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STRESS SINGULARITIES AT A RIM OF CIRCULAR CYLINDERS*

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1. INTRODUCTION

Stress singularities at a corner of an elastic sheet where boundary conditions undergo abrupt changes have been studied in numerous investigations since the publication of [5]. Corner stress singularities may also exist at the upper and lower rims of a flat plate. For applications to certain plate problems (see [1] for example), we consider in this paper the stress singularity problem for a semi-infinite homogeneous circular cylinder occupying the region $\{Z < 0, X^2 + Y^2 < R_0^2\}$ in axisymmetric deformation. In the neighborhood of the rim, the end face $Z = 0$ of the cylinder is traction free while the cylindrical surface $X^2 + Y^2 = R_0^2$ is constrained from displacements. The equation for determining the severity of the stress singularity will be obtained and numerical results needed in applications [4] will be generated for a range of material parameter values for both isotropic and transversely isotropic cylinders.

2. AXISYMMETRIC LINEAR ELASTOSTATICS OF SEMI-INFINITE CIRCULAR CYLINDERS

Consider a homogeneous elastic circular cylinder occupying the region $\{Z < 0, R^2 = X^2 + Y^2 < R_0^2\}$ and free of distributed external loads in

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the cylinder interior. The elasticity of the cylinder is characterized by the following orthotropic stress strain relations [3].

$$\varepsilon_{rr} = a_{11}\sigma_{rr} + a_{12}\sigma_{\theta\theta} + a_{13}\sigma_{zz}\ , \quad \varepsilon_{\theta\theta} = a_{21}\sigma_{rr} + a_{22}\sigma_{\theta\theta} + a_{23}\sigma_{zz}\ ,$$
$$\varepsilon_{zz} = a_{31}\sigma_{rr} + a_{32}\sigma_{\theta\theta} + a_{33}\sigma_{zz}\ , \quad \varepsilon_{rz} = \varepsilon_{zr} = a_{44}\sigma_{rz} \tag{2-1}$$

relevant to axisymmetric deformations. For simplicity, we limit our consideration to the special case of transverse isotropy with $a_{ij} = a_{ji}, a_{11} = a_{22}$ and

$$a_{11} = \frac{1}{E}, a_{12} = -\frac{\nu}{E}, a_{13} = a_{23} = -\frac{\nu'}{E'}, a_{33} = \frac{1}{E'}, a_{44} = \frac{1}{G'}\ . \tag{2-2}$$

For infinitesimally small axisymmetric deformations the strain components in (2-1) are defined in terms of the radial and axial displacement components u_r and u_z by

$$\varepsilon_{rr} = u_{r,R}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{R}, \quad \varepsilon_{zz} = u_{z,Z}, \quad \varepsilon_{rz} = u_{r,Z} + u_{z,R}\ . \tag{2-3}$$

with $(\ ),_t = \partial(\ )/\partial t$.

Static equilibrium of the cylinder is satisfied by the following stress function representation:

$$\begin{aligned}
\sigma_{rr} &= -\frac{\partial}{\partial Z}\left[\frac{\partial^2}{\partial R^2} + \frac{\alpha_2}{R}\frac{\partial}{\partial R} + \alpha_1\frac{\partial^2}{\partial Z^2}\right]\psi, \\
\sigma_{zz} &= \frac{\partial}{\partial Z}\left[\alpha_3\Delta_R + \alpha_4\frac{\partial^2}{\partial Z^2}\right]\psi, \\
\sigma_{\theta\theta} &= -\frac{\partial}{\partial Z}\left[\alpha_2\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \alpha_1\frac{\partial^2}{\partial Z^2}\right]\psi, \\
\sigma_{rz} = \sigma_{zr} &= \frac{\partial}{\partial R}\left[\Delta_R + \alpha_1\frac{\partial^2}{\partial Z^2}\right]\psi
\end{aligned} \tag{2-4}$$

where $\Delta_R = (\quad)_{,RR} + R^{-1}(\quad)_{,R}$ and

$$\beta\alpha_1 = a_{13}(a_{11} - a_{12}), \quad \beta\alpha_2 = a_{13}(a_{13} + a_{44}) - a_{12}a_{33}\ ,$$
$$\beta\alpha_3 = a_{13}(a_{11} - a_{12}) + a_{11}a_{44}, \quad \beta\alpha_4 = a_{11}^2 - a_{12}^2 \tag{2-5}$$

with $\beta = a_{11}a_{33} - a_{13}^2 > 0$ and $\alpha_4 > 0$ by the positive definiteness of the strain energy. The stress function ψ is required by the compatibility conditions of the strain measures to satisfy

$$\left[\Delta_R + \frac{1}{s_1^2}\frac{\partial^2}{\partial Z^2}\right]\left[\Delta_R + \frac{1}{s_2^2}\frac{\partial^2}{\partial Z^2}\right]\psi = 0 \tag{2-6}$$

where

$$\{s_1^2, s_2^2\} = \frac{1}{2\alpha_4}\{(\alpha_1+\alpha_3) \pm \sqrt{(\alpha_1+\alpha_3)^2 - 4\alpha_4}\} . \qquad (2\text{-}7)$$

