Computers & Structures Vol. 35, No. 4, pp. 349–359, 1990
Printed in Great Britain.

0045-7949/90 \$3.00 + 0.00
Pergamon Press plc

BENDING AND FLEXURE OF SEMI-INFINITE CANTILEVERED ORTHOTROPIC STRIPS

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Abstract—The bending and flexure problems of a semi-infinite cantilevered orthotropic strip are reduced to a Fredholm integral equation of the first kind with a generalized Cauchy kernel. This preliminary reduction leads to a more efficient numerical solution. Moments of stresses for these problems are evaluated accurately; they appear naturally in the properly formulated boundary conditions of plate theories for prescribed displacement edge data. Applications are illustrated by the determination of the correct interior solution for a sheared orthotropic block. The corresponding flexibility coefficients are compared with the upper and lower bounds of the same problem available in the literature.

1. INTRODUCTION

This study is concerned with the bending and flexure of an orthotropic semi-infinite cantilevered strip within the framework of the linear theory of elastostatics. The extension, bending and flexure problems for isotropic semi-infinite strips have been the subjects of numerous investigations (see [1–5] for examples). In this paper, we obtain analogous solutions for the more general case of bending and flexure problems for orthotropic strips by the efficient numerical solution process developed recently in [6].

In our solution process, the relevant boundary value problem in linear plane elasticity is reduced to a single Fredholm integral equation of the first kind. (In contrast, the reduction in [2] is to two simultaneous integral equations for two unknowns.) A significant step toward demonstrating the existence of its solution and formulating an efficient numerical procedure for that solution is the recasting of this equation in the form of a singular integral equation with the singular part of the kernel being a linear combination of several Cauchy-type kernels. A simple extension of the theory for Cauchy-type singular integral equations developed in [7, 8] has been made in [6] to handle our generalized Cauchy integral equation.

Beyond their importance in linear elasticity theory, the bending and flexure (as well as extension) problems for semi-infinite strips also have a fundamental role in the linear plate theory solution for thin plates with displacement edge data in plane strain deformation. It is known from [9–12] that the solution of Kirchhoff's linear thin plate theory corresponds to the leading term of an interior (outer) asymptotic expansion of the exact solution for the same problem in linear three-dimensional elasticity theory. This leading term interior solution is generally insufficient for fitting the prescribed data at a cylindrical edge of the plate. For isotropic plates, it has been shown that the correct asymptotic interior solution fits the given edge data in a certain prescribed weighted average sense [13, 14]. When the edge data prescribed in terms of displacements give rise to a state of plane strain, the weighting functions in the averaging process are the stress components of the three canonical problems for cantilevered semi-infinite strips [15]. To the extent that the reciprocal theorem continues to hold, the same is true for orthotropic plates in plane strain deformation. Certain moments of the stresses of our strip problems will be calculated as they appear naturally in the appropriate displacement boundary conditions for plate theory solutions (to be formulated for the first time in Sec. 6 of this paper).

The proper displacement boundary conditions for plate theories will, for example, enable us to obtain the correct interior solution for a sheared orthotropic block. The corresponding flexibility coefficient of the block will be compared with the upper and lower bounds for the same coefficient available in the literature [16–18].

2. TWO CANONICAL PROBLEMS FOR A CLAMPED SEMI-INFINITE STRIP

We consider in this paper linear plane elasticity problems for orthotropic materials with the stress–strain relations

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{zz} \\ \epsilon_{xz} \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{pmatrix}$$

$$\equiv \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{12}}{E_2} & 0 \\ -\dfrac{\nu_{21}}{E_1} & \dfrac{1}{E_2} & 0 \\ 0 & 0 & \dfrac{1}{G} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{pmatrix}, \tag{1}$$

where $a_{12} = a_{21} (\nu_{12}/E_2 = \nu_{21}/E_1)$. For an isotropic material, we have $E_1 = E_2 = E$, $\nu_{12} = \nu_{21} = \nu$ and $G = E/2(1+\nu)$.

The strain components are related to the displacement components u_x and u_z by

$$\begin{aligned} \epsilon_{xx} &= u_{x,x} \\ \epsilon_{zz} &= u_{z,z} \\ \epsilon_{xz} &= \epsilon_{zx} = u_{z,x} + u_{x,z}. \end{aligned} \tag{2}$$

Throughout this paper, $(\ )_{,t}$ indicates partial differentiation of $(\ )$ with respect to t.

The stress components must be in equilibrium with interior and surface loads. In the absence of interior loading, the differential equations of equilibrium are satisfied identically by expressing the stress components in terms of a stress function ϕ as follows:

$$\begin{aligned} \sigma_{xx} &= \phi_{,zz} \\ \sigma_{zz} &= \phi_{,xx} \\ \sigma_{xz} &= \sigma_{zx} = -\phi_{,xz}. \end{aligned} \tag{3}$$

To maintain strain compatibility, ϕ must satisfy the fourth-order linear partial differential equation

$$\phi_{,yyyy} + (2+\delta_0)\phi_{,yyzz} + \phi_{,zzzz} = 0, \tag{4}$$

where

$$\begin{aligned} y &= x/\beta_0 \\ \beta_0 &= \left(\frac{E_1}{E_2}\right)^{1/4} \\ \delta_0 &= \frac{E}{G} - 2(1+\nu) \\ E &= \sqrt{(E_1 E_2)} \\ \nu &= \sqrt{(\nu_{21}\nu_{12})}. \end{aligned} \tag{5}$$

The parameter δ_0 vanishes for an isotropic material and is taken to be non-negative throughout this paper.

We are interested in problems involving the semi-infinite strip $\{|z| \leqslant h, 0 \leqslant x < \infty\}$ with two traction-free edges so that

$$z = \pm h: \quad \sigma_{zz} = 0, \quad \sigma_{xz} = 0 \quad (y > 0). \tag{6}$$

The end $x = 0$ of the strip is fixed so that

$$y = 0: \qquad u_x = 0, \quad u_z = 0 \quad (|z| < h). \tag{7}$$

The strip is loaded at infinity in one of the following ways:

(*B*): *the bending problem*

$$\sigma_{xx} \longrightarrow \frac{3z}{2h^3}, \quad \sigma_{xz} \longrightarrow 0 \quad \text{as } y \longrightarrow \infty \tag{8}$$

(*F*): *the flexure problem*

$$\sigma_{xx} \longrightarrow \frac{3xz}{2h^3}, \quad \sigma_{xz} \longrightarrow \frac{3}{4h}\left(1 - \frac{z^2}{h^2}\right) \quad \text{as } y \longrightarrow \infty. \tag{9}$$

As we shall see later, these are canonical problems whose solution will be needed in the plate theory solution of plane (strain) elasticity problems with displacement edge data. This same observation has already been noted in [15] for isotropic materials.

