll 1$, the Föppl theory applies. On the other hand, if $k^2 = 0(1)$ and $b^2 \ll 1$, we have $y^2 \ll k^2$ near $x = 1$ and therefore from (2.3), $\varepsilon = 0$, Problem S, approximately

$$y'' + \frac{3}{x}y' \doteq \frac{4}{k^2x^2}\left(\frac{y}{kx} - 1\right) = -\frac{4}{k^2x^2} + 0(b) \quad \text{as} \quad 1 - x \to 0.$$

Hence, there is no boundary layer effect in that range of parameters. The only new situation where a boundary layer occurs is when $0 < b^2 \ll k^2 \ll 1$, which again amounts to small rotations. Hence, the corrections due to $k \neq 0$ can be expected to be small. In fact, the equation for the leading term outer solution will effectively be the same as in the Föppl theory. Near the edge, we have $y = 0(b)$. Slightly away from the edge, when $y = 0(k)$, $y(x)$ still exhibits boundary layer behavior as $k \ll 1$. It is this layer which should be matched with the outer solution. Thus we set $y(x) = kY(s)$, and $s = (1 - x)/\lambda$ for $\lambda \ll 1$. From (2.3) we find $\lambda = k^{3/2}$ for the distinguished limit. Hence, the leading term inner solution is obtained from

$$\frac{1}{4}\ddot{Y}_0(s) = \frac{Y_0(s)}{[1 + Y_0^2(s)]^{1/2}} - 1, \quad Y_0(0) = \frac{b}{k}.$$

This equation can be solved, after carrying out one integration, by substituting $Y_0 = \sinh\varphi$. The result is (see [14] for details)

$$\frac{1}{3}e^{3\varphi/2} - e^{-\varphi/2} = 2\sqrt{2}(s + C_1), \quad C_1 = \frac{e^{2\varphi_0} - 3}{6\sqrt{2}\,e^{\varphi_0/2}},$$

where φ_0 is given by $\sinh\varphi_0 = Y_0(0) = b/k$. These expressions can be simplified because of $b/k \ll 1$. The matching of $Y_0(s)$ with the outer solution then proceeds along the same lines as in subsection 3.3. We merely state the result obtained for y near the edge $x = 1$, to leading order

$$y = k\,Y_0 \doteq b + 2\sqrt{2/k}\,(1 - x), \quad \text{for } 1 - x = 0(k^{3/2}).$$

4. The annular membrane problem

Numerical solutions of BVP (2.2) show a boundary layer effect near the hole when $\varepsilon \ll 1$. It can be proved [14] that the limit problem (2.2) with $\varepsilon = 0$ and $Q(x) = R(x, 0) \neq 0$ does not have a positive $C^1(0, 1]$ solution. Therefore, any solution of the ODE in (2.2) with $\varepsilon = 0$ ceases to be valid as $x \to 0$, and a leading term outer solution can be made to satisfy only the boundary condition at $x = 1$. It is to be matched to some inner solution near the edge $x = \varepsilon$. In this section we indicate how the method of matched asymptotic expansions can be used to construct an asymptotic solution to this problem for small ε.

The region adjacent to $x = \varepsilon$ is stretched by the variable $s = (x/\varepsilon) - 1$. The form of the resulting ODE for $Z(s) = z(\varepsilon(1 + s))$ then suggests an inner expansion

$$Z(s; \varepsilon) = Z_0(s) + \varepsilon Z_1(s) + \varepsilon^2 Z_2(s) + \cdots. \tag{4.1}$$

The functions $Z_i(s)$, $i = 0, 1, \ldots$ all satisfy linear ODE's which can be solved in closed form [14]. They also satisfy $Z_i(0) = 0$ and are determined up to constants of integration C_i to be found by matching $Z(s; \varepsilon)$ with an outer expansion of $z(x)$ in an overlap domain. The leading term outer solution (for $\bar{p} = 1$) must satisfy

