A Follower Load Buckling Problem for Rectangular Plates

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Introduction

In what follows, we consider a rectangular elastic plate with two opposite edges simply supported and with the other two edges acted upon by a uniform distribution of equal and opposite in-plane normal edge stress resultants. In order to determine the buckling values of these edge stress resultants, it is necessary to stipulate their direction during the process of buckling. One possible assumption is that they remain parallel to the plane of the unbuckled plate, the same as in the determination of the Euler load for a cantilever beam. The solution for this "non-follower" edge load plate problem has been given by Woinowsky-Krieger (1951).

An alternate stipulation for the applied edge loads is that they are of the follower type, with their directions remaining tangent to the plate surface during the process of buckling. It is well known that there is no static buckling load for the corresponding follower load cantilever *beam* buckling problem. The follower load plate buckling problem with which we are concerned here is mentioned in Woinowsky-Krieger (1951), with a statement which reads in free translation: *"It would not be difficult to show that there are no static follower type buckling loads for this plate problem, similar to the corresponding result for the cantilever beam buckling problem."* The results in Woinowsky-Krieger (1951) for the non-follower load problem are reproduced in Timoshenko and Gere (1961) without mention of the possibility, or impossibility, of follower load instabilities.

In this Note we show that the indicated static follower load problem—which may be of intrinsic rather than of practical interest—is in fact associated with finite buckling loads, and we determine numerical values and asymptotic expressions for these loads.

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Differential Equations and Boundary Conditions

We consider a uniform isotropic plate with midplane coordinates x and y, with simply supported edges $x = 0, a$ and with the edges $y = \pm b/2$ acted upon by uniform thrusts N. The differential equation for this plate buckling problem is

$$D\nabla^2\nabla^2 w + N w_{,yy} = 0. \tag{1}$$

The associated conditions of simple support are

$$x=0,\ a:\quad w=0,\quad D(w_{,xx}+\nu w_{,yy})=0. \tag{2}$$

The conditions at the loaded edges of the plate are

$$y=\pm\frac{b}{2}:\ \begin{cases} D(w_{,yy}+\nu w_{,xx})=0 \\ D[w_{,yyy}+(2-\nu)w_{,yxx}]+\epsilon N w_{,y}=0. \end{cases} \tag{3}$$

In these equations, D is the plate-bending stiffness factor, ν is Poisson's ratio, and ϵ has the value 0 or 1. When $\epsilon = 1$, we have the non-follower load case with the edge loads N remaining parallel to the undeflected midplane of the plate. When $\epsilon = 0$, we have the follower load case, with the edge loads remaining tangent to the deflected midsurface of the plate. We do not, in this Note, concern ourselves with problems corresponding to other values of ϵ.

The Condition of Buckling

We satisfy (1) and (2) by stipulating

$$w(x,y)=\sin(\pi x/a)f(\pi y/a). \tag{4}$$

With $\pi y/a = \xi$ and $(\ )_{,y} = (\pi/a)(\ )_{,\xi} \equiv (\pi/a)(\ )'$ we then obtain from (1), as a differential equation for $f(\xi)$,

$$f'''' - (2-k)f'' + f = 0, \tag{5}$$

where $k = Na^2/\pi^2 D$.

The boundary conditions at $y = \pm b/2$ become conditions for $\xi = \pm(\pi/2)(b/a) \equiv \pm\lambda$, of the form

$$f''(\pm\lambda) - \nu f(\pm\lambda) = f'''(\pm\lambda) - \rho f'(\pm\lambda) = 0 \tag{6}$$

where $\rho = 2 - \nu - \epsilon k$.

The solution of (5) can be written in the form

$$f = c_o \sinh(r\xi) + \bar{c}_o \sinh(\bar{r}\xi) + c_e \cosh(r\xi) + \bar{c}_e \cosh(\bar{r}\xi) \tag{7}$$

where $(\bar{\ })$ denotes the complex conjugate of ( ) and

$$r^2 = 1 - \frac{1}{2}k + i\sqrt{1-\left(1-\frac{1}{2}k\right)^2}. \tag{8}$$

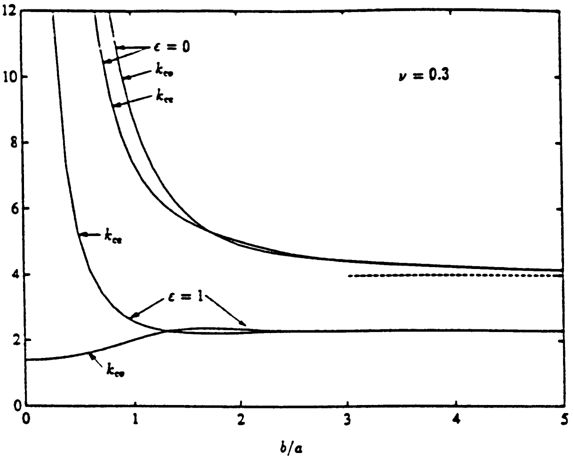

Fig. 1 The dependence of the critical loads k_{ce} and k_{co} on the aspect ratio b/a for $\nu = 0.3$