For an isotropic medium, we have $2\alpha_4 = \alpha_1 + \alpha_3 = 2$ and therewith $s_1 = s_2 = 1$ so that equation (2-6) reduces correctly to a biharmonic equation with axisymmetry. With $\alpha_1 = \alpha_2 = -\nu/(1-\nu)$ and $\alpha_3 = (2-\nu)/(1-\nu)$, the representation (2-4) also reduces correctly to the corresponding representation for the isotropic case (cf. (5-2) in [1]).

We are interested here in a semi-infinite circular cylinder which is free of traction on the face $Z = 0 (0 < R_1 < R < R_0)$ and is constrained against any displacement along the cylindrical surface $R = R_0(Z_0 < Z < 0)$. In that case we have

$$Z = 0 : \begin{cases} \frac{\partial}{\partial Z}\left[\alpha_3\Delta_R + \alpha_4\frac{\partial^2}{\partial Z^2}\right]\psi = 0, \quad (R_1 < R < R_0) . \\ \frac{\partial}{\partial R}\left[\Delta_R + \alpha_1\frac{\partial^2}{\partial Z^2}\right]\psi = 0, \end{cases} \qquad (2\text{-}8)$$

$$R = R_0 : u_r = 0, \quad u_z = 0 \quad (Z_0 < Z < 0) . \qquad (2\text{-}9)$$

The displacement conditions (2-9) require

$$R = R_0 : \varepsilon_{zz} = u_{z,Z} = 0, \quad \varepsilon_{rz,Z} - \varepsilon_{zz,R} = u_{r,ZZ} = 0 \quad (Z_0 < Z < 0) \qquad (2\text{-}10)$$

which we will use in the subsequent development.

The cylinder is loaded elsewhere but we will not specify the precise nature of the external load except that it should be axisymmetric. It is known (see [2] for example) that the abrupt change from the traction-free conditions (on the end surface) to the no displacement conditions (on the cylindrical edge) across the geometrically non-smooth rim may give rise to a stress singularity along the rim $\{R = R_0, Z = 0\}$. We are interested in the nature of the singularity there. The determination of the severity of the singularity along the rim does not require a precise specification of the load.

3. LOCAL POLAR COORDINATES AND STRESS SINGULARITIES

For the study of stress singularities, we are only concerned with the solution behavior in a narrow region near $R = R_0$ and $Z = 0$, so that $R_0 - R \leq l$ and $Z \leq -l$ with $\varepsilon \equiv l/R_0 << 1$. Let $x = (R_0 - R)/l$ and $y = -Z/l$. We have then

$$\Delta_R = l^{-2}\left[\frac{\partial^2}{\partial x^2} - \frac{\varepsilon}{1-\varepsilon x}\frac{\partial}{\partial x}\right] = l^{-2}\left[\frac{\partial^2}{\partial x^2} + 0(\varepsilon)\right] \qquad (3\text{-}1)$$

so that the leading term of the asymptotic expansion for ψ as $\varepsilon \to 0$ satisfies

$$\Psi_{,xx} + s_1^{-2}\Psi_{,yy} = 0 \ , \quad \psi_{,xx} + s_2^{-2}\psi_{,yy} = \Psi \ . \tag{3-2a,b}$$

The corresponding leading term approximations for the stress components are given by

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{l^3}\frac{\partial}{\partial y}\left[\frac{\partial^2}{\partial x^2} + \alpha_1\frac{\partial^2}{\partial y^2}\right]\psi \ , \\
\sigma_{\theta\theta} &= \frac{1}{l^3}\frac{\partial}{\partial y}\left[\alpha_2\frac{\partial^2}{\partial x^2} + \alpha_1\frac{\partial^2}{\partial y^2}\right]\psi \\
\sigma_{zz} &= -\frac{1}{l^3}\frac{\partial}{\partial y}\left[\alpha_3\frac{\partial^2}{\partial x^2} + \alpha_4\frac{\partial^2}{\partial y^2}\right]\psi, \\
\sigma_{rz} = \sigma_{zr} &= -\frac{1}{l^3}\frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial x^2} + \alpha_1\frac{\partial^2}{\partial y^2}\right]\psi \ .
\end{aligned} \tag{3-3}$$

In (3-2) and (3-3), we have used the same symbols to denote the corresponding leading term asymptotic expressions. We will continue to do so in the subsequent development as only leading term approximations will be involved.