$$z_0'' + \frac{3}{x}z_0' + \frac{2}{z_0^2} = 0, \quad B\,z_0(1) = b \tag{4.2}$$

which is the circular membrane BVP, provided that $z_0'(0) = 0$. The matching process via an intermediate variable $\eta = x/\delta(\varepsilon)$, where $\eta = 0(1)$ and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that $\varepsilon/\delta(\varepsilon) \to 0$, then requires that $z_0'(0) = 0$ and $z_0(0) = C_0$. Hence, $z_0(x)$ is indeed the solution for the circular membrane, while the condition $z_0(0) = C_0$ determines $Z_0(s)$, which supplies the correction to $z_0(x)$ in the boundary layer near $x = \varepsilon$ to leading order. This process can be continued, resulting in appropriate boundary conditions for $z_m(x)$, $m \geqq 1$, at $x = 0$, so that $z_m(x)$ can be calculated without reference to $Z_m(s)$. On the other hand the constants C_m cannot be found without computing the outer solution term $z_m(x)$.

A quantity of particular interest is the *stress concentration factor* $S_c := \sigma_\theta/\sigma_0$ at the inner edge, where σ_0 refers to the hoop stress of the circular membrane at the center. Both σ_θ and σ_0 refer to solutions of Problem S (or H) with the same edge tension (or displacement) at $x = 1$. Thus we have $\sigma_\theta(r_i) = c_1(x z'(x) + z(x))$ evaluated at $x = \varepsilon$, and $\sigma_0 = c_1 z_0(0)$. Using the inner expansion valid in the layer of width $0(\varepsilon)$ near $x = \varepsilon$, we have obtained [14]

$$S_c = \frac{1}{C_0}\dot{Z}(0;\varepsilon) = 2\left[1 + \frac{\varepsilon^2}{C_0^3}\left(C_0^2 C_2 - \frac{1}{2}\right) + \frac{\varepsilon^4}{C_0^6}\left(C_0^5 C_4 + C_0^2 C_2 - \frac{3}{8}\right) + \cdots\right]. \tag{4.3}$$

Schwerin concluded from his formal power series solution for $\bar{p} = 1$ and fixed outer edge (Problem H, $b = 0$) that $S_c = 2$ in the limit of an "infinitesimally small hole" [11]. Our result confirms his conclusion on the basis of our asymptotic solution for both Problem S and Problem H. Moreover, explicit correction terms for small holes are provided by (4.3) in terms of the constants C_m. We also mention that a rigorous proof of $\lim_{\varepsilon \to 0} S_c(\varepsilon) = 2$ has been obtained recently [15], but these results do not provide information on how $S_c(\varepsilon)$ depends on ε. In all cases computed (asymptotically and numerically), we find $S_c(\varepsilon) \leqq 2$ for small ε. This result is at variance with numerical calculations of $S_c(\varepsilon)$ in [16].

The method described above for the Föppl annular membrane carries over to the Reissner membrane with few changes. In fact, the inner solutions $Z_i(s)$ for $i = 0, 1, 2, 3$ turn out to be the same, that is, the finite rotation effect in the boundary layer manifests itself only in $Z_m(s)$, $m \geqq 4$. As a consequence, $S_c(\varepsilon)$ is again given by (4.3), except that in the ε^4-term $-3/8$ should be replaced by $(k^2 C_0 - 3)/8$. But the finite rotation effect does change the outer solution significantly if k is of order unity.

5. Numerical examples

The asymptotic solutions for the circular membrane have been calculated for $b = 10^{-2}$ and $b = 10^{-4}$. In order to confirm the accuracy of these approximate solutions, accurate numerical solutions of BVP (2.1) have been obtained by the general purpose BVP-solver COLSYS [17]. The results are shown in Table 1. The numerical part of the outer asymptotic solution for $x < 0.98$ has been calculated by using the value $y(0.98)$ obtained from the local series expansion of $H(\xi)$. With $y(0.98) \doteq 0.13869$ and $y'(0) = 0$, an accurate numerical solution of the ODE in (2.1) (by any BVP-solver) is immediate. The values of y and $(x y)'$

Table 1
Circular membrane (Problem S): Comparison of asymptotic solutions and accurate numerical solutions for membranes with $b = 10^{-2}$ and 10^{-4} (uniform load).