For the numerical solutions of these canonical problems, it is more convenient to subtract off a particular solution (different) for each case and work with the *residual* problems denoted by $\{\tilde{\phi}, \tilde{\sigma}, \tilde{u}\}$. For the bending problem, denoted by a superscript B, we let

$$\begin{aligned} \sigma^B_{xx} &= \tilde{\sigma}^B_{xx} + \frac{3z}{2h^3} \\ \sigma^B_{xz} &= \tilde{\sigma}^B_{xz} \\ \sigma^B_{zz} &= \tilde{\sigma}^B_{zz} \\ u^B_x &= \tilde{u}^B_x + \frac{3xz}{2h^3 E_1} - \omega_0 z \\ u^B_z &= \tilde{u}^B_z - \frac{3}{4h^3 E_1}(x^2 + \nu_{21} z^2) + \omega_0 x + w_0. \end{aligned} \tag{10}$$

For the flexure problem, denoted by a superscript F, we set

$$\begin{aligned} \sigma^F_{xx} &= \tilde{\sigma}^F_{xx} + \frac{3xz}{2h^3} \\ \sigma^F_{xz} &= \tilde{\sigma}^F_{xz} + \frac{3}{4h}\left(1 - \frac{z^2}{h^2}\right) \\ \sigma^F_{zz} &= \tilde{\sigma}^F_{zz} \\ u^F_x &= \tilde{u}^F_x + \frac{1}{4h^3}\left(\frac{3x^2 z}{E_1} + \frac{\nu_{21} z^3}{E_1} - \frac{z^3}{G}\right) - \omega_0 z \\ u^F_z &= \tilde{u}^F_z + \frac{1}{4h^3}\left(\frac{3h^2 x}{G} - \frac{x^3}{E_1} - \frac{3\nu_{21} x z^2}{E_1}\right) \\ &\quad + \omega_0 x + w_0. \end{aligned} \tag{11}$$

In both cases, the residual problem is governed by eqn (4) and the traction-free conditions (6) along $z = \pm h$ with the stress components tending to zero as $y \to \infty$. At the root of the strip, we have for the bending problem

$$y = 0: \quad \tilde{u}_x^B = \omega_0 z, \quad \tilde{u}_z^B = \frac{3\nu z^2}{4h^3 E} - w_0 \tag{12a}$$

and for the flexure problem

$$y = 0: \quad \tilde{u}_x^F = \frac{1}{4h^3}\left(\frac{1}{G} - \frac{\nu}{E}\right) z^3 + \omega_0 z$$
$$\tilde{u}_z^F = -w_0. \tag{12b}$$

In subsequent developments, we will work with the residual problems to analyze their solution. However, numerical results will be given for the actual solution of the physical boundary value problems unless explicitly stated otherwise.

3. THE RESIDUAL PROBLEM

We work with the *residual* problem defined by (i) the partial differential equation (4); (ii) the edge conditions (6) written in terms of the stress functions $\tilde{\phi}$ as

$$z = \pm h: \quad \tilde{\phi}_{,yy} = -\tilde{\phi}_{,yz} = 0; \tag{6'}$$

(iii) the end conditions (12a) or (12b) written as

$$y = 0: \quad \tilde{u}_x = \tilde{u}(z), \quad \tilde{u}_z = \tilde{w}(z), \tag{12'}$$

where $\tilde{u}(z)$ and $\tilde{w}(z)$ are given in eqn (12a) for the bending problem and in eqn (12b) for the flexure problem; and (iv) the limiting conditions at infinity

$$y \to \infty: \quad \tilde{\sigma}_{xx} \to 0, \quad \tilde{\sigma}_{xz} \to 0. \tag{13}$$

We have omitted the superscript B or F which denotes quantities associated with the bending or flexure problems as the development of this section applies to both (as well as the problem of stretching analyzed in [6]).

For isotropic materials, we know from the discussion in [15] that the solution of this residual problem must be an exponentially decaying state. In other words, we have as $y \to \infty \, (x \to \infty)$

$$\{\tilde{u}, \tilde{\sigma}\} = O(\mathrm{e}^{-\alpha x/h}) \tag{14}$$

for some positive $O(1)$ constant α. The interior (outer) solution and boundary layer (residual) solution for orthotropic plane elasticity problems obtained in [19] suggest that the same is true for our residual orthotropic strip problem. The method of Fourier cosine and sine transforms used for the stretching problem in [6] is therefore also applicable to the present problem. Let

$$\Phi(s, z) = \int_0^\infty \tilde{\phi}(y, z)\cos(sy)\,\mathrm{d}y \tag{15}$$

and transform eqn (4) into

$$\Phi_{,zzzz} - s^2(2 + \delta_0)\,\Phi_{,zz} + s^4\Phi = g(s, z), \tag{16}$$

where

$$g(s, z) = \tilde{\phi}_{,yyy}(0, z) - s^2\tilde{\phi}_{,y}(0, z) + (2 + \delta_0)\,\tilde{\phi}_{,yzz}(0, z). \tag{17}$$

For boundary conditions, we note that the Fourier sine transform of the boundary condition (6b) is

$$\Phi_{,z}(s, \pm h) = 0, \tag{18a}$$

whereas the Fourier cosine transform of (6a) is $-s^2\Phi(s, \pm h) - \tilde{\phi}_{,y}(0, \pm h) = 0$ or

$$\Phi(s, \pm h) = 0, \tag{18b}$$

with $\tilde{\sigma}_{xz}(y, \pm h) = 0$ implying $\tilde{\phi}_{,y}(y, \pm h) = 0$ [19].

We now rewrite the expression for $g(s, z)$ in terms of $\tilde{\sigma}_{xz}(0, z) \equiv \tilde{\tau}(z)$ and known quantities. For exponentially decaying states, the following relations hold:

$$\tilde{\phi}_{,y}(0, z) = -\beta_0 \int_{-h}^{z} \tilde{\tau}(t)\,\mathrm{d}t = \beta_0 \int_z^h \tilde{\tau}(t)\,\mathrm{d}t \tag{19a}$$

$$\tilde{\phi}_{,yzz}(0, z) = -\beta_0 \tilde{\tau}'(z) \tag{19b}$$

$$\tilde{\phi}_{,yyy}(0, z) = -\beta_0 E\tilde{u}''(z) + \beta_0\left(\frac{E}{G} - \nu\right)\tilde{\tau}'(z), \tag{19c}$$

where a prime indicates differentiation with respect to the argument of the function. To obtain the third relation in eqn (19), we use the relation for the transverse normal strain ϵ_{zz} in the form

$$E_2 \tilde{u}_{z,zx} = (\tilde{\sigma}_{zz} - \nu_{12}\tilde{\sigma}_{xx})_{,x} = \frac{1}{\beta_0^3}\tilde{\phi}_{,yyy} - \nu_{12}\tilde{\sigma}_{xx,x}$$

or

$$\tilde{\phi}_{,yyy}(y, z) = \beta_0^3[E_2\tilde{u}_{z,xz} + \nu_{12}\tilde{\sigma}_{xx,x}]$$
$$= -\beta_0 E\tilde{u}_{x,zz} + \beta_0\left(\frac{E}{G} - \nu\right)\tilde{\sigma}_{xz,z},$$

where we have used the equilibrium equation $\tilde{\sigma}_{xx,x} + \tilde{\sigma}_{xz,z} = 0$ to eliminate $\tilde{\sigma}_{xx,x}$ and the relations (1c) and (2c) for shear strain to eliminate $\tilde{u}_{z,x}$. Now, eqn (19c) follows upon setting $y = 0 (x = 0)$ and keeping in mind $\tilde{u}_x(0, z) \equiv \tilde{u}(z)$ and $\tilde{\sigma}_{xz}(0, z) \equiv \tilde{\tau}(z)$. With eqn (19), we can write eqn (17) as

$$\frac{1}{\beta_0} g(s, z) = \nu\tilde{\tau}'(z) - E\tilde{u}''(z) - s^2 \int_z^h \tilde{\tau}(t)\,\mathrm{d}t. \tag{20}$$