For (3-2a), we set $x = \rho\cos\phi$ and $s_1 y = \rho\sin\phi$ and transform the equation for Ψ into Laplace's equation in the polar coordinates (ρ, ϕ) with solutions of the form

$$\Psi = \rho^\lambda\{\cos(\lambda\phi), \sin(\lambda\phi)\} \tag{3-4}$$

for any constant λ. For (3-2b), we set $x/s_2 = r\cos\theta, y = r\sin\theta$ and transform the equation for ψ into Poisson's equation in the polar coordinates (r, θ):

$$\psi_{,rr} + r^{-1}\psi_{,r} + r^{-2}\psi_{,\theta\theta} = s_2^2\Psi \ . \tag{3-5}$$

With (r, θ) and (ρ, ϕ) related by

$$\rho = r\sqrt{s_1^2\sin^2\theta + s_2^2\cos^2\theta} \equiv rA_\theta \ , \quad \phi = \tan^{-1}\left(\frac{s_1}{s_2}\tan\theta\right) \equiv \omega_\theta \ , \tag{3-6}$$

we have

$$\rho^\lambda\{\cos(\lambda\phi), \sin(\lambda\phi)\} = r^\lambda A_\theta^\lambda\{\cos(\lambda\omega_\theta), \sin(\lambda\omega_\theta)\} \ . \tag{3-7}$$

A particular solution ψ_c of (3-2b) for $\Psi = \rho^\lambda\cos(\lambda\phi)$ may be taken in the form $r^{\lambda+2}f_\lambda(\theta)$ with f_λ determined by

$$f_\lambda'' + (\lambda+2)^2 f_\lambda = A_\theta^\lambda\cos(\lambda\omega_\theta) \ , \quad (\ )' \equiv \frac{d(\ )}{d\theta} \ . \tag{3-8}$$

It follows that

$$\psi_c = s_2^2 r^{\lambda+2} f_\lambda(\theta), \quad f_\lambda(\theta) = \int_0^\theta A_t^\lambda \cos(\lambda\omega_t)\sin((\lambda+2)(\theta-t))dt\,. \quad (3\text{-}9)$$

Similarly, we have as a particular solution ψ_s for (3-2b) with $\Psi = \rho^\lambda \sin(\lambda\phi)$

$$\psi_s = s_2^2 r^{\lambda+2} g_\lambda(\theta), \quad g_\lambda(\theta) = \int_0^\theta A_t^\lambda \sin(\lambda\omega_t)\sin((\lambda+2)(\theta-t))dt\,. \quad (3\text{-}10)$$

The complementary solution ψ_h of (3-2b) is of the form $r^\mu\{\cos(\mu\theta), \sin(\mu\theta)\}$. While the general solution for ψ is a superposition of these three types of solutions $\{\psi_h, \psi_c, \psi_s\}$ for all λ and μ, it suffices for our analysis of rim-corner singularities to consider one typical term in this general solution in the form:

$$\psi(r,\theta) = r^{\lambda+2}\{c_1\cos((\lambda+2)\theta) + c_2\sin((\lambda+2)\theta) + c_3 f_\lambda(\theta) + c_4 g_\lambda(\theta)\} \quad (3\text{-}11)$$

where λ and c_i are constants to be determined by the boundary conditions. For an isotropic medium, we have $s_1 = s_2 = A_\theta = 1$ and $\omega_\theta = \theta$ so that f_λ and g_λ reduce to linear combinations of $\cos(\lambda\theta), \sin(\lambda\theta), \cos((\lambda+2)\theta)$ and $\sin((\lambda+2)\theta)$.