		Numerical solution (by COLSYS)		Outer solution		Inner solution	
b	x	$y = c_1 \sigma_r$	$(x y)' = c_1 \sigma_\theta$	y	$(x y)'$	y	$(x y)'$
	0.000	0.80809	0.80809	0.80792	0.80792	–	–
	0.500	0.70367	0.47466	0.70346	0.47433	–	–
	0.800	0.48527	−0.41894	0.48490	−0.42022	–	–
	0.920	0.30556	−1.58328	0.30492	−1.58786	–	–
	0.960	0.20786	−2.64507	0.20695	−2.65660	–	–
10^{-2}	0.980	0.13869	−3.89454	0.13744	−3.92293	0.13097	−3.30456
	0.990	0.09156	−5.36882	0.08974	−5.85878	0.08898	−4.96898
	0.995	0.06038	−7.10075	(0.05794)	(−7.55792)	0.05950	−6.83245
	0.998	0.03553	−9.78817	(0.03211)	(−10.37628)	0.03531	−9.62785
	1.000	0.01000	−19.54183	–	–	0.01000	−19.49000
	0.000000	0.80791	0.80791	0.80792	0.80792	–	–
	0.500000	0.70341	0.47429	0.70347	0.47433	–	–
	0.800000	0.48485	−0.42038	0.48490	−0.42022	–	–
	0.920000	0.30485	−1.58841	0.30492	−1.58780	–	–
	0.980000	0.13729	−3.92635	0.13744	−3.92293	(0.12977)	(−3.34212)
10^{-4}	0.999000	2.0429E-2	−13.34792	2.0427E-2	−13.35280	2.0373E-2	−13.25506
	0.999900	4.4724E-3	−29.58756	4.4626E-3	−29.62247	4.4712E-3	−29.56741
	0.999990	9.8508E-4	−63.54309	(9.6459E-4)	(−64.24481)	9.8586E-4	−63.53889
	0.999996	5.5258E-4	−84.96481	(5.2388E-4)	(−87.26819)	5.5258E-4	−84.96280
	1.000000	0.00010	−199.95022	–	–	0.00010	−199.95001

Table 2
Annular membrane: Stress concentration factors according to asymtotic solutions and accurate numerical solutions (by COLSYS) for Problem S with $b = 0.5$, and for Problem H with $b = 0$, $\nu = 1/3$.

Problem S	Numerical solution	Asymptotic solution
$\varepsilon = 0.2$	1.997266	1.996192
$\varepsilon = 0.1$	1.999111	1.999048
$\varepsilon = 0.05$	1.999766	1.999762
$\varepsilon = 0.01$	1.999990	1.999990
Problem H		
$\varepsilon = 0.2$	1.944443	1.942572
$\varepsilon = 0.1$	1.985767	1.985643
$\varepsilon = 0.05$	1.996419	1.996411
$\varepsilon = 0.01$	1.999856	1.999856

of the outer solution for $x \geqq 0.98$ are calculated by the series solution truncated after 12 terms (retaining 4 more terms changed the solution only in the sixth significant figure).

Asymptotic solutions for the annular membrane problem may be found in [14]. Again, accurate numerical solutions by COLSYS confirm the increasing accuracy of the asymptotic solutions as ε decreases. We give in Table 2 a comparison of asymptotic and numerical results for the stress concentration factor (see (4.3)), obtained for uniform load.

While accurate numerical solutions may be obtained by advanced software designed specifically for BVP's exhibiting boundary layer behavior, the asymptotic solutions show the dominant structure of the solution in the layer region more explicitely, in particular with regard to the increase of the stresses near the edge of the membrane.

6. Acknowledgement

This research is supported in part by a U.S.-NSF Grant No. MCS-8306592 and a Canadian NSERC Grant No. A 9259. The second author was a Visiting Professor of Applied Mathematics at the University of Washington during the Spring Quarter of 1985. The authors are grateful to Beth Ong for her assistance in the numerical calculations.