The solution of the boundary value problem eqns (16)–(18) may be obtained by the method of variation of parameters. We take it in the form of a Green's function representation

$$\Phi(s, z) = \int_0^h \bar{G}(s, z; t)g(s, t)\,\mathrm{d}t, \tag{21}$$

where the Green's function $\bar{G}(s, z; t)$ is given by

$$\bar{G}(s,z;t)=\frac{\delta}{(\delta^4-1)s^3\Delta(\delta)}\begin{cases} S_1(\delta)+\delta^4 S_1(1/\delta)+\delta^2[T_1(\delta)+T_1(1/\delta)] & (0\leqslant t\leqslant z)\\ S_2(\delta)+\delta^4 S_2(1/\delta)+\delta^2[T_2(\delta)+T_2(1/\delta)] & (z\leqslant t\leqslant h)\end{cases} \tag{22}$$

with

$$\begin{aligned}
&\delta^2=1+\tfrac{1}{2}\delta_0+\sqrt{((1+\tfrac{1}{2}\delta_0)^2-1)}\geqslant 1\\
&\Delta(\xi)=\sinh(s\xi h)\cosh(sh/\xi)\\
&\qquad -\xi^2\cosh(sh\xi)\sinh(sh/\xi)\\
&S_1(\xi)=\cosh(sh/\xi)\sinh(s\xi t)\sinh(s\xi[z-h])\\
&S_2(\xi)=\cosh(sh/\xi)\sinh(s\xi z)\sinh(s\xi[t-h])\\
&T_1(\xi)=\sinh(s\xi t)\{\sinh(sh/\xi)\cosh(s\xi[h-z])\\
&\qquad -\sinh(sz/\xi)\}\\
&T_2(\xi)=\sinh(s\xi z)\{\sinh(sh/\xi)\cosh(s\xi[h-t])\\
&\qquad -\sinh(st/\xi)\}.
\end{aligned} \tag{23}$$

Upon substituting eqn (20) into eqn (21) and integrating the resulting expression by parts, we obtain

$$\frac{1}{\beta_0}\Phi(s,z)=-E\int_0^h \bar{G}\tilde{u}''(t)\,\mathrm{d}t - \int_0^h\left\{v\bar{G}_{,t}+s^2\int_0^t \bar{G}(s,z;\xi)\,\mathrm{d}\xi\right\}\tilde{\tau}(t)\,\mathrm{d}t. \tag{24}$$

By the inversion formula, we obtain from eqn (24)

$$\begin{aligned}
\tilde{\phi}(y,z)&=\frac{2}{\pi}\int_0^\infty \cos(sy)\Phi(s,z)\,\mathrm{d}s\\
&=-\frac{2}{\pi}\beta_0 E\int_0^h G_0(y,z;t)\tilde{u}''(t)\,\mathrm{d}t\\
&\quad -\frac{2\beta_0}{\pi}\int_0^h\{vG_1(y,z;t)\\
&\quad +G_2(y,z;t)\}\,\tilde{\tau}(t)\,\mathrm{d}t,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
G_0(y,z;t)&=\int_0^\infty \cos(sy)\bar{G}(s,z;t)\,\mathrm{d}s\\
G_1(y,z;t)&=\int_0^\infty \cos(sy)\bar{G}_{,t}(s,z;t)\,\mathrm{d}s\\
G_2(y,z;t)&=\int_0^\infty \cos(sy)s^2\left[\int_0^t \bar{G}(s,z;\xi)\,\mathrm{d}\xi\right]\mathrm{d}s.
\end{aligned} \tag{26}$$

The expression (25) is not yet a solution of our problem as we do not know $\tilde{\tau}(z)\equiv\tilde{\sigma}_{xz}(0,z)$. To determine $\tilde{\tau}(z)$, we use the known end data for $\tilde{u}_z$ as follows. From the expression for the transverse normal strain component ϵ_{zz} in eqn (2), we obtain

$$\tilde{u}_{z,z}=\frac{1}{E}\left\{\frac{\partial^2}{\partial y^2}-v\frac{\partial^2}{\partial z^2}\right\}\tilde{\phi}\equiv\frac{1}{E}L[\tilde{\phi}]. \tag{27}$$

Upon substituting eqn (25) into the right-hand side of the above relation, we obtain

$$\frac{1}{\pi}\int_0^h\{vL[G_1]+L[G_2]\}\,\tilde{\tau}(t)\,\mathrm{d}t = -\frac{E}{2\beta_0}\tilde{u}_{z,z}-\frac{E}{\pi}\int_0^h L[G_0]\tilde{u}''(t)\,\mathrm{d}t. \tag{28}$$

With $\tilde{u}_z$ and $\tilde{\phi}$ both being even functions of y, we may set $y=0$ to obtain

$$\frac{1}{\pi}\int_0^h K(z;t)\tilde{\tau}(t)\,\mathrm{d}t=F(z), \tag{29a}$$

where

$$K(z;t)=\{vL[G_1(y,z;t)]+L[G_2(y,z;t)]\}\big|_{y=0} \tag{29b}$$

$$F(z)=-\frac{E}{2\beta_0}\tilde{w}'(z) -\frac{E}{\pi}\int_0^h L[G_0(y,z;t)]\big|_{y=0}\,\tilde{u}''(t)\,\mathrm{d}t. \tag{29c}$$

Equation (29a) is a Fredholm integral equation of the first kind for $\tilde{\tau}(z)\equiv\tilde{\sigma}_{xz}(0,z)$ as the right-hand side is a known function. For the bending problem we have

$$F(z)=-\frac{3vz}{4\beta_0 h^3} \tag{29d}$$

and for the flexure problem,

$$\begin{aligned}
F(z)&=-\frac{3}{2\pi h^3}\left(\frac{E}{G}-v\right)\int_0^h L[G_0]\big|_{x=0}t\,\mathrm{d}t\\
&=\frac{3}{4\pi h^3}\left(\frac{E}{G}-v\right)\int_0^h L[G_1]\big|_{x=0}t^2\,\mathrm{d}t.
\end{aligned} \tag{29e}$$

For the purpose of formulating the proper boundary conditions for plate theories with displacement edge data, we need only $\tilde{\sigma}(z)\equiv\tilde{\sigma}_{xx}(0,z)$ in addition to $\tilde{\tau}(z)$.