For the boundary conditions, we note the relations

$$\frac{\partial}{\partial x} = \frac{1}{s_2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right), \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \quad (3\text{-}12)$$

which may be applied repeatedly to get expressions for second and third order partial derivatives for ψ. These expressions are needed for the application of the boundary conditions but will not be listed here. With (3-3) and (3-11), the traction-free conditions (2-8) become

$$\beta_1(\lambda+1)c_1 + s_2^\lambda c_3 = 0\,, \quad \beta_2(\lambda+1)c_2 + s_1 s_2^{\lambda-1} c_4 = 0 \quad (3\text{-}13, 14)$$

where

$$\beta_1 = \frac{1}{\alpha_1 s_2^2} - 1\,, \quad \beta_2 = \frac{\alpha_3}{\alpha_4 s_2^2} - 1\,. \quad (3\text{-}15)$$

The boundary conditions (2-10) on the strain measures can also be expressed in terms of ψ by way of the stress strain relation for ε_{ZZ} and the stress function representation. With (3-11), these conditions become

$$\begin{aligned} &c_1\beta_3(\lambda+1)\cos\left((\lambda+2)\frac{\pi}{2}\right) + c_2\beta_3(\lambda+1)\sin\left(+(\lambda+2)\frac{\pi}{2}\right) \\ &+ c_3\left\{\beta_3(\lambda+1)f_\lambda\left(\frac{\pi}{2}\right) + s_1^\lambda\cos\left(\lambda\frac{\pi}{2}\right)\right\} \\ &+ c_4\left\{\beta_3(\lambda+1)g_\lambda\left(\frac{\pi}{2}\right) + s_1^\lambda\sin\left(\lambda\frac{\pi}{2}\right)\right\} = 0 \end{aligned} \quad (3\text{-}16)$$

$$c_1 \sin\left((\lambda+2)\frac{\pi}{2}\right) - c_2 \cos\left((\lambda+2)\frac{\pi}{2}\right) - c_3 \widehat{f}_\lambda\left(\frac{\pi}{2}\right) - c_4 \widehat{g}_\lambda\left(\frac{\pi}{2}\right) = 0 \quad (3\text{-}17)$$

where

$$\beta_3 = \frac{2\alpha_1 a_{13} - \alpha_4 a_{33}}{(\alpha_2+1)a_{13} - \alpha_3 a_{33}} s_2^2 - 1 \, , \quad (3\text{-}18)$$

$$\widehat{f}_\lambda(\theta) = \int_0^\theta A_t^\lambda \cos(\lambda\omega_t)\cos((\lambda+2)(\theta-t))dt,$$

$$\widehat{g}_\lambda(\theta) = \int_0^\theta A_t^\lambda \sin(\lambda\omega_t)\cos((\lambda+2)(\theta-t))dt \quad (3\text{-}19)$$

For the linear system (3-13), (3-14), (3-16), and (3-17) to have a nontrivial solution, the determinant of the coefficient matrix must vanish giving an equation for the remaining unknown λ in the form $F(\lambda) = 0$ where

$$\begin{aligned} F(\lambda) = & \frac{1}{2}(\lambda+1)^2 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} A_\psi^\lambda A_\phi^\lambda \sin(\lambda[\omega_\psi - \omega_\phi])\sin([\lambda+2][\phi-\psi])d\phi d\psi \\ & + \frac{1}{\beta_1} s_2^\lambda(\lambda+1)\int_0^{\frac{\pi}{2}} A_\psi^\lambda \sin(\lambda\omega_\psi)\cos([\lambda+2]\psi)d\psi \\ & - \frac{1}{\beta_2} s_1 s_2^{\lambda-1}(\lambda+1)\int_0^{\frac{\pi}{2}} A_\psi^\lambda \cos(\lambda\omega_\psi)\sin([\lambda+2]\psi)d\psi \\ & + \frac{1}{\beta_3} s_1^\lambda(\lambda+1)\int_0^{\frac{\pi}{2}} A_\psi^\lambda \sin\left(\lambda\left[\frac{\pi}{2}-\omega_\psi\right]\right)\cos\left([\lambda+2]\left[\frac{\pi}{2}-\psi\right]\right)d\psi \\ & + \frac{1}{\beta_2\beta_3} s_1^{\lambda+1} s_2^{\lambda-1}\cos\left(\lambda\frac{\pi}{2}\right)\cos\left([\lambda+2]\frac{\pi}{2}\right) \\ & + \frac{1}{\beta_1\beta_3} s_1^\lambda s_2^\lambda \sin\left(\lambda\frac{\pi}{2}\right)\sin\left([\lambda+2]\frac{\pi}{2}\right) - \frac{1}{\beta_1\beta_2} s_1 s_2^{2\lambda-1} \end{aligned} \quad (3\text{-}20)$$

