ON THE EQUATIONS OF FLEXURAL MOTIONS FOR SHEAR-DEFORMABLE PLATES

Frederic Y.M. Wan
Department of Applied Mathematics
University of Washington, FS-20
Seattle, WA 98195

ABSTRACT

Mindlin's reduction of the equations of flexural motions for shear-deformable plates to a single fourth order equation for the transverse displacement function w is supplemented by a companion derivation of a second order equation for a stress function χ similar to that in the Reissner's theory for the static case. Initial and boundary conditions are also obtained in terms of w and χ.

INTRODUCTION

Equations of flexural motions of shear-deformable plates including the effect of rotary inertia were formulated in [1] analogous to the corresponding static equations first developed in [2,3]. A single fourth order equation (in both space and time) for the transverse displacement w of the (mid-plane of the) plate was derived from these equations in [1]. In the absence of surface loads and assuming simple harmonic motion in time, the equations of motion were alternatively uncoupled into three (reduced) "wave equations" in the spatial variables x_1 and x_2. However, we cannot recover from the results of this alternative reduction a corresponding set of space-time wave equations by replacing the negative of the square of the frequency by the second time derivative. In this note, we supplement Mindlin's fourth order equation for w by deriving a second order space-time equation for a stress function χ. The solutions for w and χ from the two uncoupled partial differential equations are coupled through the initial and boundary conditions expressed in terms of w and χ. These conditions, not discussed in [1], are also obtained in this paper. Applications of the results will be given in a future publication.

AN EQUATION FOR THE MID-PLANE DISPLACEMENT

We take the stress-strain relations for the transverse shear strains in the form $\gamma_j = \phi_j + w_{,j} = BQ_j$, where $(\ )_{,j} = \partial(\ )/\partial x_j$ and $B = 6/5GH$ for a transversely isotropic plate with plate thickness h and transverse shear modulus G. They may be used to express ϕ_j, and hence the curvature changes k_{ij}, in terms of w and Q_j:

$$\phi_j = - w_{,j} + BQ_j \,, \qquad k_{ij} = - w_{,ij} + BQ_{j,i} \qquad (i,j = 1,2) \;. \qquad (1)$$

From the relations between moment resultants and curvature changes $M_{11} = D(k_{11} + \nu k_{22})$, etc., we have

$$M_{11} = - D(w_{,11} + \nu w_{,22}) + DB(Q_{1,1} + \nu Q_{2,2}) \,, \qquad (2\text{-a})$$

$$M_{22} = \dots \,, \qquad M_{12} = M_{21} = - 2D_s[w_{,12} - B(Q_{1,2} + Q_{2,1})] \qquad (2\text{-b,c})$$

where in terms of Young's modulus E and Poisson's ratio ν , we have $D = Eh^3/12(1 - \nu^2)$ and $D_s = D(1 - \nu)/2$.

The two moment equilibrium equations relevant to transverse motions are

$$M_{1j,1} + M_{2j,2} - Q_j + q_j = \frac{1}{12}\rho_m h^3 \phi_{j,tt} \;, \quad (j = 1,2) \;, \quad (\;)_{,t} = \partial(\;)/\partial t \qquad (3)$$

where the q_j's are the distributed moment loads and $\rho_m h^3/12$ is the appropriate moment of inertia. Equation (3) may be written in terms of a stress function

$$\chi = Q_{2,1} - Q_{1,2} \qquad (4)$$

with the help of (2) as

$$Q_j + \frac{1}{12}\rho_m h^3 BQ_{j,tt} = f_j$$

$$\equiv - D(\nabla^2 w)_{,j} + q_j + DB(\rho h w_{,tt} - p_3)_{,j} + \frac{1}{12}\rho_m h^3 w_{,ttj} + D_s B e_{kj3} \chi_{,k} \qquad (5)$$

where ∇^2 is Laplace's operator in two dimensions, $e_{123} = 1$, $e_{213} = - 1$ and $e_{kj3} = 0$ otherwise.

Differentiate (5-a) for $j = 1$ with respect to x_1 and (5-b) for $j = 2$ with respect to x_2 and add. After the transverse force equilibrium equation $Q_{1,1} + Q_{2,2} + p_3 = \rho h w_{,tt}$ is used to eliminate the expression $(Q_{1,1} + Q_{2,2})$ from the resulting expression, we get

$$\{[D\nabla^2 - \frac{1}{12}\rho_m h^3 \frac{\partial^2}{\partial t^2}][\nabla^2 - \rho h B \frac{\partial^2}{\partial t^2}] + \rho h \frac{\partial^2}{\partial t^2}\} w$$

$$= [1 - DB\nabla^2 + \frac{1}{12} \rho_m h^3 B \frac{\partial^2}{\partial t^2}] p_3 + q_{1,1} + q_{2,2} \;. \qquad (6)$$

where p_3 is the distributed transverse surface load. Equation (6) is identical to that obtained in [1] and reduces to that of [2,3] for the static case.