To obtain $\tilde{\sigma}_{xx}(0,z)$, we note the relation

$$\begin{aligned}
\tilde{\sigma}_{xx}(0,z)&=\lim_{y\to 0}\tilde{\phi}_{,zz}(y,z)\\
&=-\frac{2}{\pi}\beta_0 E\int_0^h G_{0,zz}(y,z;t)\big|_{y=0}\tilde{u}''(t)\,\mathrm{d}t\\
&\quad -\frac{2}{\pi}\beta_0\int_0^h\{vG_{1,zz}(y,z;t)\\
&\quad +G_{2,zz}(y,z;t)\}\big|_{y=0}\,\tilde{\tau}(t)\,\mathrm{d}t.
\end{aligned} \tag{30}$$

For the bending problem, the above relation simplifies to read

$$\tilde{\sigma}(z) = -\frac{2}{\pi}\beta_0\int_0^h \{\nu G_{1,zz} + G_{2,zz}\}_{y=0}\tilde{\tau}(t)\,\mathrm{d}t. \quad (31)$$

In either case, the right-hand side is a known quantity once we have $\tilde{\tau}(z)$ from the solution of eqn (29).

Knowing $\tilde{\tau}(z)$, we actually have the complete solution of the problem as eqn (25) gives $\tilde{\phi}(x, z)$ and the displacement components can be obtained from the strain–displacement relations (2).

4. A GENERALIZED CAUCHY INTEGRAL EQUATION

The expressions for G_1 and G_2 defined in eqn (26) will now be cast in a form convenient for a further analysis of the kernel $K(z; t)$ in the integral equation (29). Let

$$\begin{aligned}
\bar{P}_1(s, z; t; \delta) &= \cosh(sh/\delta)\cosh(s\delta t)\sinh(s\delta[z-h])\\
\bar{P}_2(s, z; t; \delta) &= \cosh(sh/\delta)\sinh(s\delta z)\cosh(s\delta[t-h])\\
\bar{Q}_1(s, z; t; \delta) &= \cosh(st/\delta)\{\sinh(s\delta h)\cosh(s[h-z]/\delta) - \sinh(s\delta z)\}\\
\bar{Q}_2(s, z; t; \delta) &= \sinh(sz/\delta)\sinh(s\delta h)\sinh(s[t-h]/\delta) - \sinh(s\delta z)\cosh(st/\delta)
\end{aligned} \quad (32)$$

and

$$\{P_k(y,z; t; \delta), Q_j(y, z; t; \delta)\} = \int_0^\infty \frac{\cos(sy)}{s^2\Delta(\delta)}\{\bar{P}_k(s, z; t; \delta), \bar{Q}_j(s, z; t; \delta)\}\,\mathrm{d}s \quad (33)$$

for $k,j = 1, 2$. We may then write G_1 and G_2 as

$$G_1(y, z; t; \delta) = \frac{\delta^2}{\delta^4 - 1}\begin{cases} [P_1(y, z; t; \delta) - Q_1(y, z; t; 1/\delta)] \\ \qquad - [P_1(y, z; t; 1/\delta) - Q_1(y, z; t; \delta)] & (0 \leqslant t \leqslant z) \\ [P_2(y, z; t; \delta) - Q_2(y, z; t; 1/\delta)] \\ \qquad - [P_2(y, z; t; 1/\delta) - Q_2(y, z; t; \delta)] & (z \leqslant t \leqslant h) \end{cases} \quad (34)$$

$$G_2(y, z; t; \delta) = \frac{1}{\delta^4 - 1}\begin{cases} [P_1(y, z; t; \delta) - Q_1(y, z; t; 1/\delta)] \\ \qquad - \delta^4[P_1(y, z; t; 1/\delta) - Q_1(y, z; t; \delta)] & (0 \leqslant t \leqslant z) \\ [P_2(y, z; t; \delta) - Q_2(y, z; t; 1/\delta)] \\ \qquad - \delta^4[P_2(y, z; t; 1/\delta) - Q_2(y, z; t; \delta)]. & (z \leqslant t \leqslant h) \end{cases} \quad (35)$$

It is important to note that, in studying the convergence of the improper integrals for G_1 and G_2, the entire integrand, and not individual terms such as those given by eqn (33), should be analyzed. A particular improper integral may converge even if some of its parts do not individually.

The asymptotic expressions for $\bar{P}_k$ and $\bar{Q}_j$ for $s \geqslant \eta \gg 1$ have been worked out in [6]. They enable us to split up the kernel $K(z; t)$ of the integral equation for $\tilde{\tau}(z)$ in eqn (29) into a nonsingular part $K_\eta(z; t)$, a singular part, and an exponentially small part for $\eta \gg 1$, so that

$$\begin{aligned}
&\frac{1}{\pi}\int_0^h\left[\frac{\lambda_1}{t-z} + \frac{\lambda_2}{t+z} + \frac{\lambda_3}{2h-t-z} + \frac{\lambda_4}{h-z+\delta^2(h-t)} + \frac{\lambda_5}{\delta^2(h-z)+(h-t)}\right]\tilde{\tau}(t)\,\mathrm{d}t \\
&\quad + \frac{1}{\pi}\int_0^h K_\eta(z; t)\tilde{\tau}(t)\,\mathrm{d}t \\
&= F(z) + 0(\mathrm{e}^{-\eta h/\delta}),
\end{aligned} \quad (36)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_5$ are independent of η and are defined by

$$\lambda_1 = -\lambda_2 = [\delta^2(\delta^2+\nu)^2 - (1+\nu\delta^2)^2]/[2\delta(\delta^4-1)] \quad (37)$$

$$\lambda_3 = [(1+\nu\delta^2)^2 + \delta^2(\nu+\delta^2)^2]/[2\delta(\delta^2-1)^2] \quad (38)$$

$$\lambda_4 = -\delta(1+\nu\delta^2)(\nu+\delta^2)/[(\delta^4-1)(\delta^2-1)]$$

$$\lambda_5 = -\delta^3(\nu+\delta^2)(1+\nu\delta^2)/[(\delta^4-1)(\delta^2-1)]. \quad (39)$$

