Asymptotic and Computational Analysis

Conference in Honor of Frank W. J. Olver's 65th Birthday

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Finite Axial Extension and Torsion of Elastic Helicoidal Shells

Frederic Y. M. Wan Professor, Department of Applied Mathematics, University of Washington, Seattle, Washington

ABSTRACT

A finite deformation and infinitesimal strain formulation of the problem of axial extension and torsion of helicoidal shells is deduced from a general nonlinear shell theory for infinitesimal strain problems. A consistent application of the infinitesimal strain assumption further simplifies the relevant boundary value problem effectively to a linear problem; only the unknown stretch and twist parameters appear nonlinearly. The effects of finite deformations are clearly shown by examining the perturbation solutions for slightly pretwisted strips and (shallow) shells with a small pitch. In the presence of the core of the helicoid, the Poincaré-Lighthill perturbation technique is needed to avoid the spurious singularity associated with the regular perturbation solution.

I. INTRODUCTION

Many widely used solid structures, such as turbine blades and coil springs with thin cross-sections, are special cases of helicoidal shells. A study of their structural integrity in the elastic range may be formulated as boundary value problems in shell theory. Among the problems fundamental to the understanding of the elastostatics of these structures is the problem of a helicoidal shell acted upon by equal and opposite axial forces and torques (Figure 1). This paper presents an analysis of this fundamental problem within the framework of an infinitesimal strain-finite deflection shell theory. The Saint Venant solution for this problem obtained by our analysis reduce to the known result for shells with a small pitch [3].

In circular cylindrical coordinates (r,θ,z), the middle surface of a helicoidal shell is described by $z = a\theta$ where the constant $2\pi a$ is the pitch of the right helicoid. The special case of axial extension and torsion of pretwisted elastic strips has been investigated previously by way of a linear shell theory in [2, 4, 6, 7]. The principal qualitative conclusions obtained were:

(1) While the stress and strain distributions in the shell are rotationally symmetric, the displacement field is multi-valued in the polar angle θ of the form

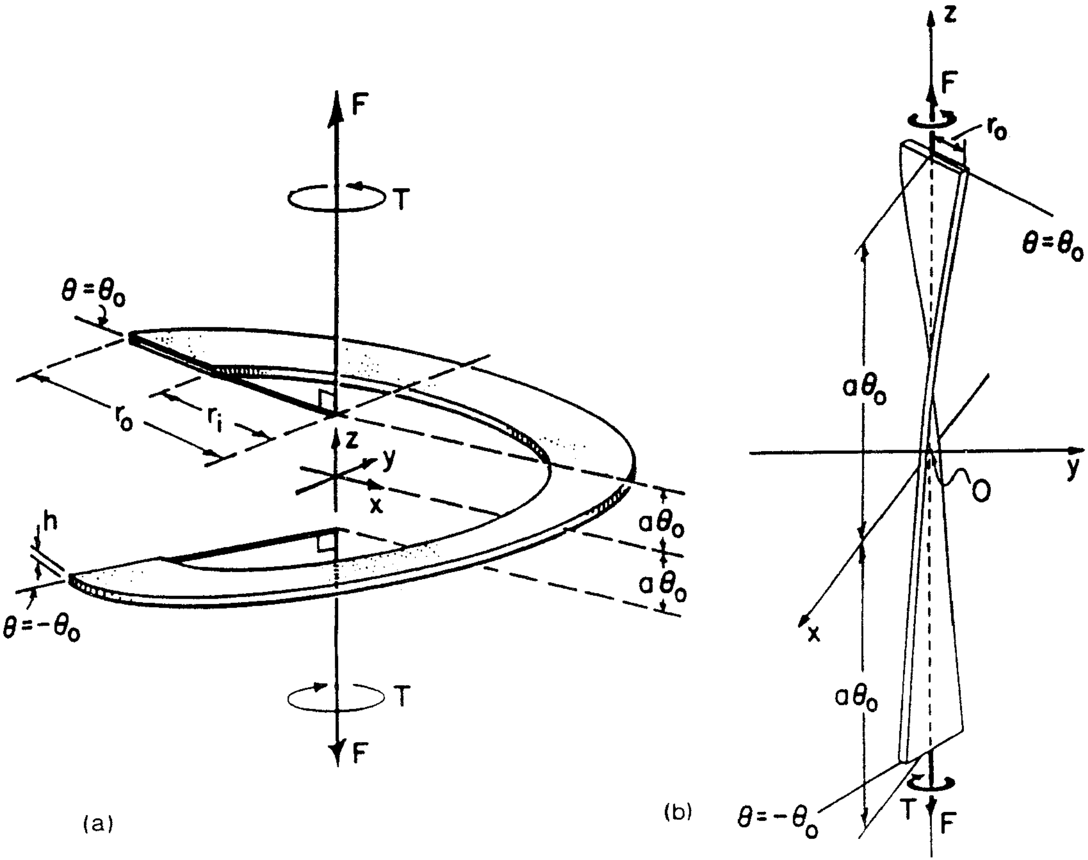
$$u_r = u(r) \quad , \quad u_\theta = \psi r a\theta \quad , \quad u_z = ka\theta \tag{1.1}$$

where u_r, u_θ and u_z are the radial, circumferential and axial displacement component of the middle surface of the shell, respectively, and k and ψ are two parameters corresponding to the axial strain and the angle of twist for a flat strip.

(2) The elastostatic problem may be reduced to a two point boundary value problem involving a second order (linear) ODE for the only unknown function $u(r)$ with k and ψ appearing as *forcing terms*.

(3) The membrane and inextensional bending actions of the shell are uncoupled in a well-defined way.

(4) There is no edge effect due to edge bending along the helical (r = constant) edges of the shell.



1. Helicoidal shells under axial forces and torques: (a) a ring shell sector; (b) a pretwisted strip.

The analysis for pretwisted strips was extended to a larger class of problems which includes the axial extension and torsion of helicoidal springs with wide rectangular cross-sections [6]. For a shell with small pitch-to-width ratio, a/r_0 (where r_0 is the radial distance from the axis of the shell to its outer helical edge), the latter problem was investigated earlier by a nonlinear shallow shell theory in [3].

The present paper considers the more general problem of rotationally symmetric stress state of helicoidal shells within the framework of a finite deflection-infinitesimal strain theory for shells with arbitrary pitch-to-width ratios. We will adopt also a semi-inverse procedure, used successfully in the earlier work, in the sense that the loading is assumed to be such that the stress and strain distributions in the shell will be independent of θ. Among the several nonlinear shell theories available in the literatures, we chose to follow the approach of [5] to make apparent the meaning of the parameters which appear in our semi-inverse procedure.

The requirement of rotationally symmetric strain distributions restricts the possible displacement fields and gives rise to a set of nonlinear strain displacement relations allowing for the rotationally symmetric stretching, twisting, bending and shearing actions of the shell. The subsequent analysis will be concerned exclusively with the stretching and twisting actions and the strain displacement relations are simplified by the restriction of small strain. For this class of problems, we show that the strain displacement relations must be linear in the radial displacement component u. The resulting boundary value problem has a governing differential equation which is linear in the only unknown function u; nonlinearity appears only in the *forcing terms*. In that case, the analysis of the relevant boundary value problem can be carried out exactly as that for a linear theory. Upon casting the present finite deformation problem in an appropriate form, the solution for the radial displacement, stress resultants and stress couples, can be obtained directly from the linear analysis of [6] without another set of independent calculations. The leading term perturbation solutions for the extreme cases of small and large pitch-to-width ratio are identical to the results of [3] and to the corresponding results for a shallow hyperbolic paraboloidal shell based on Marguerre's nonlinear shallow shell theory, respectively. While the principal qualitative conclusions for a linear theory carry over to our *quasilinear theory*, the latter gives additional information concerning certain critical values of the applied forces and torques when the deformation is no longer infinitesimal.

