CANADIAN APPLIED
MATHEMATICS QUARTERLY
Volume 1, Number 1, Winter 1993

PERTURBATION SOLUTIONS FOR THE SECOND-BEST LAND USE PROBLEM

FREDERIC Y.M. WAN

ABSTRACT. A perturbation solution is obtained for the *second-best* land use problem in residential land economics for cities with a fixed boundary and with an absentee landlord. When the transportation cost increases linearly with traffic density and is a relatively small fraction of the total household income, the perturbation solution shows that the cost-benefit criterion based on market land price allocates more land for roads throughout the city than the second-best allocation and results in a lower (common) utility for the households. The perturbation method also allows us to make other explicit comparisons between the two types of urban land allocation. A spline-collocation solution for the same boundary value problems validates the accuracy of the perturbation solution for a small transportation cost and provides numerical results to a prescribed accuracy for other cases. The perturbation and numerical methods used in this paper apply also to the public ownership case and for a city with a free boundary determined by a prescribed agricultural rent.

1. Introduction. The pioneering work of R.M. Solow and W.S. Vickery [**9**] and R.M. Solow [**7,8**] showed that a conventional cost-benefit analysis based on the market value of urban land (called the *market allocation*) may be misleading as far as the allocation of residential land for roads and housing is concerned. When the social costs of traffic congestion are not priced by some sort of congestion tolls, the market land price does not reflect the true social value of the land. Hence, the market allocation is expected to lead to an excessive amount of land for roads. This qualitative conclusion was substantiated quantitatively "on the average" in [**8**] using only an equilibrium model. Such an approach does not give a direct pointwise comparison between the market rent and the shadow rent of the optimal configuration. Actual solutions for optimal allocation models must be investigated to obtain the direction of the bias of a cost-benefit analysis based on market rent without social costs of congestion.

Received by the editors on December 30, 1991, and in revised form on February 27, 1992.
This research was supported by NSF Grant DMS-8904845.

A number of papers (see [**5**, **6**] and references therein) pertaining to the general problem of optimal land use appeared in the literature after the publication of [**9**, **7**, **8**]. On residential land models with a fixed homogeneous population, two kinds of problems were considered in these papers. One is the comparison between the (*socially*) *optimal allocation* and the *market allocation* with the latter characterized by more roads until market benefit from additional land for roads is equal to the market rent for the land [**6**]. The study of optimal versus market allocation is quite complete since explicit solutions for both types of allocations are possible for typical utility functions and congestion cost functions.

The other class of problems is the comparison between the market allocation and the so-called "second best solution" which is the optimal allocation in the absence of congestion tolls. Because of political or technical difficulties, it is usually impossible to levy congestion tolls which correctly reflect the social costs of traffic congestion. In fact, there are very few city roads where congestion tolls are levied and there is no city where congestion tolls are adopted in the whole city.

Except for results by numerical simulations [**1**], a direct comparison of the second-best and market allocation has not been carried out "*because the second-best solution is too complicated*" (see [**5**, p. 484]). Some progress has been made in the form of a modified problem where the relative magnitude of the market rent and market benefit is deduced for a small change in the second-best road width. This modified problem was analyzed qualitatively in [**5**, **6**] and is essentially a more general version of the approach used in [**9**].

In this paper, a perturbation method of solution will be developed for the determination of the difficult second-best solution and actual second-best allocations will be obtained for some typical situations. The solution scheme is based on the observation that the total transportation cost is only a small fraction of the total income of any household. Various unknown quantities (such as the distributions of road width, population, and transportation cost over the urban geography) may be expanded as perturbation series in powers of a small parameter ε_t, which is (some power of) a typical magnitude of this fraction. With these perturbation series, the relevant nonlinear boundary value problem (BVP) for a fourth order system of ordinary differential equations with an integral constraint, which determines the second-best allocation, is reduced to a sequence of linear problems for the coefficients of the various expansions. The solutions of these simple linear problems are obtained successively in terms of elementary functions.