Let

$$\alpha_1 = -\alpha_2 = -\frac{1}{2\delta}(1+\nu\delta^2)$$

$$\alpha_3 = \frac{(1+\delta^2)}{2\delta(\delta^2-1)}(1+\nu\delta^2) \quad (40)$$

$$\alpha_4 = -\frac{\delta}{(\delta^2-1)}(\delta^2+\nu)$$

$$\alpha_5 = -\alpha_6 = \frac{1}{2\delta}(\nu+\delta^2) \quad (41)$$

$$\alpha_7 = \frac{(1+\delta^2)}{2\delta(\delta^2-1)}(\delta^2+\nu)$$

$$\alpha_8 = -\frac{\delta}{(\delta^2-1)}(1+\nu\delta^2). \quad (42)$$

Then, the nonsingular portion of the kernel $K(z;t)$ is given by

$$K_\eta(z;t) = \nu K_{1\eta} + K_{2\eta} + \frac{\nu\delta^2}{\delta^4-1}\sum_{i=1}^{8}\alpha_i f_{iR} + \frac{1}{\delta^4-1}\left[\sum_{i=1}^{4}\alpha_i f_{iR} + \delta^4\sum_{i=5}^{8}\alpha_i f_{iR}\right], \quad (43)$$

where the functions $f_{iR}(z;t;\eta,\delta)$ are as previously given in [6] and

$$K_{j\eta}(z;t) = \begin{cases} K_{j\eta 1}(z;t) & (0 \leqslant t \leqslant z) \\ K_{j\eta 2}(z;t) & (z \leqslant t \leqslant h) \end{cases} \quad (44a)$$

with

$$K_{1\eta k}(z;t) = -\frac{\delta^2}{\delta^4-1}\left\{\int_0^\eta\left[\bar{P}_k(\delta)+\bar{Q}_k(\delta) + \delta^2\bar{P}_k\left(\frac{1}{\delta}\right)+\delta^2\bar{Q}_k\left(\frac{1}{\delta}\right)\right]\frac{ds}{\Delta(s,\delta)} + \nu\int_0^\eta\left[\delta^4\tilde{Q}_k\left(\frac{1}{\delta}\right)+\delta^2\bar{P}_k(\delta) + \bar{P}_k\left(\frac{1}{\delta}\right)+\frac{1}{\delta^2}\tilde{Q}_k(\delta)\right]\frac{ds}{\Delta(s,\delta)}\right\} \quad (44b)$$

$$K_{2\eta k}(z;t) = -\frac{1}{\delta^4-1}\left\{\int_0^\eta\left[\bar{P}_k(\delta)+\delta^2\bar{Q}_k\left(\frac{1}{\delta}\right) + \delta^4\bar{Q}_k(\delta)+\delta^6\bar{P}_k\left(\frac{1}{\delta}\right)\right]\frac{ds}{\Delta(s,\delta)} + \nu\int_0^\eta\left[\delta^2\bar{P}_k(\delta)+\delta^2\tilde{Q}_k(\delta) + \delta^4\bar{P}_k\left(\frac{1}{\delta}\right)+\delta^4\tilde{Q}_k\left(\frac{1}{\delta}\right)\right]\frac{ds}{\Delta(s,\delta)}\right\}. \quad (44c)$$

In eqns (44b) and (44c), the functions $\tilde{Q}_1$ and $\tilde{Q}_2$ are defined by

$$\tilde{Q}_1(\xi) \equiv \tilde{Q}_1(s,z;t;\xi) = \cosh(st/\xi)\{(\sinh(s\xi h)\cosh(s[h-z]/\xi) - \xi^4\sinh(s\xi z)\}$$

$$\tilde{Q}_2(\xi) \equiv \tilde{Q}_2(s,z;t;\xi) = \sinh(s\xi h)\sinh(sz/\xi)\sinh(s[t-h]/\xi) - \xi^4\sinh(s\xi z)\cosh(st/\xi); \quad (45)$$

we have also suppressed the dependence of $\bar{P}_k$ and $\bar{Q}_k$ on s, z and t [as indicated in eqn (32)] so that the equations do not become unwieldy. Note that the regular kernel $K_\eta(z;t)$ contains contributions from both the proper integrals over $(0,\eta)$, $\nu k_{1\eta}+k_{2\eta}$ in eqn (43), and the improper integrals over (η,∞), the rest of the right-hand-side of eqn (43).

Our Fredholm integral equation of the first kind [eqn (29)] is now seen to be singular of the form of eqn (36). While the singularity of $K(z;t)$ is of the Cauchy type, the integral equation (36) is more general than those treated in [8] and [7]. With a slight modification, the method of [7] may also be used for extracting the corner singularity of our orthotropic strip. The details are effectively contained in [6] and will not be given here (see also [20]).

5. NUMERICAL SOLUTION FOR THE BENDING AND FLEXURE PROBLEM

The singular integral eqn (36) is a special case of the more general eqn (I.1) in Appendix I of [6] with $a=0$, $b=h$, $x=z$, $J=1$, $M=3$ and $c_0=\lambda_1$. The remaining constants are identified as follows:

$$\begin{array}{lll} c_1=\lambda_2, & k_1=1, & \theta_1=\pi \\ d_1=-\lambda_3, & h_1=1, & \omega_1=0 \\ d_2=-\lambda_4/\delta^2, & h_2=1/\delta^2, & \omega_2=0 \\ d_3=-\lambda_5, & h_3=\delta^2, & \omega_3=0. \end{array} \quad (46)$$

Therefore, the singularity exponent β at $z=h$ is a root of the equation (cf. eqn (I.14) of [6])

$$f(\beta) \equiv \lambda_1\cos(\pi\beta) - \lambda_3 - \lambda_4\delta^{-2+2\beta} - \lambda_5\delta^{-2\beta} = 0. \quad (47)$$

The coefficients, λ_i, of eqn (47) are not sensitive to the directional preference of the material; hence neither is the solution β. In the limit as $\delta \to 1$, eqn (47) reduces to the well known equation for isotropic material. With $f'(\beta)<0$ for $0<\beta<1$, $f(0)>0$ and $f(1)<0$, there is a unique real solution of eqn (47) in the interval $0<\beta<1$. It has been proved that, for an isotropic strip, eqn (47) has no complex solution in $0<\mathrm{Re}\{\beta\}<1$ [21]. As indicated in the last section,

Table 1. Elastic moduli for six plate materials

Material	E_1	E_2	G	ν_{12}
(1) Pine wood 1	100,000 kg/cm^2	4200 kg/cm^2	7500 kg/cm^2	0.01
(2) Plywood 1	120,000 kg/cm^2	64,400 kg/cm^2	7200 kg/cm^2	0.044
(3) Pine wood 2	4200 kg/cm^2	100,000 kg/cm^2	7500 kg/cm^2	0.238
(4) Plywood 2	64,400 kg/cm^2	120,000 kg/cm^2	7200 kg/cm^2	0.082
(5) Isotropic 1	15,732 kg/cm^2	15,732 kg/cm^2	7500 kg/cm^2	0.0488
(6) Isotropic 2	15,265 kg/cm^2	15,265 kg/cm^2	7200 kg/cm^2	0.06

the singularity of orthotropic strips can also be analyzed by a method similar to that of [20]. In fact, there is an excellent agreement between the numerical solutions by these two methods obtained in [21].