II. ANALYSIS OF STRAINS

Let ξ_1 and ξ_2 be general orthogonal surface coordinates, and $\vec{r}(\xi_1,\xi_2)$ and $\vec{n}(\xi_1,\xi_2)$ be the position vector and the unit normal of the midsurface of the shell before deformation. After deformation, $\vec{r}$ changes to $\vec{\rho}(\xi_1,\xi_2)$ with a new unit normal $\vec{v}$. In terms of $\vec{r}$ and $\vec{\rho}$ the expressions for midsurface strain and curvature change measures of an infinitesimal strain, large deformation shell theory proposed in [5] are

$$\varepsilon_1 = \frac{1}{2}\left(\frac{\vec{\rho}\,'\cdot\vec{\rho}\,'}{\alpha^2} - 1\right), \qquad \varepsilon_2 = \frac{1}{2}\left(\frac{\vec{\rho}\,^{\cdot}\cdot\vec{\rho}\,^{\cdot}}{\alpha_2^2} - 1\right), \qquad \gamma = \frac{\vec{\rho}\,'\cdot\vec{\rho}\,^{\cdot}}{\alpha_1\alpha_2} \tag{2.1}$$

$$\kappa_1 = \frac{\vec{\rho}\,'\cdot\vec{v}\,^{\cdot}}{\alpha_1^2} - \frac{1}{R_{11}}, \qquad \kappa_2 = \frac{\vec{\rho}\,^{\cdot}\cdot\vec{v}\,'}{\alpha_2^2} - \frac{1}{R_{22}},$$

$$\tau = \frac{\vec{\rho}\,'\cdot\vec{v}\,^{\cdot} + \vec{\rho}\,^{\cdot}\cdot\vec{v}\,'}{\alpha_1\alpha_2} - \frac{2}{R_{12}} \tag{2.2}$$

where primes and dots indicate differentiation with respect to ξ_1 and ξ_2, respectively, $\alpha_1 = |\vec{r}\,'|$, $\alpha_2 = |\vec{r}\,^{\cdot}|$ and R_{ij} are radii of curvature of the undeformed midsurface.

To specialize these general expressions to the case of a helicoidal shell, we take $\xi_1 = r$ and $\varepsilon_2 = \theta$. With $z = a\theta$, we have $\alpha_1 = 1$, $\alpha_2 = (a^2+r^2)^{1/2} \equiv \alpha$, $1/R_{11} = 1/R_{22} = 0$ and $1/R_{12} = a/\alpha^2 \equiv 1/R$. The position vector $\vec{r}$ of the undeformed middle surface of a helicoidal shell is given by

$$\vec{r} = r\vec{i}_r + a\theta\vec{i}_z, \qquad \vec{i}_r = \cos\theta\,\vec{i}_x + \sin\theta\,\vec{i}_y \tag{2.3}$$

where $\vec{i}_r$, $\vec{i}_x$, $\vec{i}_y$ and $\vec{i}_z$ are unit vectors in directions indicated by the subscripts. Analogous to these, we write the position vector $\vec{\rho}$ of the deformed midsurface as

$$\vec{\rho} = U_\rho(r,\theta)\,\vec{i}_\rho + U_z(r,\theta)\,\vec{i}_z, \qquad \vec{i}_\rho = \cos\eta\,\vec{i}_x + \sin\eta\,\vec{i}_y \tag{2.4}$$

where $\eta = \eta(r,\theta)$. Observe that $d(\vec{i}_\rho)/d\eta = -\sin\eta\,\vec{i}_x + \cos\eta\,\vec{i}_y \equiv \vec{i}_\eta$.

We are interested here in the class of deformation given by

$$U_\rho = U(r) \quad , \qquad U_z = W(r) + c_z\theta \ , \qquad \eta = \phi(r) + \phi_0\theta \tag{2.5}$$

where c_z and ϕ_0 are two unknown constants. The associated strain and curvature change measures are

$$\varepsilon_r = \frac{1}{2}\left[(U')^2 + (\phi')^2U^2 + (W')^2 - 1\right] \ , \quad \varepsilon_\theta = \frac{1}{2}\left[\frac{\phi_0^2U^2 + c_z^2}{\alpha^2} - 1\right]$$

$$\gamma = \frac{\phi_0\phi' U^2 + c_zW'}{\alpha} \tag{2.6}$$

$$\kappa_r = \frac{c_zU'}{\alpha}\left[\phi''U + 2\phi'U'\right] - \frac{U}{\alpha}\left[(U'' - (\phi')^2U)(c_z\phi' - \phi_0W') + \phi_0U'W''\right]$$

$$\kappa_\theta = -\frac{\phi_0^2U^2}{\alpha^3}(\phi_0W' - c_z\phi') \tag{2.7}$$

$$\frac{1}{2}\tau = \frac{\phi_0\phi'U^2}{\alpha^2}(c_z\phi' - \phi_0W') + \frac{c_z\phi_0(U')^2}{\alpha^2} - \frac{a}{\alpha^2} \ .$$

Note that they are all independent of θ.

For stretching and twisting shell actions, we take $W \equiv \phi \equiv 0$. In that case we have $\gamma = \kappa_r = \kappa_\theta = 0$. Upon setting

$$U = r + u(r) \ , \ c_z = a(1 + k) \ , \ \phi_0 = 1 + \psi a \ , \tag{2.8}$$

the remaining strain measures become

$$\varepsilon_\theta = \frac{r^2}{\alpha^2}\left\{\frac{u}{r}\left(1 + \frac{u}{2r}\right) + \left(\psi a + \frac{1}{2}\psi^2a^2\right)\left(1 + \frac{u}{r}\right)^2 + \frac{ka^2}{r^2}\left(1 + \frac{1}{2}k\right)\right\}$$

$$\varepsilon_r = u' \ , \qquad \tau = \frac{2a}{\alpha^2}\{k + \psi a + k\psi a\} \tag{2.9}$$

where we have omitted $u' \equiv \varepsilon_r$ compared to unity in the expression for τ to be consistent with the assumption of infinitesimal strains. Note that τ does not depend on u.

III. A QUASILINEAR THEORY FOR FINITE STRETCHING AND TWISTING

The expression for ε_θ in (2.9) is quadratic in (u/r). We solve this equation for u/r *to get*

$$\frac{u}{r} = -1 \pm \left[\frac{1 + 2\alpha^2\{\varepsilon_\theta - ka^2(1 + k/2)/\alpha^2\}/r^2}{(1 + \psi a)^2}\right]^{1/2} \tag{3.1}$$

Given that $ka^2(1 + k/2)/\alpha^2$ *and* $\psi ar^2(1 + \psi a/2)/\alpha^2$ are necessarily of the order of the hoop strain ε_θ, only the positive square root is acceptable in the above expression for u/r. (This is so even for $\psi a + 1 << 1$.) Otherwise, we have $\varepsilon_r = u$ not small compared to unity which violates the assumption of infinitesimal strains. It follows that we have $u/r = 0(\varepsilon_\theta)$ and $1 + u/r = 1$. In that case, the strain displacement relations given by (2.9) can be further simplified to

$$\varepsilon_r = u', \quad \varepsilon_\theta = \frac{r^2}{\alpha^2}\left[\frac{u}{r} + \psi a\left(1 + \frac{1}{2}\psi a\right) + \frac{ka^2}{r^2}\left(1 + \frac{1}{2}k\right)\right] \tag{3.2a,b}$$

$$\tau = \frac{2a}{\alpha^2}[k + \psi a + k\psi a] . \tag{3.2c}$$

The strain measures are related to the stress resultants and couples of the shell by a system of stress strain relations. We will take them int he form

$$\varepsilon_r = A(N_r - \nu N_\theta), \quad \varepsilon_\theta = A(N_\theta - \nu N_r) \tag{3.3}$$

$$M_{r\theta} = M_{\theta r} = \frac{1}{2}D(1 - \nu)\tau \tag{3.4}$$

where in terms of Young's modulus E, Poisson's ratio ν, and the shell thickness h, we have $A = 1/Eh$ and $D = Eh^3/12(1 - \nu)^2$. Equations (3.4) and the expression for τ in (3.2) can be combined to give

$$M_{r\theta} = M_{\theta r} = \frac{D(1-\nu)}{R}(k + \psi a + k\psi a) \tag{3.5}$$

where the unknown function u does not appear explicitly. Without the nonlinear term, the expression (3.5) is exactly the same as what we found for a linear theory [4,6].