A SUPPLEMENTARY EQUATION FOR THE STRESS FUNCTION

We now differentiate (5-b) for $j = 2$ with respect to x_1 and (5-a) for $j = 1$ with respect to x_2 and substract. The result is an equation for χ alone:

$$D_s B\nabla^2\chi - \chi - \frac{1}{12}\rho_m h^3 B\chi_{,tt} = q_{1,2} - q_{2,1} \quad . \tag{7}$$

The two uncoupled equations for w and χ form a sixth order system in both temporal and spatial variables. The determination of w and χ however is coupled through the boundary conditions.

If we neglect the effect of rotary inertia (by setting $\rho_m = 0$), the time derivative term in (7) drops out and equation (6) for w becomes second order in time. Correspondingly, equations (5) simplify to

$$Q_j = -D(\nabla^2 w)_{,j} + q_j + DB(\rho h w_{,tt} - p_3)_{,j} + D_s B e_{kj3}\chi_{,k} \; , \quad (j = 1,2) \quad . \tag{8}$$

It follows that Q_1 and Q_2 are completely determined once we know w and χ. Equations (2) then determine M_{ij} and equations (1) give ϕ_1 and ϕ_2. Hence only the initial conditions on w and $w_{,t}$ are needed for the determination of w and χ. This is consistent with the fact that the governing system of equations of motion is now only second order in time. The situation is more complex if $\rho_m \neq 0$.

THE INITIAL-BOUNDARY VALUE PROBLEM FOR w AND χ

For $\rho_m \neq 0$, the system (6) and (7) is sixth order in time (as well as in space). The determination of the behavior of the plate for $t > 0$ now requires six initial conditions: the initial values of w, ϕ_1 and ϕ_2 and their first time derivatives. The homogeneous version of these conditions are:

$$\begin{aligned} w(x_1,x_2,0) &= w_{,t}(x_1,x_2,0) = 0 \\ \phi_1(x_1,x_2,0) &= \phi_{1,t}(x_1,x_2,0) = 0 \\ \phi_2(x_1,x_2,0) &= \phi_{2,t}(x_1,x_2,0) = 0 \quad . \end{aligned} \tag{9-a,b,c}$$

It follows from equations (1) and (9) that

$$Q_j(x_1,x_2,0) = Q_{j,t}(x_1,x_2,0) = 0 \tag{10}$$

and as a consequence of (4) and (10)

$$\chi(x_1,x_2,0) = \chi_{,t}(x_1,x_2,0) = 0 \quad . \tag{11}$$

The two conditions in (11) serve as the initial conditions for equation (7).

On the other hand, equation (6) for w is fourth order in time. We need four initial conditions on w, $w_{,t}$, $w_{,tt}$ and $w_{,ttt}$. We have the first two, e.g., equation (9-a) for the homogeneous case. The remaining two follows from the force equilibrium equation, taken in the form

$$w_{,tt} = (Q_{1,1} + Q_{2,2} + p_3)/\rho h \qquad (12\text{-a})$$

and its time derivative

$$w_{,ttt} = (Q_{1,1} + Q_{2,2} + p_3)_{,t}/\rho h \quad . \qquad (12\text{-b})$$

The right-hand side of both expressions of (12) are known, e.g., Q_j and $Q_{j,t}$ are known from equation (10) for the homogeneous initial conditions (9).

For boundary conditions on χ and w, we get Q_1 and Q_2 in terms of w and χ from equations (5-a) and (5-b) by the method of variation of parameters

$$Q_j = \mu\int_0^t f_j(\tau)\sin[\mu(t-\tau)]d\tau \quad , \quad \mu = [\tfrac{1}{12}\rho_m h^3 B]^{-1/2} \qquad (13)$$

where $f_1(x_1,x_2,t)$ and $f_2(x_1,x_2,t)$ are the right-hand members of these two equations. The two constants of integration in each case are fixed (in the case of the homogeneous conditions (9)) by the initial conditions (10). The remaining stress and deformation measures are calculated from equations (1) and (2). As such, stress and displacement boundary conditions can all be expressed in terms of w and χ.

ACKNOWLEDGMENT

The research is supported by National Science Foundation Grant DMS-8743445.

REFERENCES

1. Mindlin, R.D., Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates, J. Appl. Mech., Vol. 18, pp. 31-38 (1951).

2. Reissner, E., On the theory of bending of elastic plates, J. Math. & Phys., Vol. 23, pp. 184-191 (1944).

3. Reissner, E., The effect of transverse shear deformation on the bending of elastic plates, J. Appl. Mech., Vol. 12, pp. 69-77 (1945).