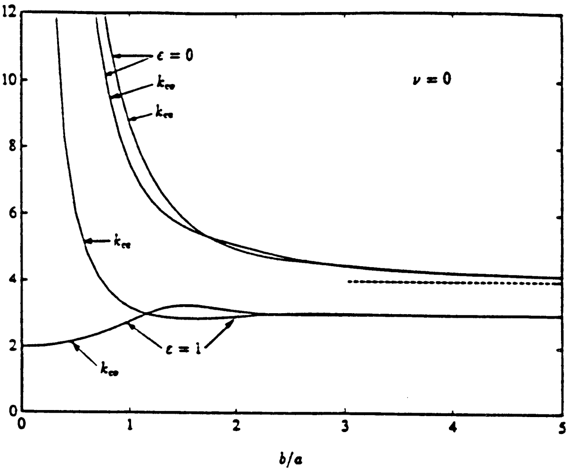

Fig. 2 The dependence of the critical loads k_{ce} and k_{co} on the aspect ratio b/a for $\nu = 0$

When $4 < k$, which turns out to be the range of k-values of interest here, it is preferable to write

$$f = c_{1o}\sin({}_1\xi) + c_{2o}\sin(p_2\xi) + c_{1e}\cos(p_1\xi) + c_{2e}\cos(p_2\xi) \tag{9}$$

where

$$\begin{pmatrix} p_1^2 \\ p_2^2 \end{pmatrix} = \frac{1}{2}k - 1 \pm \sqrt{\left(\frac{1}{2}k-1\right)^2 - 1}. \tag{10}$$

Introduction of (9) into (6) and a separate consideration of the even and odd buckling modes give the following conditions for the determination of the critical values of k in the range $4 < k$. For the even modes, we have

$$G_e(k) \equiv 2\begin{vmatrix} (p_1^2+\nu)\cos(\lambda p_1) & (p_2^2+\nu)\cos(\lambda p_2) \\ (p_1^3+\rho p_1)\sin(\lambda p_1) & (p_2^3+\rho p_2)\sin(\lambda p_2) \end{vmatrix}$$
$$= \sqrt{k-4}\,[(1-\nu)^2 - \nu k - \epsilon(1-\nu)k]\sin(\lambda\sqrt{k})$$
$$-\sqrt{k}\,[(1-\nu)(3+\nu) + \nu k - \epsilon(1+\nu)k]\sin(\lambda\sqrt{k-4}) = 0. \tag{11}$$

The corresponding condition for the odd modes[3] comes out to be

$$G_o(k) \equiv \sqrt{k}\,[(3+\nu)(1-\nu) + \nu k - \epsilon(1+\nu)k]\sin(\lambda\sqrt{k-4})$$
$$+\sqrt{k-4}\,[(1-\nu)^2 - \nu k - \epsilon(1-\nu)k]\sin(\lambda\sqrt{k}) = 0. \tag{12}$$

Similarly, introduction of (7) into (6) gives as the condition for the determination of the even-mode critical values of k in the range $0 < k < 4$

$$F_e(k) \equiv 2i\begin{vmatrix} (r^2-\nu)\cosh(\lambda r) & (\bar{r}^2-\nu)\cosh(\lambda\bar{r}) \\ (r^3-\rho r)\sinh(\lambda r) & (\bar{r}^3-\rho\bar{r})\sinh(\lambda\bar{r}) \end{vmatrix}$$
$$= \sqrt{k}\,[(3+\nu)(1-\nu) + \nu k - \epsilon k(1+\nu)]\sinh(\lambda\sqrt{4-k})$$
$$-\sqrt{4-k}\,[(1-\nu)^2 - \nu k - \epsilon k(1-\nu)]\sin(\lambda\sqrt{k}) = 0 \tag{13}$$

and for the odd-mode critical values,

$$F_o(k) \equiv \sqrt{k}\,[(3+\nu)(1-\nu) + \nu k - \epsilon k(1+\nu)]\sinh(\lambda\sqrt{4-k})$$
$$+\sqrt{4-k}\,[(1-\nu)^2 - \nu k - \epsilon k(1-\nu)]\sin(\lambda\sqrt{k}) = 0. \tag{14}$$

Equations (11), (13), and, in less explicit form (14), with $\epsilon = 1$, have previously been derived in Woinowsky-Krieger (1951). The equation corresponding to (12) for $\epsilon = 1$ is omitted there.