The stress resultants and couples, together with the external radial surface load intensity p_r , and edge loads are subject to the conditions of force and moment equilibrium. Within the framework of a small strain theory we have, as a consequence of equilibrium and strain displacement relations, the following virtual work equation for the nonvanishing stress, strain and displacement measures:

$$\int_{r_i}^{r_0} (N_r \delta\varepsilon_r + N_\theta \delta\varepsilon_\theta + M_{r\theta}\delta\tau)\,\alpha\, dr$$

$$= \int_{r_i}^{r_0} p_r \delta u \;\alpha dr + P\delta[ka] + T\delta[\psi a] + N_0\delta[u(r_0)] - N_i\delta[u(r_i)] \;. \tag{3.6}$$

In (3.6) P and T are the resultant force and torque at a *radial* (θ = constant) edge, and N_0 and N_i are the applied radial edge resultant at the *helical* edges, $r = r_0$ and $r = r_i$, respectively. Note that the constraint of no transverse shear deformation normally introduces fictitious corner forces. Their effects should be included in the virtual work expression (3.6). However, the contribution of these effects to the final results is of the same order as that from terms neglected in the approximations already made. Therefore, it is consistent not to include the effects of corner forces in (3.6).

From (3.2), (3.5) and (3.6), we get as conditions of local and overall equilibrium, one differential equation

$$(\alpha N_r)' - \frac{r}{\alpha} N_\theta + \alpha p_r = 0 \tag{3.7}$$

two boundary conditions at the two helical edges

$$N_r(r_0) = N_0 \ , \ N_r(r_i) = N_i \tag{3.8}$$

and two integrated conditions at the radial edges

$$\int_{r_i}^{r_0} \left\{ \frac{a}{\alpha}(1+k)\, N_\theta + D(1-\nu)\, \frac{2a}{\alpha^3}(1+\psi a)\,(k+\psi a+k\psi a) \right\} dr = P \tag{3.9}$$

$$\int_{r_i}^{r_0} \left\{ \frac{r^2}{\alpha}(1+\psi a)\, N_\theta + D(1-\nu)\, \frac{2a^2}{\alpha^3}(1+k)\,(k+\psi a+k\psi a) \right\} dr = T \ . \tag{3.10}$$

Other conditions at the helical edges can also be prescribed. For example, we may prescribe the displacement component u instead.

The expressions for ε_r and ε_θ in equations (3.2) may be combined with (3.3) to give N_r and N_θ in terms of the unknown function u :

$$AN_r = u' + \frac{\nu r^2}{\alpha^2}\left[\frac{u}{r} + \psi a\left(1+\frac{1}{2}\psi a\right) + \frac{ka^2}{r^2}\left(1+\frac{1}{2}k\right)\right] \tag{3.11}$$

$$AN_\theta = \nu u' + \frac{r^2}{\alpha^2}\left[\frac{u}{r} + \psi a\left(1+\frac{1}{2}\psi a\right) + \frac{ka^2}{r^2}\left(1+\frac{1}{2}k\right)\right] \ . \tag{3.12}$$

Upon substituting these expressions into (3.7), we get a second order differential equation for u :

$$u'' + \frac{r}{a^2+r^2}u' + \frac{\nu a^2 - r^2}{(a^2+r^2)^2}u = \frac{(1+\nu)a^2rk\left(1+\frac{1}{2}k\right) + \psi ar\left(1+\frac{1}{2}\psi a\right)\left[(1+\nu)r^2 - 2\nu a^2\right]}{\left(a^2+r^2\right)^2} - Ap_r \ . \tag{3.13}$$

Equation (3.13) and (3.8), expressed in terms of u by (3.11), define a two-point boundary value problem for u . It determines u up to two unknown parameters k

and ψ. These two parameters are then related to P and T by the two integral relations (3.9) and (3.10).

Equation (3.13) is the only differential equation to be solved for the problem of finite stretching and twisting given that u/r is of the order of the midsurface strains; it is *linear* in the unknown u. The effect of finite deformation appears only through the nonlinear terms involving the parameters k and ψ. With these nonlinear terms omitted from the right-hand side of the equation, we recover the governing differential equation for small deformation as obtained in [2] and elsewhere. It is also significant that the parameters k and ψ do not appear in the coefficients on the left side of the differential equation.

IV. A STRESS FUNCTION FORMULATION

We consider here a shell free of surface loads so that $p_r = 0$ and the two helical edges are free of edge tractions so that $N_i = N_0 = 0$. The two radial edges of the shell $\theta = \pm \theta_0$ are subject to equal and opposite resultant axial forces and torques. With the condition at the two helical edges prescribed in terms of N_r, the corresponding two-point boundary value problem for an infinitesimal deflection theory has been reformulated in [6] for the stress variable N_r. Given the form of the boundary value problem for the quasilinear theory, we will also use a similar stress function formulation to take advantage of the results already obtained in [6].

We begin by using the equilibrium equation (3.7) with $p_r \equiv 0$ to write

$$N_\theta = \frac{\alpha}{r}(\alpha N_r)' . \tag{4.1}$$

The strain measures given in terms of a single unknown function u satisfy the compatibility equation

$$\left(\frac{\alpha^2}{r}\varepsilon_\theta\right)' - \varepsilon_r = \psi a\left(1 + \frac{1}{2}\psi a\right) - \frac{a^2}{r^2}k\left(1 + \frac{1}{2}k\right) . \tag{4.2}$$

Upon expressing ε_r and ε_θ in terms of N_r by way of the stress strain relations, we get

$$\varepsilon_r = A\left[N_r - \nu\frac{\alpha}{r}(\alpha N_r)'\right] , \quad \varepsilon_\theta = A\left[\frac{\alpha}{r}(\alpha N_r)' - \nu N_r\right] . \tag{4.3}$$

Introduction of these expressions into the compatibility equation (4.2) gives us a second order differential equation for N_r:

$$N_r^{\cdot\cdot} - \frac{2a^2 - 3r^2}{r(a^2+r^2)} N_r^{\cdot} - \frac{(1-\nu)a^2}{(a^2+r^2)^2} N_r = \frac{\psi a\left(1+\frac{1}{2}\psi a\right)r^2 - a^2 k\left(1+\frac{1}{2}k\right)}{A\,(a^2+r^2)^2} . \qquad (4.4)$$

The boundary conditions at the helical edges are now simply (3.8) with $N_0 = N_i = 0$:

$$N_r(r_i) = N_r(r_0) = 0 . \qquad (4.5)$$

Having determined N_r by (4.4) and (4.5), we get N_θ from (4.1), while $M_{r\theta}$ is known from (3.5). We then use the second equation of (3.2) to determine u . Finally, the parameters k and ψ are related to P and T by way of (3.9) and (3.10).