The approximate analytical solution will be validated by the general BVP solver COLSYS [**2**] which is based on a spline-collocation solution scheme. In order to use COLSYS, we undertake an unorthodox reformulation of the boundary value problem with an integral constraint as a standard boundary value problem (without the integral constraint). This standard nonlinear boundary value problem is then solved numerically by the general boundary value problem solver COLSYS developed by U. Ascher et al. [**2**]. All numerical results for this paper were obtained using double precision (14 hexadecimal digits) with a prescribed relative error tolerance of 10^{-5} (and usually with a much smaller actual estimated error) for all solution components.

To gain some preliminary insight into the second best allocation, we will work with a restricted urban land use model of the type used in [**7, 8, 10**] and with a specific class of utility functions and transportation cost functions. It will be evident that the same methods of solution can be used for the more general model formulated in [**5, 6**] and other classes of utility and transportation cost functions. We will also confine ourselves herein to closed cities with a prescribed city limit and an absentee landlord. Results for models with a free boundary and public ownership will be reported elsewhere.

Perturbation and asymptotic methods of solution have been used previously in [**10**] for equilibrium models of the household behavior with the road width prescribed in advance. The perturbation solution for the second-best model differs qualitatively and in a mathematically significant way from the equilibrium solution. The appropriate perturbation series for the second-best problem is in powers of a small parameter ε_t while the series for the equilibrium model with a prescribed fraction of land for housing analyzed in [**10**] is in a higher power of ε_t.

For a sufficiently small value of ε_t, a two-term perturbation solution provides an adequate approximation of the second-best allocation in typical settings. Such a solution shows that *the fraction of land for roads is, to a good approximation, a simple monotone increasing function of the radial distance from the city center.* The following comparisons between the market allocation and the second-best allocation found from simple perturbation solutions should be of interest:

(i) The market allocation always assigns more land for roads at any location than the second-best allocation and therewith less housing space per household.

(ii) The transportation cost for any household away from the edge of the CBD is higher and the consumption of commodities is correspondingly lower in the second-best world.

(iii) The unit land rent at the edge of the CBD is lower in the second-best world and the common utility for all households is therefore higher.

Other properties of the second-best allocation can also be easily deduced from the simple perturbation solution.

The second-best city and the market city being compared are of the same size. We do not insist that this is an appropriate comparison; but whatever the appropriate comparison may be, perturbation solutions similar to that obtained in section (6) and the Appendix readily provide us with important information such as those listed above. In fact, the perturbation solution for the market allocation is more informative and useful in this regard than the known exact solution.

Numerical simulations have been used for urban land use problems in [**8**, **1**] but not for the second-best problem. The particular numerical method we use differs significantly from all those previously employed in that it gives the solution of the problem to a specified level of accuracy (e.g., a relative error of 10^{-5} for all locations in the city) or, in exceptional cases, indicates that the desired accuracy cannot be attained with the available machine time and CPU capacity. In this sense, we have in effect an exact solution of our problem which can be used to confirm all the conclusions based on the perturbation solution and to study similar problems when ε_t is not so small compared to unity for which the perturbation method is not practical.

2. Consumer's optimum and locational equilibrium. In the conventional urban economic model for land use problems, the abstract city is taken to be circular with a circular Central Business District (CBD) of radius R_i and an annular residential area extending from $X = R_i$ to $X = R_o$ where X is the radial distance from the city center. To facilitate later comparison with the results of [**5**, **6**], the city is divided into pie-shaped sectors. For one of these sectors with a sectoral angle of θ radians ($\theta \leq 2\pi$), the residential area is inhabited by N_0 identical households travelling only to and from the CBD. Each household has the same annual income y to be used for housing, for consumption goods and for transportation. We may take the unit price of the composite consumption goods to be unity. When saving is not an option, a household at distance X from the city center has as its budget equation

$$c + rs + t = y \tag{2.1}$$

where $c(X)$ and $s(X)$ are the amount of consumption goods and residential land for that household per annum, respectively, and where $r(X)$ and $t(X)$ are the per annum unit land rent and total transportation cost for the household, respectively. Upon introducing a dimensionless after-transportation income function $w(X)$, the budget equation (2.1) may be written as