[3]These odd modes are also the buckling modes for a plate with sides a and $b/2$ and three simply-supported edges.

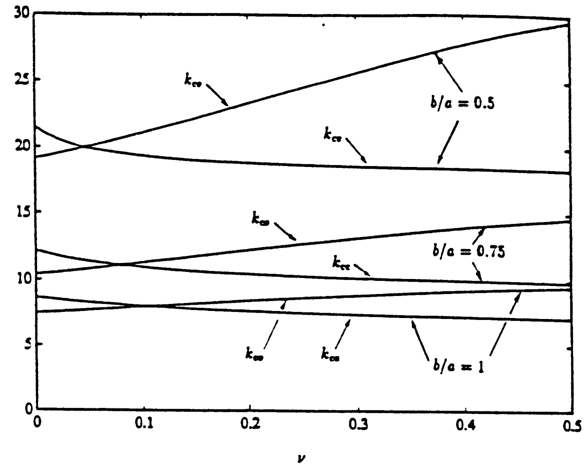

Fig. 3 The dependence of the critical loads k_{ce} and k_{co} on Poisson's ratio ν for $b/a = 0.5$, 0.75, and 1.0

The Buckling Load for the Follower Load Case

The numerical determination of the critical values of k for the three conditions (11), (13), and (14) for the non-follower load case with $\epsilon = 1$ has been carried out in Woinowsky-Krieger (1951). We limit ourselves here to the evaluation of the follower load case $\epsilon = 0$. While we do not find follower buckling loads in the range $0 \le k < 4$ as $F_e(k)$ and $F_o(k)$ do not change sign in this range of k, we do find follower buckling loads in the range $4 < k$ on the basis of Eqs. (11) and (12). Graphs of $G_e(k)$ and $G_o(k)$ give us estimates of the critical values k_c. Newton's iteration is then employed to obtain k_c accurate to four significant figures. Our numerical results are shown in Figs. 1 and 2 for the range $0 < b/a < 5$. The corresponding previously known results for the non-follower load case $\epsilon = 1$ are also shown for comparison.

As might be expected, k_c decreases as b/a increases and appears to be asymptotic to the value 4 as b/a approaches infinity. As might also be expected, for a given value of b/a, the values of k_c are larger for the follower load case than for the non-follower load case. The results for the two cases differ in the range by at most not much more than a factor of two, $2 < b/a$.

Asymptotic Behavior for Large and Small Aspect Ratios

The numerical results for sufficiently large values of b/a

Table 1 Variation of the computed k_{co}/k_{ce} ratio with aspect ratio and Poisson's ratio

b/a \ ν	0.5	0.3	0
0.5	1.614...	1.379...	0.8896...
0.3	1.664...	1.404...	0.9451...
0	1.685...	1.404...	1

indicate that k_c approaches the value four from above as λ tends to infinity. Setting in (11) and (12)

$$k_c \approx 4 + \frac{c^2}{\lambda^2}, \tag{15}$$

we find that these equations effectively reduce to the form $\sin c = 0$ with the smallest positive root $c = \pi$, and therewith

$$k_{ce} \approx k_{co} \approx 4\left(1 + \frac{a^2}{b^2}\right) \tag{16}$$

for sufficiently large values of b/a.

In the range $b/a \ll 1$, it is suggested by the form of (11) and (12) that

$$k_c \approx \frac{c^2}{\lambda^2}. \tag{17}$$

However, we now find that the results for Eqs. (11) and (12) differ from each other. Introduction of (17) into the even-mode formula (11), in conjunction with stipulating $\lambda \ll 1$, leads again to the simple relation $\sin c = 0$, so that in this range

$$k_{ce} \approx \frac{\pi^2}{\lambda^2} = \frac{4a^2}{b^2}. \tag{18}$$

Introduction of (17) into the odd-mode formula (12) leads to a somewhat less simple asymptotic result. We find, on the basis of the two terms with νk here having opposite signs, that the coefficient c^2 in (17) is now determined by the relation

$$(2 - \nu) \sin c = \nu c \cos c. \tag{19}$$

Evidently, we have $k_{co} = k_{ce}$ when $\nu = 0$. For $\nu > 0$, however, we have $c = c(\nu) > \pi$ and therewith the asymptotic values of k_{co} are larger than the corresponding values of k_{ce}. Specifically, we have $k_{co} \approx 1.404\, k_{ce}$ for $\nu = 0.3$ and $k_{co} \approx 1.685\, k_{ce}$ for $\nu = 0.5$.

The above asymptotic results are (as they must be) consistent with our numerical results for the effect of Poisson's ratio on the values of k_{ce} and k_{co} as may be seen from Table 1.

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