To obtain the solution of the boundary value problem for N_r and the corresponding overall load-deformation relations, we introduce the following dimensionless quantities:

$$s = r/r_0 \;,\; s_i = r_i/r_0 \;,\; \lambda = r_0/a \;,\; \mu = h/r_0$$
$$(n_r, n_\theta) = A(N_r, N_\theta) \;,\; m = \frac{r_0 M_{r\theta}}{D(1-\nu)} \;,\; \overline{u} = \frac{u}{r_0} . \qquad (4.6)$$

In terms of these quantities, the differential equation (4.4) becomes

$$n_r^{\cdot\cdot} - \frac{2-3\lambda^2 s^2}{s(1+\lambda^2 s^2)} n_r^{\cdot} - \frac{(1-\nu)\lambda^2}{(1+\lambda^2 s^2)^2} n_r = \frac{\psi r_0 \lambda^2 s^2\left(\lambda + \frac{1}{2}\psi r_0\right) - k\lambda^2\left(1+\frac{1}{2}k\right)}{(1+\lambda^2 s^2)^2} \qquad (4.7)$$

where dots now indicate differentiation with respect to s , and the boundary conditions (4.5) become

The auxiliary equations (4.1), (3.5) and (3.2b) may be written as

$$n_\theta = \frac{1+\lambda^2 s^2}{\lambda^2 s} n_r^{\cdot} + n_r \quad , \quad m = \frac{k\lambda + \psi r_0(1+k)}{1+\lambda^2 s^2} \tag{4.9a,b}$$

$$\bar{u} = \frac{(1+\lambda^2 s^2)^2}{\lambda^4 s^2}\left[n_r^{\cdot} + \frac{(1+\nu)\lambda^2 s}{1+\lambda^2 s^2} n_r - \frac{k\left(1+\frac{1}{2}k\right)\lambda^2 s + \psi r_0\left(\lambda + \frac{1}{2}\psi r_0\right)\lambda^2 s^3}{(1+\lambda^2 s^2)^2}\right] \tag{4.10}$$

and the two integral relations (3.10) and (3.11) may be written as

$$\frac{PA}{r_0} = (1+k)\int_{s_i}^{1} \frac{\sqrt{1+\lambda^2 s^2}}{\lambda^2 s^2} n_r ds + \frac{\mu^2(\lambda + \psi r_0)\,[k\lambda + \psi r_0(1+k)]}{6(1+\nu)} \int_{s_i}^{1} \frac{ds}{(1+\lambda^2 s^2)^{3/2}} \tag{4.11}$$

$$\frac{TA}{r_0^2} = -\frac{\lambda + \psi r_0}{\lambda^2}\int_{\rho_i}^{1} \sqrt{1+\lambda^2 s^2}\, n_r ds + \frac{\mu^2(1+k)[k\lambda + \psi r_0(1+k)]}{6(1+\nu)} \int_{s_i}^{1} \frac{ds}{(1+\lambda^2 s^2)^{3/2}} . \tag{4.12}$$

The boundary value problem defined by (4.7) and (4.8) is *linear* and differs from the corresponding boundary value problem for infinitesimal deformation in [6] only in the appearance of nonlinear terms in the parameters k and ψ. We may therefore write the solution of the finite deformation problem as

$$n_r = k\left(1+\frac{1}{2}k\right) n_{rk} + \psi r_0\left(\lambda + \frac{1}{2}\psi r_0\right) n_{r\psi} \tag{4.13}$$

where n_{rk} and $n_{r\psi}$ are the same as those for the linear theory. Since these quantities have already been obtained in [6], we do not have to solve the present finite deformation problem anew. In particular, the stiffness relations (4.11) and (4.12) can be written as

$$\frac{PA}{2r_0} = (1+k)\left[k\left(1+\frac{1}{2}k\right)C^M_{Fk} + \psi r_0\left(\lambda + \frac{1}{2}\psi r_0\right)C^M_{F\psi}\right]$$
$$+\frac{\mu^2}{12(1+\nu)}(\lambda+\psi r_0)\left[k\lambda+\psi r_0(1+k)\right]\left[\frac{s}{\sqrt{1+\lambda^2 s^2}}\right]^1_{s_i} \tag{4.14a}$$

$$\frac{TA}{2r_0^2} = (\lambda+\psi r_0)\left[k\left(1+\frac{1}{2}k\right)C^M_{Tk} + \psi r_0\left(\lambda + \frac{1}{2}\psi r_0\right)C^M_{T\psi}\right]$$
$$+\frac{\mu^2}{12(1+\nu)}(1+k)\left[k\lambda+\psi r_0(1+k)\right]\left[\frac{s}{\sqrt{1+\lambda^2 s^2}}\right]^1_{s_i} \tag{4.14b}$$

where

$$C^M_{Fk} = \frac{1}{2}\int_{s_i}^1 \frac{n_{\theta k}}{\sqrt{1+\lambda^2 s^2}}\,ds \quad , \quad C^M_{T\psi} = \frac{1}{2}\int_{s_i}^1 \frac{\lambda s^2 n_{\theta\psi}}{\sqrt{1+\lambda^2 s^2}}\,ds$$
$$C^M_{F\psi} = \frac{1}{2}\int_{s_i}^1 \frac{n_{\theta\psi}}{\sqrt{1+\lambda^2 s^2}}\,ds = \frac{1}{2}\int_{s_i}^1 \frac{\lambda s^2 n_{\theta k}}{1+\lambda^2 s^2}\,ds = C^M_{Tk} \tag{4.14c}$$

(with $n_\theta + k\left(1+\frac{1}{2}k\right)n_{\theta k} + \psi r_0\left(\lambda+\frac{1}{2}\psi r_0\right)n_{\theta\psi}$) are the stiffness coefficients associated with the linear membrane shell action obtained in [6].

For a shell with a large pitch-to-width ratio so that $\lambda^2 << 1$, a perturbation solution in powers of λ^2 for the quasilinear theory of finite stretching and twisting is appropriate. We consider here only the special case of a pretwisted strip with $s_i = -1$. (Here, $r_i = -r_0$ corresponds to the image point of r_0 with respect to the z-axis.) Upon setting

$$\{n_r(s;\lambda), n_\theta(s;\lambda), \bar{u}(s;\lambda)\} \sim \sum_{i=0}^{\infty}\{n_{ri}(s), n_{\theta i}(s), \bar{u}_i(s)\}\lambda^{2i} \tag{5.1}$$

we get from the results of [6] for $\lambda << 1$ the following leading term solution of the membrane resultants and radial displacement:

$$n_r \sim \frac{1}{2} k\left(1 + \frac{1}{2} k\right)\underline{\underline{\lambda^2}}\{-1 + s^2\} + \frac{1}{4} \psi r_0\left(\lambda + \frac{1}{2} \psi r_0\right)\underline{\underline{\lambda^2}}\{-1 + s^4\}$$

$$n_\theta \sim k\left(1 + \frac{1}{2} k\right) + \psi r_0\left(\lambda + \frac{1}{2} \psi r_0\right)s^2 \tag{5.2}$$

$$\bar{u} \sim k\left(1 + \frac{1}{2} k\right)\{-\nu s\} + \psi r_0\left(\lambda + \frac{1}{2} \psi r_0\right)\left\{-\frac{1}{3} \nu s^3\right\}$$

and from the expression for m in (4.9b) for the twisting stress couple

$$m \sim k\underline{\underline{\lambda}} + \psi r_0(1 + \underline{k}) \ . \tag{5.3}$$

For $\lambda << 1$, k is effectively the axial strain of the strip. Consistent with the infinitesimal strain approximation already made in our formulation, we should set $1 + ck = 1$ *for any* $0(1)$ quantity c. Also, we have

$$n_{rk} = 0(\lambda^2 n_{\theta k}) \quad , \quad n_{r\psi} = 0(\lambda^2 n_{\theta\psi}) \ ;$$

therefore, n_r is of order λ^2 compared to n_θ. Finally, the twisting stress σ_t induced by m_k is negligibly small relative to the direct hoop stress σ_d induced by $n_{\theta k}$, i.e.,

$$\frac{\sigma_{tk}}{\sigma_{dk}} = \frac{h}{2(1+\nu)r_0} \frac{m_k}{n_{\theta k}} = 0(\mu\lambda) \ ,$$

so that to leading order, we should consistently omit the $k\lambda$ term in m. Altogether, these observations allow us to simplify (5.2) and (5.3) to get the following leading term solution for the stress resultants, twisting couple and radial displacement:

$$n_r \sim 0 \quad , \quad n_\theta \sim k + \psi r_0\left(\lambda + \frac{1}{2} \psi_0 r\right)s^2$$

$$m \sim \psi r_0 \quad , \quad \bar{u} \sim -\nu k s - \frac{1}{3} \nu \psi r_0\left(\lambda + \frac{1}{2} \psi r_0\right)s^3 \ . \tag{5.4}$$

Correspondingly the leading term overall load-deformation relations for $\lambda^2 << 1$ consistent with the simplified leading term resultants and couples in (5.4) are

$$\frac{PA}{2r_0} \sim k + \frac{1}{3}\,\psi r_0\left(\lambda + \frac{1}{2}\,\psi r_0\right) \tag{5.5a}$$

$$\frac{TA}{2r_0^2} \sim \frac{1}{3}\,k(\lambda + \psi r_0) + \frac{1}{5}\,\psi r_0\left(\lambda + \psi r_0\right)\left(\lambda + \frac{1}{2}\,\psi r_0\right) + \frac{\mu^2}{6(1+\nu)}\,\psi r_0\,. \tag{5.5b}$$

We see from (5.5) that the shell stiffness comes mainly from its membrane action; bending is important only in the torque-twist relation and only when $|\lambda + \psi r_0| << \mu^2$ and $\left|\lambda + \frac{1}{2}\,\psi r_0\right| << \mu^2$.