$$c + rs = y\left(1 - \frac{t}{y}\right) \equiv yw(X), \tag{2.2}$$

with

$$t = y(1 - w). \tag{2.3}$$

Each household in the model city treated herein has the same utility function

$$U(c, s) = U(\xi/\xi_0), \qquad \xi = c^{1-\sigma}s^{\sigma}, \qquad 0 < \sigma < 1, \tag{2.4}$$

where ξ_0 is a parameter having the same units as ξ so that ξ/ξ_0 is dimensionless (e.g., ξ_0 is the value of ξ at the edge of CBD). $U(\cdot)$ is assumed to be monotone increasing and strictly concave. Note that the class of utility function (2.4) includes the logarithmic-additive utility used in [**7**, **8**, **10**] (with $U(\cdot) = \ln(\cdot)$) and the Cobb-Douglas type utility functions used in [**5**, **6**]. We have not allowed U to depend on X explicitly but we expect it to depend on X indirectly through c and s which will be seen to vary with the household location.

Each household chooses c and s to maximize U subject to the budget constraint (2.1) (or (2.2)). The first order necessary conditions for a maximum U yield the stationary point $(\bar{c}, \bar{s})$ given in terms of t and r by

$$\bar{c} = (1-\sigma)(y-t) = (1-\sigma)yw, \qquad r\bar{s} = \sigma(y-t) = \sigma yw. \tag{2.5}$$

Evidently, σ is the fraction of household income after travel cost spent on housing. The corresponding stationary value of the utility function is

$$\overline{U} \equiv (\bar{c}, \bar{s}) = U(\sigma^\sigma(1-\sigma)^{1-\sigma}(y-t)/\xi_0 r^\sigma). \tag{2.6}$$

The monotone increasing and strictly concave properties of U ensure that the stationary value $\overline{U}$ is a maximum.

In locational equilibrium, $\overline{U}$ must be independent of X. In particular, $\overline{U}$ at an arbitrary location X must be the same as $\overline{U}$ at the edge of the CBD:

$$U(\sigma^\sigma(1-\sigma)^{1-\sigma}(y-t)/\xi_0 r^\sigma) = U(\sigma^\sigma(1-\sigma)^{1-\sigma}(y-t_i)/\xi_0 r_i^\sigma) \equiv \overline{U}, \tag{2.7}$$

where $t_i \equiv t(R_i)$ and $r_i \equiv r(R_i)$. We assume henceforth that transportation is free within the CBD so that $t_i = 0$. Since $U(\cdot)$ is monotone increasing, it follows from (2.7) that

$$(y-t)/r^\sigma = y/r_i^\sigma \quad \text{or} \quad r = r_i w^{\alpha+1}, \qquad \alpha + 1 = 1/\sigma. \tag{2.8}$$

With (2.8), the consumer's behavior is completely determined once the per annum transportation cost $t(X)$ and the fraction of land $b(X)$ within an incremental annular sector at X allocated for housing are known. The latter determines the unknown parameter r_i.

3. Transportation costs. In an annular sector of the residential area extending from X to $X+dX$, a fraction $b(X)$ of the land is used for housing and the remaining land area ($= [1-b(X)]\theta X\, dX$) is for roads. The total per annum transportation cost $t(X)$ of a household at location X is taken to be the sum of a *distance cost* $t_d(X)$ with $t_d(R_i) = 0$ (since travel within the CBD is assumed free) and a congestion cost $t_c(X)$ which depends on the traffic density ($= N/\theta X(1-b)$). We take t in the form [**7, 8, 10**]

$$\begin{aligned} t(X) &= t_d(X) + t_c(X) \\ &= \int_{R_i}^{X} \left[\tau + a_0 \left\{ \frac{N(z)}{\theta z[1-b(z)]} \right\}^k \right] dz, \end{aligned} \tag{3.1}$$

where $N(X)$ is the number of households located outside the ring X, $\tau(X) = dt_d/dx$ is the known fixed unit distance travel cost within the residential area (per household per annum), and k and a_0 are prescribed positive constants. We will work with the differentiated form of (3.1):

$$\frac{dt}{dX} = \tau + a_0 \left\{ \frac{N(X)}{\theta X[1-b(X)]} \right\}^k = -y\frac{dw}{dX}. \tag{3.2}$$