For a numerical solution of eqn (36), the singular integral equation is discretized by the method of orthogonal collocation [22]. With the singularity exponent β determined by eqn (47), the unknown function $\tilde{\tau}(z)$ can be written as $(h-z)^{-\beta}\bar{\tau}(z)$, where $\bar{\tau}(z)$ is a bounded function and represented by a linear combination of Jacobi polynomials $J_n^{(0,-\beta)}(z)$. The solution can be made to satisfy the global equilibrium conditions and be evaluated at the zeros of the Jacobi polynomials [23].

The parameter η which splits the improper integral eqn (33) [cf. also eqn (44)] is taken to be 50; a larger η does not noticeably change the numerical results. Various integrals are evaluated numerically by Gaussian quadratures in Jacobi polynomials whose weights and roots are given by known procedures [24]. By taking advantage of the symmetry property of $\tilde{\tau}$, a system of eight equations is sufficiently accurate for all the cases treated in this paper.

The above numerical solution process has been implemented on the Amdahl 5850 in the Computing Centre of the University of British Columbia. A typical run for the complete solution with 10 collocation points required less than 23.3 sec of machine time. The FORTRAN code written for this purpose was tested by comparing the results for an effectively isotropic strip (with $E_2 = 20{,}000$ kg/cm^2, $E_1 = 20{,}000.00001$ kg/cm^2, $G = 7500$ kg/cm^2 and $\nu = \frac{1}{3}$ corresponding to a Poisson's ratio of $\frac{1}{4}$) with the corresponding results in [3]. The agreement is excellent. In fact, with eight collocation constants, the moments of stress [as to be defined in eqn (55)] are close to (within 2% of) the results of [3] except for the quantity n_3^F; the latter is close to (within 7% of) the result of [3] with 10 collocation constants.

New results are reported here for six different materials whose elastic parameter values are as given in Table 1. Evidently, materials (3) and (4) are just materials (1) and (2), respectively, rotated in the plane by 90°. The value of ν for materials (5)/(6) is equal to $\sqrt{(\nu_{12}\nu_{21})}$ for pine woods/plywoods.

The distributions of $\sigma_{xz}^B(0, z/h)/(M/h^2)$, where

$$M = \int_{-h}^{h} \sigma_{xx}^B z \, dz,$$

and $\sigma_{xx}^B(0, z/h)/(M/h^2)$ for bending, are shown in Figs 1 and 2. The distributions for $\sigma_{xx}^B/(M/h^2)$ at the root of the strip for all six materials shown in Fig. 2 are nearly indistinguishable and linear. This suggests that the corresponding St. Venant solution for $\sigma_{xx}^B/(M/h^2)$ is applicable to a wide range of materials, except near the corner.

The distributions of $\sigma_{xz}^F(0, z/h)/(Q/h)$, where

$$Q = \int_{-h}^{h} \sigma_{xz}^F \, dz$$

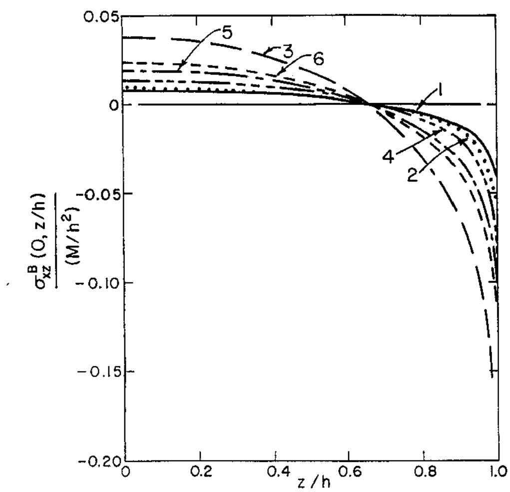

Fig. 1. Distributions of transverse shear stress at the root of semi-infinite cantilevered strips in bending.

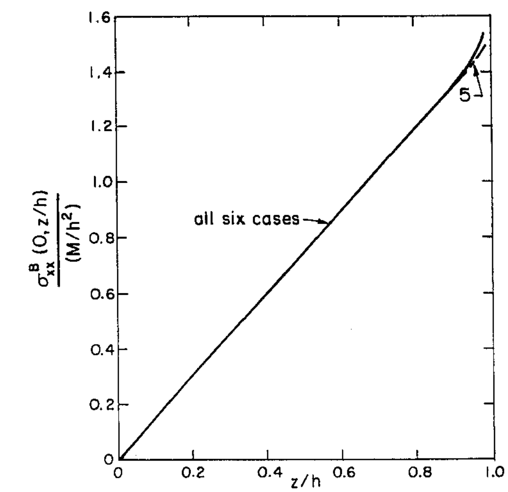

Fig. 2. Distributions of axial stress at the root of semi-infinite cantilevered strips in bending.

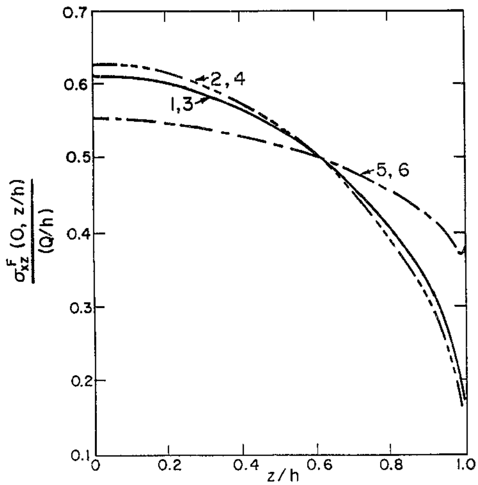

Fig. 3. Distributions of transverse shear stress at the root of semi-infinite cantilevered strips in flexure.

and $\sigma^F_{xx}(0, z/h)(Q/h)$ for flexure are shown in Figs 3 and 4. It is evident from Fig. 3 that $\sigma^F_{xz}(0, z/h)/(Q/h)$ is insensitive to the direction of the orthotropy. Figure 4, on the other hand, shows that $\sigma^F_{xx}(0, z/h)/(Q/h)$ for the cantilevered strip in flexure depends on the direction of orthotropy through the parameter β_0.

The accuracy of these results for orthotropic strips has been verified by a second (less efficient) method of solution described in Appendix II of [6].