The leading term solution (5.4) and (5.5) for $\lambda^2 << 1$ is identical to the exact solution of Marguerre's shallow shell equations for axial extension and torsion of pretwisted rectangular plates (with midsurface equation $z = \beta xy$). This exact solution is given by $w = \psi xy$ and $F = \left[12ky^2 + \psi(2\beta + \psi)y^4\right]/24A$ where w is the transverse midplane displacement of the plate, F the dual stress function and $\beta r_0 = \lambda$. Observe that there is a reciprocal relation inherent in the overall load-deformation relations in the sense that the expressions for P and T may be written as

$$P = \frac{\partial \Pi}{\partial k}\ , \quad T = \frac{\partial \Pi}{\partial \psi} \tag{5.6a,b}$$

where

$$\Pi = \frac{r_0}{A}\left\{k^2 + \frac{1}{3}\,k\psi r_0(2\lambda + \psi r_0) + \frac{1}{5}\left[(\psi r_0)^2\lambda^2 + (\psi r_0)^3\lambda + \frac{1}{4}(\psi r_0)^4\right] + \frac{\mu^2(\psi r_0)^2}{6(1+\nu)}\right\} \tag{5.6c}$$

with $\partial^2\Pi/\partial k\partial\psi = \partial^2\Pi/\partial\psi\partial k$. The following conclusions may be drawn from the overall load-deformation relations (5.5):

(1) For the limiting case of a flat strip with $\lambda = 0$, we have

$$n_r = 0 \quad , \quad n_\theta = k + \frac{1}{2}(\psi r_0)^2 s^2 \ ,$$

$$m = \psi r_0 \quad , \quad \bar{u} = -\nu k s - \frac{1}{6}\nu(\psi r_0)^2 s^3 \tag{5.7}$$

and

$$\frac{PA}{2r_0} = k + \frac{1}{6}(\psi r_0)^2 \ , \qquad \frac{TA}{2r_0^2} = \left[\frac{1}{3}k + \frac{1}{10}(\psi r_0)^2 + \frac{\mu^2}{6(1+\nu)}\right](\psi r_0) \ . \tag{5.8}$$

Unlike the linear case, the twisting and stretching actions of the shell remain coupled for a flat strip undergoing finite deformations.

(2) If the deformation due to the applied forces and torques is such that $\psi r_0 = -\lambda$, then the overall load-deformation relations become

$$\frac{PA}{2r_0} = k + \frac{\lambda^2}{6} \quad , \quad \frac{TA}{2r_0^2} = -\frac{\mu^2 \lambda}{6(1+\nu)} \ . \tag{5.9}$$

The expression for T in this case implies the somewhat unexpected results that the shell cannot be flattened by axial forces alone!

(3) If on the other hand, the deformation is such that $\psi r_0 = -2\lambda$, then (5.8) becomes

$$\frac{PA}{2r_0} = k \quad , \quad \frac{TA}{2r_0^2} = -\frac{\lambda}{3}\left[k + \frac{\mu^2}{1+\nu}\right] \ . \tag{5.10}$$

We have then another somewhat unexpected result. If the shell is deformed into its image shape by axial torques alone, then there would be no axial displacement associated with the deformation. In fact, there would be no membrane shell action at all in the sense that $n_\theta = n_r = \bar{u} = 0$.

(4) If $P = 0$, then we have $k = -\psi r_0\left(\lambda + \frac{1}{2}\psi r_0\right)/3$ and therewith

$$\frac{TA}{2r_0^2} \sim \psi r_0\left[\frac{2}{45}(2\lambda + \psi r_0)(\lambda + \psi r_0) + \frac{\mu^2}{6(1+\nu)}\right] \tag{5.11}$$

$$= \lambda^3(\psi a)\left[\frac{4}{45}(1+\psi a)\left(1+\frac{1}{2}\psi a\right) + \frac{(\mu/\lambda)^2}{6(1+\nu)}\right].$$

For an initially flat strip ($\lambda = 0$), we have

$$\frac{TA}{2r_0^2} \sim \psi r_0\left[\frac{2}{45}(\psi r_0)^2 + \frac{\mu^2}{6(1+\nu)}\right] \tag{5.12}$$

so that T is a monotone increasing function of ψ. For strips with an initial pretwist, a plot of T vs ψ is given in Figure (2) for different values of the thickness parameter $\mu = h/r_0$. For a fixed value of μ, and $\lambda \leq \mu$, we see that T continues to be a monotone increasing function of ψ for all ψ. In fact, a straightforward calculation shows that T has no stationary value for $(\mu/\lambda)^2 > 4(1+\nu)/15$ and two stationary values for smaller values of (μ/λ) attained at

$$\psi a = -1 \pm \left[\frac{1}{3} - \frac{5}{4}\frac{(\mu/\lambda)^2}{1+\nu}\right]^{1/2} < 0\,. \tag{5.13}$$

There are in principle three different possible (negative) values of ψ corresponding to the same value of the end torque T. The corresponding graph for the strain energy of the shell (Figure 3) with two stationary points at

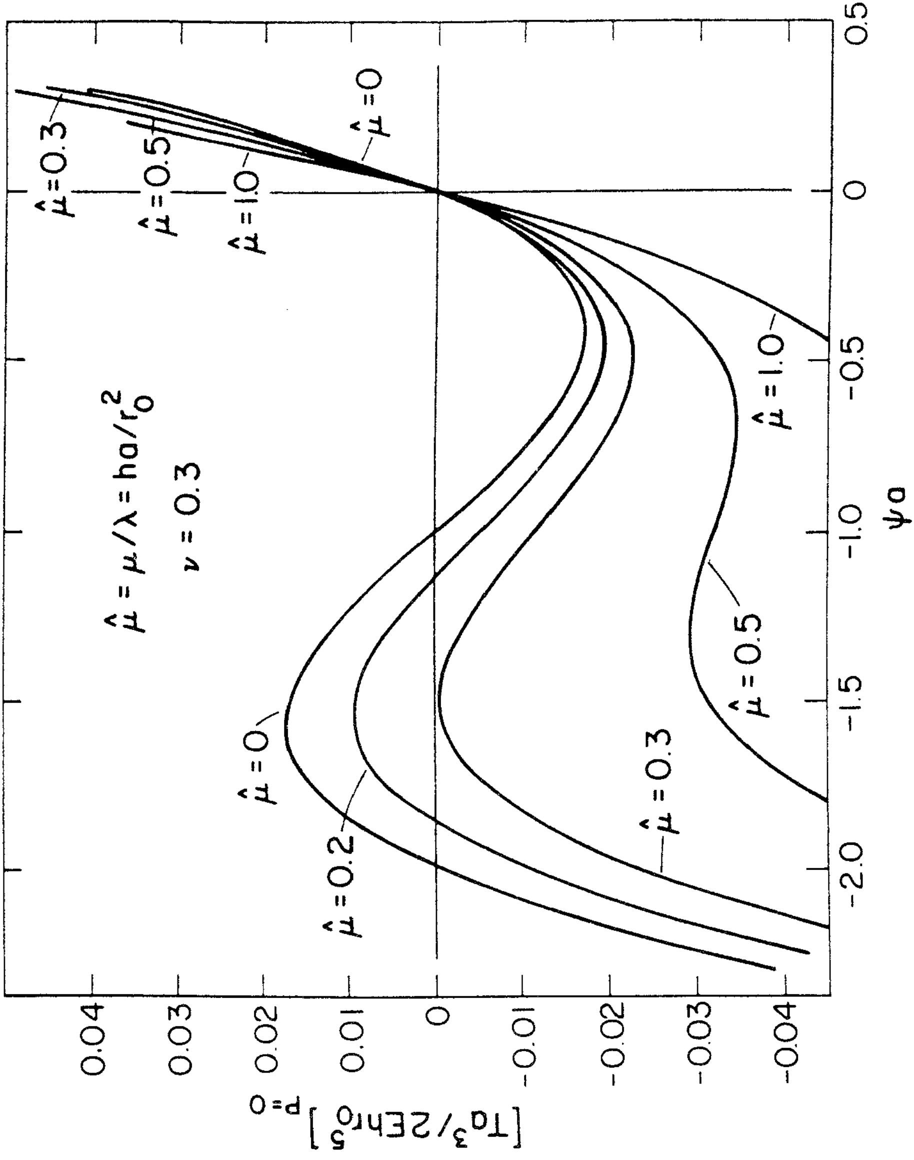
$$\psi a = \frac{1}{2}\left\{-3 \pm \left[1 - \frac{15}{1+\nu}\left(\frac{\mu}{\lambda}\right)^2\right]^{1/2}\right\} \tag{5.14}$$