Since $N(X)$ is also an unknown, we need another condition to determine w and N. This condition comes from a conservation law for space: the amount of space occupied by the households in the annular sector (of area $\theta X\, dX$) must equal the total amount of space in the sector allocated for housing so that $-s dN = b\theta X\, dX$. We may write this relation as

$$\frac{dN}{dX} = -\frac{\theta X b(X)}{\bar{s}(X)} = -\frac{\theta r_i(\alpha+1)}{y} X b w^{\alpha} \equiv -n(X) \tag{3.3}$$

where $n(X)$ is the *population density* in the incremental sector at X.

For the two first order differential equations (3.2) and (3.3), we have the initial conditions

$$w(R_i) = 1, \tag{3.4}$$

$$N(R_i) = N_0, \tag{3.5}$$

expressing the fact that households located at the edge of the CBD pay no travel expenses and that all households are located outside the CBD. For a given distribution of the fraction of land for housing $b(X)$, the initial value problem defined by (3.2), (3.3), (3.4) and (3.5) determines $w(X)$ and $N(X)$ up to the unknown parameter r_i. Finally, the condition that there is no household located outside the city limit

$$N(R_0) = 0, \tag{3.6}$$

determines r_i (and therewith the common household utility $\overline{U}$ from (2.6) and (2.7)). Note that we have

$$0 < b(X) < 1, \qquad 0 \le w(X) \le 1, \qquad 0 \le N(X) \le N_0, \tag{3.7}$$

from the definition of the three quantities.

In order to have the simplest setting for the description of the perturbation method of solution, we will confine ourselves here to closed cities with a prescribed outer boundary R_0 and with an absentee landlord to whom all rents are paid. For such cities, R_0 and y are known constants with the latter being the same wage earned by all households.

4. A prescribed distribution of land allocation. To motivate the perturbation method of solution for the second-best allocation, we consider first the two-point boundary value problem for a prescribed b, a constant $\tau(=\tau_0)$, and $k=1$ (in (3.1)). For this case, the first order ODE (3.2) may be used to eliminate $N(X)$ from (3.3). The resulting second order ODE is written in the dimensionless form

$$[p(x)w']' - v_r\varepsilon(1-\eta)q(x)w^\alpha = -\varepsilon\eta p', \tag{4.1}$$

with the help of the dimensionless quantities

$$x = \frac{X}{R_i}, \qquad R = \frac{R_0}{R_i}, \qquad v_r = \frac{r_i b_i \theta R_i^2}{\sigma y N_0}, \qquad b_i = b(R_i) \tag{4.2}$$

$$p(x) = \frac{x(1-b)}{1-b_i}, \qquad q(x) = \frac{xb}{b_i}, \qquad (\ )' \equiv \frac{d(\ )}{dx}, \tag{4.3}$$

$$\varepsilon\eta = \frac{\tau_0 R_i}{y}, \qquad \varepsilon(1-\eta) = \frac{R_i}{y}\left[\frac{a_0 N_0}{\theta R_i(1-b_i)}\right]. \tag{4.4}$$

In terms of $w(x)$, the boundary conditions (3.4)–(3.6) can be written as

$$w(1) = 1, \tag{4.5}$$

$$w'(1) = -\varepsilon, \tag{4.6}$$

$$w'(R) = -\varepsilon\eta, \tag{4.7}$$

the latter two with the help of (3.2) (with $k=1$ and $\tau=\tau_0$). For a prescribed b, two of these boundary conditions supplement (4.1) for a unique determination of $w(x; v_r, \varepsilon)$ with v_r as a parameter. The remaining condition specifies v_r. An exact solution in terms of Bessel functions is possible for $\alpha = 1$ ($\sigma = 1/2$) and a constant b ($= b_i$).