6. DISPLACEMENT BOUNDARY CONDITIONS FOR THE INTERIOR SOLUTION IN PLANE STRAIN DEFORMATIONS

We know from [19] that

$$\phi_I = \frac{N_0}{4h} z^2 + \frac{M_0}{4h^3} z^3 - \frac{3Q_0 x}{4h}\left(z - \frac{z^3}{3h^2}\right) \quad (48)$$

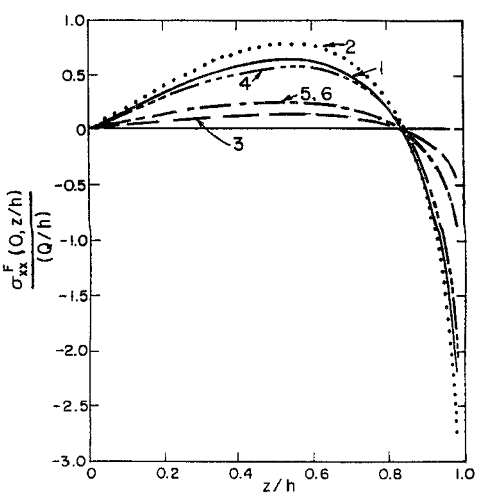

Fig. 4. Distributions of axial stress at the root of semi-infinite cantilevered strips in flexure.

is an exact solution of eqns (4) and (6). This is identical to the classical Lévy solution [15] for isotropic materials. For sufficiently thin plates, the residual solution in that case is a boundary layer effect [15]; hence eqn (48) is more commonly known as the *interior* solution. The results of [19] suggest that the same is true for the present orthotropic problem unless $\beta_0 \gg 1$. The stress and displacement components corresponding to eqn (48) have been obtained in [19]; we list here the two displacement components needed later in this section

$$\begin{aligned} E_1 u_x^I &= \frac{N_0}{2h} x + \frac{3M_0}{2h^3} xz \\ &\quad + \frac{Q_0}{4h^3}\left(3x^2 z + \nu_{21} z^3 - \frac{E_1}{G} z^3\right) \\ &\quad + E_1(d - \omega z) \\ E_1 u_z^I &= -\frac{\nu_{21} N_0}{2h} z - \frac{3M_0}{4h^3}(\nu_{21} z^2 + x^2) \\ &\quad + \frac{Q_0}{4h^3}\left(\frac{3h^2 E_1}{G} x - x^3 - 3\nu_{21} x z^2\right) \\ &\quad + E_1(c + \omega x). \end{aligned} \quad (49)$$

Let $\tilde{u}_x \equiv u_x - u_x^I$, $\tilde{u}_z \equiv u_z - u_z^I$, etc. For a strip which spans the interval $(-l, l)$ in the x-direction, we apply the reciprocal theorem to the half strip $-l \leqslant x \leqslant 0$ with state (1) taken to be the residual state $\{\tilde{\sigma}, \tilde{u}\}$ and state (2) taken to be the solution of any one of three canonical problems for a semi-infinite strip with its clamped end at $x = -l$. By an argument similar to that for isotropic materials [15], we obtain from the reciprocal theorem for $\beta_0 h \ll l$

$$\int_{-h}^{h} [\sigma_C(z)\tilde{u}_x(-l, z) + \tau_C(z)\tilde{u}_z(-l, z)]\,dz = O(e^{-\lambda_0 l/\beta_0 h}), \quad (50)$$

where λ_0 is an $O(1)$ constant, C may be E, B or F and σ_C and τ_C are the normal and shear stresses at $x = -l$ of the relevant strip problem. Upon omitting the exponentially small term, we rewrite eqn (50) as

$$\begin{aligned} &\int_{-h}^{h} [\sigma_C(z)\bar{u}(z) + \tau_C(z)\bar{w}(z)]\,dz \\ &= \int_{-h}^{h} [\sigma_C(z)u_x^I(-l, z) + \tau_C(z)u_z^I(-l, z)]\,dz. \end{aligned} \quad (51)$$

We may now use the expressions for u_x^I and u_z^I in eqn (51) and the properties of the solution of the three canonical problems to obtain

$$\frac{N_0}{2E_1}\left[\frac{x}{h} - \nu_{21} t_1^E\right] + d = \int_{-h}^{h} \{\sigma_E(z)\bar{u}_x(z) + \tau_E(z)\bar{u}_z(z)\}\,dz \quad (52)$$

$$\frac{Q_0}{4E_1}\left[\frac{3x^2}{h^2}+\left(\nu_{21}-\frac{E_1}{G}\right)n_3^B-3\nu_{21}\frac{x}{h}t_2^B\right]$$
$$+\frac{3M_0}{4E_1 h}\left[2\frac{x}{h}-\nu_{21}t_2^B\right]-\omega h$$
$$=h\int_{-h}^{h}\{\sigma_B(z)\bar{u}_x(z)+\tau_B(z)\bar{u}_z(z)\}\,\mathrm{d}z \quad (53)$$

$$\frac{Q_0}{4E_1}\left[\left(\nu_{21}-\frac{E_1}{G}\right)n_3^F+\left(\frac{3xE_1}{Gh}-\frac{x^3}{h^3}\right)-3\nu_{21}\frac{x}{h}t_2^F\right]$$
$$-\frac{3M_0}{4E_1 h}\left[\nu_{21}t_2^F+\frac{x^2}{h^2}\right]+\omega x+c$$
$$=\int_{-h}^{h}\{\sigma_F(z)\bar{u}_x(z)+\tau_F(z)\bar{u}_z(z)\}\,\mathrm{d}z, \quad (54)$$

where $x=-l$ and the dimensionless quantities t_1^E, n_3^B, t_2^B, etc. are defined by

$$n_3^F=\frac{1}{h^3Q_0}\int_{-h}^{h}z^3\sigma_F\,\mathrm{d}z$$
$$t_1^E=\frac{1}{hN_0}\int_{-h}^{h}z\tau_E\,\mathrm{d}z$$
$$t_2^F=\frac{1}{h^2Q_0}\int_{-h}^{h}z^2\tau_F\,\mathrm{d}z$$
$$n_3^B=\frac{1}{h^2M_0}\int_{-h}^{h}z^3\sigma_B\,\mathrm{d}z$$
$$t_2^B=\frac{1}{hM_0}\int_{-h}^{h}z^2\tau_B\,\mathrm{d}z, \quad (55)$$

with

$$Q_0=\int_{-h}^{h}\tau_F\,\mathrm{d}z$$
$$M_0=\int_{-h}^{h}\sigma_B z\,\mathrm{d}z$$
$$N_0=\int_{-h}^{h}\sigma_E\,\mathrm{d}z. \quad (56)$$

Three other conditions are needed to completely determine the six unknown constants N_0, Q_0, M_0, ω, c and d. They come from three necessary conditions on the difference between the interior solution and the exact solution at the second edge $x=l$. If the edge data at $x=l$ are also prescribed in terms of the displacement components, then eqns (52)–(54) for $x=l$ again apply with $\sigma_C(z)$ and $\tau_C(z)$ being for a semi-infinite strip whose end is at $x=l$ and with $\bar{u}_x$ and $\bar{u}_z$ being the data at that edge. The relevant three conditions for other sets of admissible edge data have already been discussed in [19]. As such, the interior solution of an orthotropic plate in plane strain deformation is determined (up to exponentially small terms) without any knowledge of the boundary layer solution components.

The constants $\{n_m^C, t_j^C\}$ in eqns (55) depend only on the material properties of the plate and can be calculated once and for all for a given plate material. We record in Table 2 the values of these constants for the six materials in Table 1 with t_1^E taken from [6].