for $(\mu/\lambda)^2 < (1+\nu)/15$ suggests the possibility of a snap-through at some critical value of $T\,(<0)$ for a given value of $(\mu/\lambda)^2 << 1$.

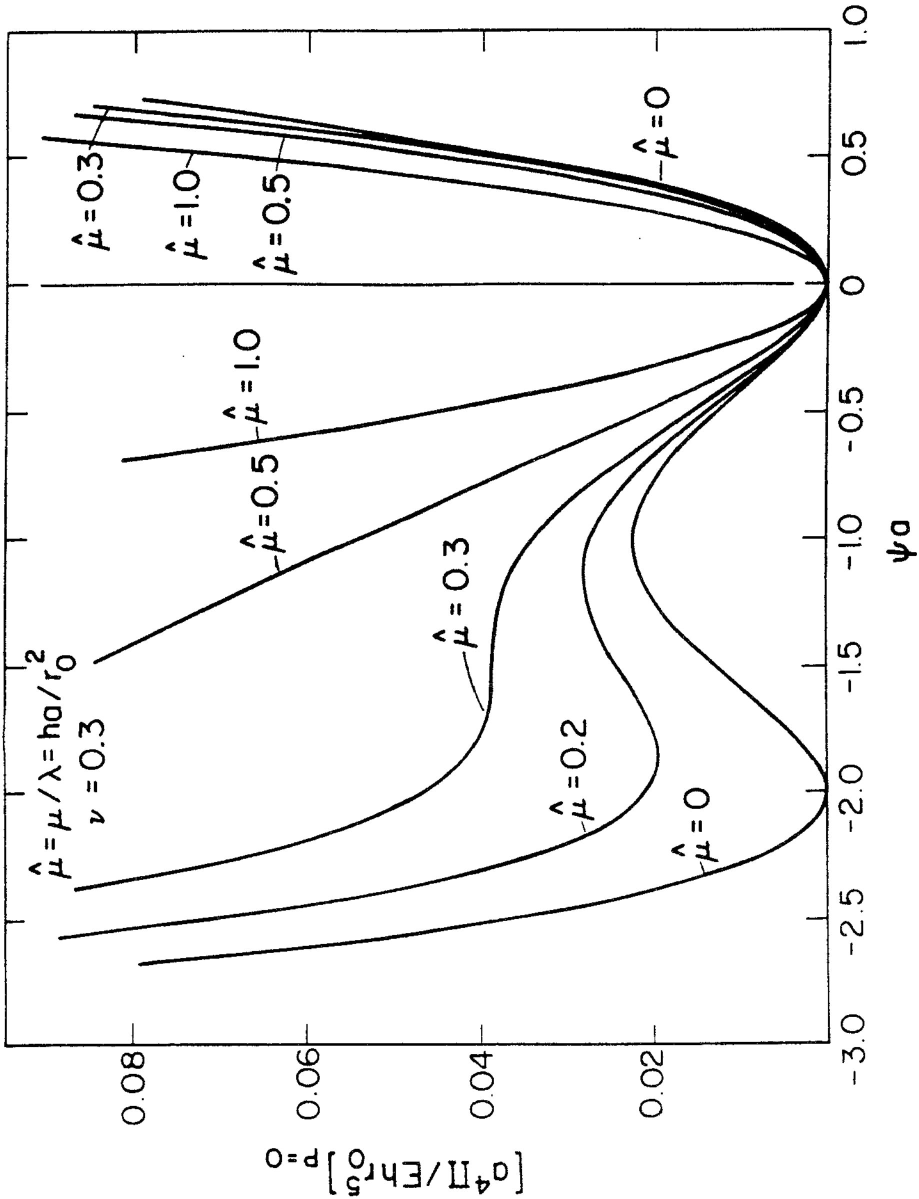
(5) If $T = 0$, we can in principle solve (5.5b) to get $\psi = \psi(k)$ and use it to eliminate ψ from (5.5a) to get $P = P(k)$. For $\mu^2/|\lambda + \psi r_0| << 1$, we have the linear stiffness relation $PA/2r_0 \sim 4k/9$ in the absence of axial torques.

VI. HELICOIDAL RING SHELLS WITH A SMALL PITCH

A helicoidal ring shell with $0 < s_i < 1$ is said to be shallow if its pitch is small so that $(\lambda s_i)^2 = (r_i/a)^2 >> 1$. For a shallow helicoidal ring shell, a



2. Torque-twist relations in the absence of axial forces.



3. Energy-twist relations in the absence of axial forces.

perturbation solution in powers of $\varepsilon^2 \equiv \lambda^{-2}$ for the finite stretching and twisting problem is appropriate. Upon setting

$$\{\bar{u}(s;\lambda), n_r(s;\lambda), n_\theta(s;\lambda)\} \sim \sum_{j=0}^{\infty} \{\hat{u}_j(s), \hat{n}_{rj}(s), \hat{n}_{\theta j}(s)\}\varepsilon^{2j} , \tag{6.1}$$

we get from the results of [6] for $(r_i/a)^2 >> 1$ the following leading term solution

$$\begin{aligned}
n_r &\sim k_\varepsilon\left(\varepsilon + \frac{1}{2}k_\varepsilon\right)\left[A_{k0} + B_{k0}s^{-2} + \frac{1}{4}s^{-2}(1 + 2\ell ns)\right] \\
&\quad + \psi_\varepsilon r_0\left(1 + \frac{1}{2}\underline{\psi_\varepsilon r_0}\right)\left[A_{\psi 0} + B_{k0}s^{-2} + \frac{1}{2}\ell ns\right] \\
n_\theta &\sim k_\varepsilon\left(\varepsilon + \frac{1}{2}k_\varepsilon\right)\left[A_{k0} - B_{k0}s^{-2} + \frac{1}{4}s^{-2}(1 - 2\ell ns)\right] \\
&\quad + \psi_\varepsilon r_0\left(1 + \frac{1}{2}\underline{\psi_\varepsilon r_0}\right)\left[A_{\psi 0} - B_{\psi 0}s^{-2} + \frac{1}{2}(1 + \ell ns)\right] \\
\bar{u}_r &\sim k_\varepsilon\left(\varepsilon + \frac{1}{2}k_\varepsilon\right)\left[A_{k0}(1-\nu)s - B_{k0}(1+\nu)s^{-1} + \frac{1}{4}s^{-1}\{(3+\nu) + 2(1+\nu)\ell ns)\}\right] \\
&\quad + \psi_\varepsilon r_0\left(1 + \frac{1}{2}\underline{\psi_\varepsilon r_0}\right)\left[A_{\psi 0}(1-\nu)s - B_{\psi 0}(1+\nu)s^{-1} + \frac{1}{2}s\{(1-\nu)\ell ns - 1\}\right]
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
A_{k0} &= \frac{\ell ns_i}{2(1 - s_i^2)} \qquad B_{k0} = -\frac{1}{4}\left[1 + \frac{2\ell ns_i}{1 - s_i^2}\right] \\
A_{\psi 0} &= -B_{\psi 0} = \frac{s_i^2 \ell ns_i}{2(1 - s_i^2)}
\end{aligned} \tag{6.3}$$