References

[1] E. Bromberg and J. J. Stoker, *Non-linear theory of curved elastic sheets.* Quart. Appl. Math. *3*, 246–265 (1945).
[2] A. Föppl, *Vorlesungen über technische Mechanik*, Vol. III. Teubner, Leipzig 1907.
[3] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics.* Springer-Verlag, New York Heidelberg Berlin 1981.
[4] E. Reissner, *On finite deflections of circular plates*, Proc. Symp. Appl. Math., Vol. I: *Nonlinear Problems in Mechanics of Continua.* AMS, Providence, 213–219 (1949).
[5] H. Hencky, *Über den Spannungszustand in kreisrunden Platten.* Z. Math. Phys. *63*, 311–317 (1915).
[6] R. W. Dickey, *The plane circular elastic surface under normal pressure.* Arch. Rat. Mech. Anal. *26*, 219–236 (1967).
[7] A. J. Callegari and E. L. Reiss, *Nonlinear boundary value problems for the circular membrane.* Arch. Rat. Mech. Anal. *31*, 390–400 (1968).
[8] H. J. Weinitschke, *Existenz- und Eindeutigkeitssätze für die Gleichungen der kreisförmigen Membran.* Meth. u. Verf. der math. Physik *3*, 117–139 (1970).
[9] H. J. Weinitschke, *Some mathematical problems in the nonlinear theory of membranes, plates and shells.* Trends in Applications of Pure Mathematics to Mechanics (Lecce-Symposium), G. Fichera (ed.), 409–424, Pitman, London 1976.
[10] H. J. Weinitschke, *On finite displacements of circular elastic membranes*, Techn. Report No. 85-7, Inst. Appl. Math., The University of British Columbia, Vancouver (1985) (to appear in Math. Meth. in the Appl. Sciences).
[11] E. Schwerin, *Über Spannungen und Formänderungen kreisringförmiger Membranen.* Z. techn. Phys. *12*, 651–659 (1929).
[12] H. J. Weinitschke, *On axisymmetric deformations of nonlinear elastic membranes.* Mechanics Today *5*, 523–542, S. Nemat-Nasser (ed.) (Reissner Anniversary Volume), Pergamon, Oxford 1980.
[13] J. G. Simmonds and A. Libai, *A simplified version of Reissner's nonlinear equations for a first-approximation theory of shells of revolution.* Appl. Math. Report No. RM-86-01, School Eng. and Appl. Sci., University of Virginia, Charlottesville, VA, USA (June 1986).
[14] F. Y. M. Wan and H. J. Weinitschke, *Boundary layer solution for some nonlinear elastic membrane problems.* Report 113, Inst. Angew. Math., Univ. Erlangen-Nürnberg, Erlangen, F.R.G. (1984); revised as Tech. Rep. No. 85-7, Dept. Appl. Math., Univ. Washington, Seattle, WA, USA (1985).

[15] H. Grabmüller and H. J. Weinitschke, *Finite displacements of annular elastic membranes*, J. Elasticity *16*, 135–147 (1986).
[16] R. Kao and N. Perrone, *Large deflections of axisymmetric circular membranes*. Int. J. Solids Struct. *7*, 1601–1612 (1971).
[17] U. Ascher, J. Christiansen und R. D. Russell, *A collocation solver for mixed order systems of boundary value problems*. Math. Comp. *33*, 659–679 (1979).

Summary

The method of matched asymptotic expansions is used to describe finite axisymmetric deformations of two thin elastic membrane problems: a circular membrane with a small radial traction at the edge, an annular membrane with a small circular hole at the center. In both problems, the surface load is assumed to be an axial pressure. There is a boundary layer at the outer edge in the first problem, and at the hole in the second problem.

Zusammenfassung

Mit Hilfe der in der singulären Störungsrechnung benutzten Methode einer inneren und äußeren asymptotischen Entwicklung werden endliche, axialsymmetrische Deformationen bei zwei Membranproblemen beschrieben: eine dünne elastische Kreismembran mit kleiner radialer Zugspannung am Rande, und eine Kreisringmembran mit einem kleinen konzentrischen Loch. In beiden Problemen wird die Belastung als ein axial gerichteter Oberflächendruck angenommen. Im ersten Problem zeigt die Lösung am äußeren Rand Grenzschichtverhalten, welches im zweiten Problem am Lochrand vorliegt.

(Received: January 24, 1986; revised: October 17, 1986)