The moments in Table 2 are for the edge $x=-l$ (or any edge with an outward normal in the negative x-direction). For an edge with an outward normal in the positive x-direction, such as $x=l$, the corresponding moments $\hat{t}_m^C$ and $\hat{n}_m^C$ are equal to t_m^C and n_m^C in magnitude but possibly with a change of sign. It is straightforward to show

$$\hat{t}_1^E=-t_1^E,\quad \hat{t}_2^B=-t_2^B,\quad \hat{t}_2^F=t_2^F$$
$$\hat{n}_3^B=n_3^B,\quad \hat{n}_3^F=-n_3^F. \quad (57)$$

7. FLEXIBILITY COEFFICIENT OF A SHEARED BLOCK

Consider a homogeneous, linearly elastic rectangular block occupying the region $-l\leqslant x\leqslant l, |z|\leqslant h$ and extending without bound in the third (spanwise) direction. The top and bottom faces $z=\pm h$ are free of tractions and the side faces $x=\pm l$ are bonded to rigid walls. The block is now placed in a state of plane strain by displacing the walls at $x=\pm l$ a distance $\pm w_0$ in the z-direction (uniformly in the third direction). What forces and moments must be applied to the wall to produce these end displacements? Also of interest is the deformed shape of the block. As an application of the results of Sec. 6, we obtain in this section the solution of this problem up to $O(e^{-\lambda_0 l/\beta_0 h})$ terms.

For the elastostatic behavior of the block, it is sufficient to consider a layer of the block in a state of plane strain as the behavior of the block is uniform in the spanwise direction. For a sufficiently thin block ($\beta_0 h/l \ll 1$), it is expected that the interior solution dominates throughout the block except in a narrow region adjacent to each edge. In the isotropic case,

Table 2. Moments of stresses for the canonical problems

Material	t_1^E	t_2^B	n_3^B	t_2^F	n_3^F
(1) Pine wood 1	−0.005975	−0.005276	0.6024	0.2685	−0.1884
(2) Plywood 1	−0.007584	−0.006534	0.6003	0.3041	−0.2258
(3) Pine wood 2	−0.02914	−0.02573	0.6024	0.2685	−0.03861
(4) Plywood 2	−0.01035	−0.008921	0.6003	0.3041	−0.1654
(5) Isotropic 1	−0.01436	−0.01276	0.6030	0.2603	−0.07205
(6) Isotropic 2	−0.01757	−0.01562	0.6037	0.2603	−0.07205

the interior solution contribution to the flexibility coefficient was obtained in [15] without any reference to the boundary layer solution components. It was also shown there that this contribution is extremely accurate, even for a nearly square block.

For an orthotropic layer, only upper and lower bounds of the flexibility coefficient have been obtained [17]. Approximate stresses and deflections of the layer are discussed in [16–18] in conjunction with some boundary layer effects. With the results of the last section, we may now obtain the interior solution contribution without a simultaneous consideration of the boundary layer solution contribution.

In the context of the displacement boundary value problem for the sheared block in our plane strain formulation, we have

$$u_x(\pm l, z) = 0$$
$$u_z(\pm l, z) = \pm w_0. \qquad (58)$$

The unknown constant applied force (per unit span width) Q_0 is given by

$$Q_0 = \int_{-h}^{h} \sigma_{xz}(l, z)\,\mathrm{d}z = \int_{-h}^{h} \sigma_{xz}(-l, z)\,\mathrm{d}z \qquad (59)$$

and the flexibility coefficient C is given by

$$C = \frac{w_0}{Q_0}. \qquad (60)$$

The necessary conditions (52) for $x = l$ and $x = -l$ require that $N_0 = d = 0$. In fact, by the antisymmetric properties of the physical problem, we expect $M_0 = c = 0$ and may work with the boundary conditions at the end of $x = -l$ alone.

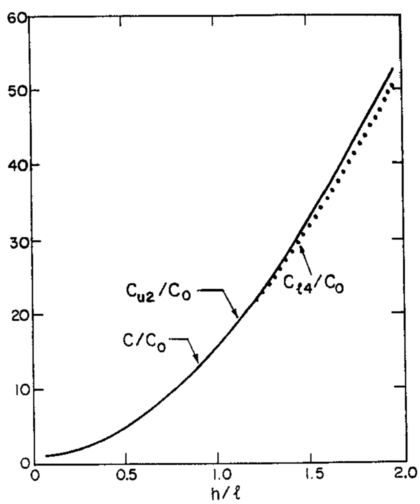

Fig. 5. Flexibility coefficients and bounds for a shear pine wood 1 block in plane strain deformation.

Table 3. Flexibility coefficient for a sheared pine wood 1 block

h/l	$C_{L4}/(l^3/2E_1h^3)$	$C/(l^3/2E_1h^3)$	$C_{U2}/(l^3/2E_1h^3)$
0.05	1.03958	1.03966	1.03969
0.55	5.57396	5.62837	5.63054
1.05	16.85403	17.20965	17.22431
1.95	50.60456	52.80735	52.90249

At $x = -l$, the remaining two necessary conditions (53) and (54) become

$$\frac{Q_0}{4E_1 h}\left[3\frac{l^2}{h^2} + \left(\nu_{21} - \frac{E_1}{G}\right)n_3^B + 3\nu_{21}\frac{l}{h}t_2^B\right] - \omega = 0 \qquad (61)$$

$$\frac{Q_0}{4E_1}\left\{\left(\nu_{21} - \frac{E_1}{G}\right)n_3^F - \left(3\frac{lE_1}{Gh} - \frac{l^3}{h^3}\right) + 3\nu_{21}\frac{l}{h}t_2^F\right\} - \omega l = -w_0. \qquad (62)$$

The results (61) and (62) can be specialized to the isotropic case and compared with the corresponding equations (5.7) and (5.8) in [15] only after we observe the fact that Poisson's ratio σ is related to ν by $\sigma = \nu/(1+\nu)$.

Upon using eqn (61) to eliminate ω from (62), we obtain

$$C = \frac{w_0}{Q_0} = \frac{l^3}{4E_1 h^3}\left\{2 + 3\nu_{21}t_2^B\frac{h}{l} + \left[n_3^B\left(\nu_{21} - \frac{E_1}{G}\right) + 3\left(\frac{E_1}{G} - \nu_{21}t_2^F\right)\right]\frac{h^2}{l^2} - n_3^F\left(\nu_{21} - \frac{E_1}{G}\right)\frac{h^3}{l^3}\right\}. \qquad (63)$$

For a block made of pine wood 1, a graph of $C/(l^3/2E_1h^3)$ is computed from eqn (63) and shown in Fig. 5. For $h/l \ll 1$, this solution is correct except for exponentially small terms. Some typical values of the normalized flexibility coefficient are given in Table 3. This table shows that the (interior) solution that we obtained lies between the upper bound C_{U2} and lower bound C_{L4} obtained in [17] and is almost indistinguishable from the upper bound on the graph.

Acknowledgements—Y.H.L. was supported by the University of British Columbia and an I. W. Killam Fellowship. F.Y.M.W. was supported by NSF Grant No. DMS-890-4845.

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