and where $k_\varepsilon \equiv k\varepsilon$ and $\psi_\varepsilon \equiv \psi\varepsilon$ remain finite as $\varepsilon \to 0$ in order for the expressions in (2.8) to be meaningful. Also, from the expression for m in (4.9b), we have the following leading term solution for the twisting couple

$$m \sim s^{-2}[k_\varepsilon(1 + \underline{\psi_\varepsilon r_0}) + \psi_\varepsilon r_0 \varepsilon] . \tag{6.4}$$

With $u_\theta = [\psi a + ur^{-1}(1 + \psi a)]r\theta = [\psi_\varepsilon r_0 + ur^{-1}(1 + \psi_\varepsilon r_0)]r\theta$ giving the circumferential displacement component and u/r of the order of the circumferential strain, we have for $\varepsilon << 1$, $u_\theta/r = 0(\psi a\theta) = 0(\psi_\varepsilon r_0\theta)$ also of the order of magnitude of the circumferential strain so that $1 + c(\psi_\varepsilon r_0) \simeq 1$ for any $0(1)$ quantity c. The underlined terms in (6.2) and (6.4) should therefore be omitted in order to be consistent with the approximation inherent in the present formulation of the problem. Also, we have

$$n_{\theta\psi} = 0(\psi_\varepsilon r_0) \quad , \quad m_\psi = 0(\psi_\varepsilon r_0 \varepsilon s^{-2}) \tag{6.5}$$

so that

$$\frac{\sigma_{t\psi}}{\sigma_{d\psi}} = \frac{h}{2(1+\nu)r_0}\,\frac{m_\psi}{n_{\theta\psi}} = 0(\mu\varepsilon)\ . \tag{6.6}$$

To the leading order, we should consistently omit the $\psi_\varepsilon r_0\varepsilon$ term in m also.

Altogether, the above two observations allow us to simplify (6.2) by omitting all underlined terms (which are nonlinear in ψ_ε) in the expressions for n_r, n_θ and $\bar{u}$ (with the resulting expression labelled (6.2'). They also allow us to simplify (6.4) to

$$m \sim k_\varepsilon s^{-2}\ . \tag{6.4'}$$

Correspondingly, the leading term overall load-deformation relations for $\varepsilon^2 << 1$, consistent with these simplified leading resultants and couple, are

$$\frac{PA}{2r_0} \sim \left\{ k_\varepsilon \frac{1-s_i^2}{s_i^2}\left[\frac{\mu^2}{24(1+\nu)}\right] + \frac{1}{16}\,k_\varepsilon(\varepsilon + k_\varepsilon)\left(\varepsilon + \frac{1}{2}\,k_\varepsilon\right)\left[1 - \left(\frac{2s_i \ell n s_i}{1-s_i^2}\right)^2\right] \right.$$

$$\left. + \frac{1}{4}\,\psi_\varepsilon r_0(\varepsilon + k_\varepsilon)\left(1 + \frac{1}{2}\,\underline{\psi_\varepsilon r_0}\right)\ell n s_i\left[1 + \frac{1+s_i^2}{1-s_i^2}\,\ell n s_i\right]\right\} \tag{6.7a}$$

$$\frac{TA}{2r_0^2} \sim \frac{1}{4}\,k_\varepsilon\left(\varepsilon + \frac{1}{2}\,k_\varepsilon\right)(1 + \underline{\psi_\varepsilon r_0})\,\ell n s_i\left[1 + \frac{1+s_i^2}{1-s_i^2}\,\ell n s_i\right]$$

$$+ \frac{1}{16}(\psi_\varepsilon r_0)\left(1 - s_1^2\right)\left[1 - \left(\frac{2s_i \ell n s_i^2}{1-s_i^2}\right)^2\right] \tag{6.7b}$$

where the underlined $\psi_\varepsilon r_0$ terms are of order of the hoop strain and are to be deleted. Again, the shell stiffness comes almost exclusively from membrane shell action; but in contrast to $\lambda^2 << 1$, bending is now only important in the force-extension relation and only when $|\varepsilon + k_\varepsilon|\left|\varepsilon + \frac{1}{2} k_\varepsilon\right| = 0(\mu^2)$.

The leading term solution (6.2'), (6.4') and (6.7) for $(\varepsilon/s_i)^2 << 1$ is identical to the exact solution of Marguerre's shallow shell equations for the problem of bending and twisting of helicoidal ring shells (see [3]). It is not difficult to see that there is a reciprocity relation inherent in (6.7) in the sense that the expressions for P and T in (4.11) may be written as

$$P = \frac{\partial \widehat{\Pi}}{\partial k_\varepsilon} \quad , \quad T = \frac{\partial \widehat{\Pi}}{\partial \psi_\varepsilon} \tag{6.8}$$

where

$$\begin{aligned} \frac{A}{r_0}\widehat{\Pi} = {} & \frac{k_\varepsilon^2}{16}\left[\left(\varepsilon^2 + k_\varepsilon\varepsilon + \frac{1}{4}k_\varepsilon^2\right) + \frac{2\mu^2}{3(1+\nu)}\right] \\ & + \frac{1}{2}k_\varepsilon\left(\varepsilon + \frac{1}{2}k_\varepsilon\right)\psi_\varepsilon r_0 \ell n s_i\left[1 + \frac{1+s_i^2}{1-s_i^2}\ell n s_i\right] \\ & + \frac{1}{16}(\psi_\varepsilon r_0)^2\left(1 - s_i^2\right)\left[1 - \left(\frac{2s_i \ell n s_i}{1-s_i^2}\right)^2\right] \end{aligned} \tag{6.9}$$

with $\partial^2\widehat{\Pi}/\partial k_\varepsilon \partial\psi_\varepsilon = \partial^2\widehat{\Pi}/\partial\psi_\varepsilon\partial k_\varepsilon$. The following conclusions may now be drawn from (6.7)

(1) For the limiting case $\varepsilon = 0$, the overall load-deformation relations (6.7) reduce to

$$\begin{aligned} \frac{PA}{2r_0} = k_\varepsilon \Bigg\{ & \frac{1-s_i^2}{s_i^2}\frac{\mu^2}{24(1+\nu)} + \frac{1}{32}k_\varepsilon^2\left[1 - \left(\frac{2s_i\ell n s_i}{1-s_i^2}\right)^2\right] \\ & + \frac{1}{4}\psi_\varepsilon r_0 \ell n s_i\left[1 + \frac{1+s_i^2}{1-s_i^2}\ell n s_i\right]\Bigg\} \end{aligned} \tag{6.10a}$$

$$\frac{TA}{2r_0^2} \sim \frac{1}{8}k_\varepsilon^2 \ell n s_i\left[1 + \frac{1+s_i^2}{1-s_i^2}\,\ell n s_i\right]$$
$$+\frac{1}{16}(\psi_\varepsilon r_0)\left(1 - s_1^2\right)\left[1 - \left(\frac{2s_i \ell n s_i}{1-s_i^2}\right)^2\right]. \tag{6.10b}$$

Unlike the linear case analyzed in [6], the twisting and stretching actions of the shell remain coupled for ring plate sector undergoing finite deformations.