For an isotropic material, the expression for $F(\lambda)$ simplifies to

$$F(\lambda) = \cos^2\left(\lambda\frac{\pi}{2}\right) - \frac{4\nu(1-\nu)}{3-4\nu} + \frac{(1-\lambda^2)}{3-4\nu} \, . \quad (3\text{-}21)$$

Observe that $F(0) = 4(1-\nu)^2/(3-4\nu) > 0$ and $F(1) = -4\nu(1-\nu)/(3-4\nu) < 0$; they imply the existence of a real root in the range $0 < \lambda < 1$ and thus a stress singularity for the problem. Upon setting $\lambda = a + ib$, we may write $F(\lambda) \equiv R(a,b) + iI(a,b)$ where R and I are the real and imaginary part of $F(\lambda)$, respectively, with $I < 0$ for $b > 0$ and $I > 0$ for $b < 0$. It follows from the Argument Principle in complex function theory (applied over a rectangular contour Γ consisting of the line segments $R_e(\lambda) = 0, R_e(\lambda) = 1$ and $I_m(\lambda) = \pm H$) that $F(\lambda)$ has no other root inside the strip $0 < R_e(\lambda) < 1$ of the complex λ-plane. We are not interested in roots of $F(\lambda) = 0$ with a real part not greater than zero or not less than unity. In the first case, the singularity is too severe and gives rise to unbounded strain energy which is not acceptable. In the second, the corresponding stress components are bounded everywhere including the rim of the cylinder end and hence are not singular.

4. NUMERICAL RESULTS

For the more general orthotropic cylinder, the nonlinear equation (3-20) has been solved numerically for real roots* between 0 and 1 by a standard library subroutine using a combination of bisection and Newton's method. After one root λ_0 has been found, the interval $(0, \lambda_0 - \epsilon)$ is searched to make sure there is no smaller root. The program is checked by substituting the root found for the isotropic case into (3-21) verifying $F(\lambda) = 0$ is satisfied to a high degree of accuracy (10^{-5}).

For the isotropic case, we have the values in Table I for the exponent of the stress singularity $\lambda - 1$. Note that λ depends only on ν and the singularity becomes increasingly more severe with increasing ν. Since R_0 does not enter into equation (3-21), it can be shown that the small region near the rim is in fact in a plane strain state as far as the stress singularity is concerned.

TABLE I: Stress Singularity Exponent $(\lambda - 1)$ for Isotropic Materials

ν	0	0.1	0.2	0.3	0.4	0.5
$\lambda - 1$	0	-0.1330	-0.2189	-0.2888	-0.3501	-0.4054

For orthotropic materials, the exponent $\lambda - 1$ is first computed for Douglas firs with $E' = 1.56 \times 10^6$ lbs/in^2, $E = 0.05E', G' = 0.078E', \nu = 0.287$ and $\nu' = 0.449$. It can be shown that $\lambda - 1$ is unchanged if the values of the elastic parameters in the transverse and in-plane direction are interchanged [4]. The exponent is then computed for several other combinations of E/E' and G'/E' (with ν and ν' kept fixed at their values for Douglas firs) to indicate the effects of the parameter changes on $\lambda - 1$. For the cases given in Table II, we have verified numerically that the plate is also in a plane strain state near the rim as in the isotropic case. Note that the singularity for $\{E/E' = 1, G'/E' = 1/3\}$ in Table II $(\lambda - 1 = -0.3510)$ is weaker than the corresponding isotropic cylinder with the same G'/E' ratio $(\lambda - 1 = -0.4054)$.

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*Complex roots with a real part in the interval (0,1) will not be given here as the corresponding highly oscillatory stress behavior in the neighborhood of the rim appears to be unrealistic.

TABLE II: Stress Singularity Exponent $(\lambda - 1)$ for Orthotropic Cylinders

G'/E \ E/E'	0.005	0.05	0.5	1.0	1.5
0.0078	-0.0851	-0.1637	-0.2434	-0.2692	-0.2860
0.078	Complex s_i	-0.1544	-0.2813	-0.3203	-0.3443
1/3	Complex s_i	Complex s_i	Complex s_i	-0.3510	-0.3840

If $u_r = 0$ at $R = R_0$ is replaced by $\sigma_{rr} = 0$, it has been shown by a similar analysis [4] that we have $F(\lambda) = \sin(\lambda\pi)$ for an isotropic cylinder in that case. Hence, there is no corner stress singularity for an isotropic plate with the particular type of mixed edge conditions. No stress singularity (with a real exponent λ) could be found numerically for the corresponding transversely isotropic cylinder [4].

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