(2) If the deformation due to the applied forces and torques is such that $k_\varepsilon = -\varepsilon$, then (6.7) becomes

$$\frac{PA}{2r_0} \sim k_\varepsilon \frac{1-s_i^2}{s_i^2}\frac{\mu^2}{24(1+\nu)} \tag{6.11a}$$

$$\frac{TA}{2r_0^2} \sim \frac{1}{8}k_\varepsilon \varepsilon \ell n s_i\left[1 + \frac{1+s_i^2}{1-s_i^2}\,\ell n s_i\right]$$
$$+\frac{1}{16}(\psi_\varepsilon r_0)\left(1 - s_1^2\right)\left[1 - \left(\frac{2s_i \ell n s_i^2}{1-s_i^2}\right)^2\right]. \tag{6.11b}$$

The expression for P in this case implies that the shell cannot be flattened by axial torques alone.

(3) If, on the other hand, the deformation is such that $k_\varepsilon = -2\varepsilon$, then we have

$$\frac{PA}{2r_0} \sim -\varepsilon\left\{\frac{1-s_i^2}{s_i^2}\,\frac{\mu^2}{24(1+\nu)} + \frac{1}{4}(\psi_\varepsilon r_0)\ell n s_i\left[1 + \frac{1+s_i^2}{1-s_i^2}\ell n s_i\right]\right\} \tag{6.12a}$$

$$\frac{TA}{2r_0^2} \sim \frac{1}{16}(\psi_\varepsilon r_0)\left(1 - s_i^2\right)\left[1 - \left(\frac{2s_i \ell n s_i}{1-s_i^2}\right)^2\right]. \tag{6.12b}$$

If the shell is deformed into its mirror image shape by axial forces alone, then there would be no circumferential displacement associated with the deformation. In fact, there would be no membrane action at all given that we have $n_\theta \equiv n_r \equiv \bar{u} \equiv 0$ for this case.

(4) If $T = 0$, then we have

$$\frac{1}{4}\psi_\varepsilon r_0 \sim -\frac{k_\varepsilon\left(\varepsilon + \frac{1}{2}k_\varepsilon\right)\ell n s_i}{1 - s_i^2}\left[\frac{1 + \dfrac{1 + s_i^2}{1 - s_i^2}\ell n s_i}{1 - \left(\dfrac{2 s_i \ell n s_i}{1 - s^2}\right)^2}\right] \tag{6.13}$$

$$\equiv -k_\varepsilon\left(\varepsilon + \frac{1}{2}k_\varepsilon\right)g(s_i)$$

and therewith

$$\frac{PA}{2r_0} \sim k_\varepsilon\left\{\frac{1 - s_i^2}{s_i^2}\frac{\mu^2}{24(1 + \nu)} + (\varepsilon + k_\varepsilon)\left(\varepsilon + \frac{1}{2}k_\varepsilon\right)f(s_i)\right\} \tag{6.14a}$$

where

$$f(s_i) = \frac{1}{16}\left[1 - \left(\frac{2s_i\ell n s_i}{1 - s_i^2}\right)^2\right] - g(s_i)\ell n s_i\left[1 + \frac{1 + s_i^2}{1 - s_i^2}\ell n s_i\right] \tag{6.14b}$$

with $g(s_i)$ defined in (6.13). This result was previously obtained and discussed in [3]. We merely note an additional observation on this stiffness relation as ε increases from zero. For $\varepsilon = 0$, P is a monotone increasing function of k_ε as $f(s_i)$ was found to be nonnegative in [3]. As ε increases from zero but remains less than μ, this monotone increasing property persists. However, for $\varepsilon > \mu$ and $k_\varepsilon < 0$, there are in principle three possible values of k_ε, which requires the same axial force P to maintain. However, the strain energy of the shell for the three different equilibrium configurations suggests the possibility of a snap-through at some critical value of $P > 0$.

(5) If $P = 0$, we can solve the first equation (6.7) for k_ε in terms of ψ_ε.

We can then eliminate k_ε from (6.7b) to get $T = T(\psi_\varepsilon)$. For $\mu^2/|\varepsilon + k_\varepsilon| <<$ $\left|\varepsilon + \frac{1}{2}k_\varepsilon\right|$, we have to obtain a good first approximation for the following linear stiffness relation:

$$\frac{TA}{2r_0^2} \sim \psi_\varepsilon r_0 \left\{ \frac{1}{16}\left(1 - s_i^2\right)\left[1 - \left(\frac{2s_i^2 \ell n s_i}{1 - s_i^2}\right)^2\right] - \left[\ell n s_i + \frac{1 + s_i^2}{1 - s_i^2}(\ell n s_i)^2\right]^2 \right\}$$

in the absence of axial forces.

7. OTHER PERTURBATION SOLUTIONS

Instead of a parametric series solution in the width-to-pitch ratio λ, we could have sought a perturbation solution in terms of the parameter

$$\eta = \frac{\lambda}{\sqrt{1 + \lambda^2}} = \frac{r_0}{\sqrt{a^2 + r_0^2}} \,. \tag{7.1}$$

Note that we have $\eta < 1$ for all $a > 0$ whether or not we have $a >> r_0$. Perturbation series of the form (5.1) with λ replaced by η (first suggested in [4]) are applicable for all (nonnegative) λ.

For a highly pretwisted strip (with $a/r_0 << 1$) which spans the range $-r_0 \le r \le r_0$ radially, a perturbation solution in $\varepsilon = 1/\lambda = a/r_0$ similar to that of section (6) is no longer appropriate. A leading term solution for that type of parametric series would correspond to neglecting a^2 terms in (4.4) leaving us with

$$n_0'' + \frac{3}{s} n_0' = \frac{1}{s^2} \psi_\varepsilon r_0\left(1 + \frac{1}{2} \psi_\varepsilon r_0\right) - \frac{1}{s^4} k_\varepsilon\left(1 + \frac{1}{2} k_\varepsilon\right) . \tag{7.2}$$

The leading term approximation for the dimensionless radial resultant AN_r has a singularity at $s = 0$, the center of the pretwisted strip. Yet we do not expect any singular behavior throughout the strip for $a > 0$. The singularity of the ODE (7.2) is in fact spuriously introduced by a perturbation solution process which is inappropriate for this type of problem. The appropriate perturbation scheme [1] requires that we seek the primary unknown $n \equiv AN_r$ and the auxiliary stress and displacement variables as functions of a new independent variable x and the parameter ε. The solution scheme requires that we take n and the independent variable s as functions of x and the parameter $\varepsilon = a/r_0$ and expand both in powers of ε:

$$n(s;\varepsilon) = \sum_{i=0}^{\infty} \tilde{n}_i(x)\varepsilon^{2i} \quad , \quad s = \sum_{i=0}^{\infty} s_i(x)\varepsilon^{2i} \; . \tag{7.3}$$

The details of this Poincaré-Lighthill type perturbation solution have already been worked out for the linear theory in [4] and will not be repeated here. We note only that for a shell-theoretic solution to be useful, we need $h/r_0 \ll a/r_0$; otherwise the stress distribution near the central axis of the helicoidal shell should be calculated by the three-dimensional theory of elasticity.

REFERENCES

1. J. Kevorkian and J.D. Cole, *Springer-Verlag, New York-Heidelberg-Berlin* (1981).

2. J.K. Knowles and E. Reissner, *Quart. Appl. Math.* 17 409-422 (1959)

3. E. Reissner, *Quart. Appl. Math.* 11, 473-483 (1954).

4. E. Reissner, *IUTAM Symp. on Shell Theory*, 434-466 (1959), Amsterdam (1960).

5. E. Reissner, *Develop. in Mech.* 3 *(Proc. 9th Midwestern Conf. on Mech.)*, 55-58 (1967), Madison, Wisconsin (1965).

6. E. Reissner and F.Y.M. Wan, *J. Math. & Phys.* 4, 1-31 (1968).

7. R.G. Sinclair, *Ph.D. Thesis, M.I.T.* (Sept. 1960).