However, few cities would require its inhabitants to spend half (or more) of its annual income on housing; we need to consider the case $\alpha > 1$.

The quantity $\varepsilon\eta$ is of the order of magnitude of the fraction of the household annual income for the distance component of the household transportation cost. The quantity $\varepsilon(1-\eta)$ is the corresponding fraction for the congestion component. Hence, $\varepsilon = \varepsilon(1-\eta) + \varepsilon\eta$ is a measure of the fraction of the household annual income for transportation; it is normally small compared to unity so that $0 < \varepsilon \ll 1$. This observation suggests that we seek a perturbation solution of (4.1), (4.5)–(4.7) in the form

$$w(x) = \sum_{k=0}^{\infty} w_k(x)\varepsilon^k, \qquad v_r = \sum_{n=0}^{\infty} v_n\varepsilon^n. \tag{4.8}$$

(As we shall see, the corresponding series for the second-best allocation will have to be in powers of $\varepsilon^{1/2}$ instead.) Since (4.1) and (4.5)–(4.7) must be satisfied identically in ε, the expansions (4.8) give rise to a sequence of linear BVP for $\{w_k(x)\}$ and $\{v_n\}$.

The 0(1) problem.

$$\begin{gathered} [p(x)w_0']' = 0, \\ w_0(1) = 1, \qquad w_0'(1) = 0, \qquad w_0'(R) = 0. \end{gathered} \tag{4.9}$$

The solution of the IVP defined by the first three equations of (4.9) is

$$w_0(x) = 1.$$

This solution also satisfies $w_0'(R) = 0$ fortuitously.

The 0(ε) problem.

$$\begin{gathered} [pw_1']' = v_0(1-\eta)qw_0^\alpha - \eta p', \\ w_1(1) = 0, \qquad w_1'(1) = -1, \qquad w_1'(R) = -\eta. \end{gathered} \tag{4.10}$$

Having $w_0(x)$ from the 0(1) problem, the right-hand side of (4.10) is known except for the constant v_0. We may integrate (4.10) to obtain

$$w_1'(x) = (1-\eta)\frac{v_0 q_0 - 1}{p(x)} - \eta, \qquad q_0(x) = \int_1^x q(z)\,dz,$$

with the help of $w'_1(1) = -1$ and $p(1) = 1$. The condition $w'_1(R) = -\eta$ requires

$$v_0 = \frac{1}{q_0(R)}.$$

From the expression for $w'_1(x)$ and the condition $w_1(1) = 0$, we obtain

$$w_1(x) = (1-\eta)[v_0 Q_0(x) - P_0(x)] - \eta(x-1),$$

with

$$Q_0(x) = \int_1^x \frac{q_0(z)}{p(z)}\,dz, \qquad P_0(x) = \int_1^x \frac{dz}{p(z)}.$$

The $0(\varepsilon^2)$ problem.

$$(4.11)\qquad \begin{gathered} (pw'_2)' = (1-\eta)q[v_1 w_0^\alpha + v_0 \alpha w_0^{\alpha-1} w_1], \\ w_2(1) = 0, \qquad w'_2(1) = 0, \qquad w'_2(R) = 0. \end{gathered}$$

The solution of the problem is

$$\begin{aligned} pw'_2 &= (1-\eta)[v_1 q_0(x) + v_0 q_1(x)], \\ q_1(x) &= \alpha \int_1^x q(z) w_1(z)\,dz, \\ v_1 &= -v_0 \frac{q_1(R)}{q_0(R)} = -v_0^2 q_1(R), \\ w_2(x) &= (1-\eta)[v_1 Q_0(x) + v_0 Q_1(x)], \\ Q_1(x) &= \int_1^x \frac{q_1(z)}{p(z)}\,dz. \end{aligned}$$

The solution of higher order terms as well as numerical results for specific choices of $b(x)$ can be found in [**10**]. (Note that we will use the same notation for $b(x)$ and $b(X)$.)

Also treated in [**10**] is the more interesting *free boundary problem* when R_0 is not fixed but the city expands until $r(R_0) = r_A$ where r_A is the known unit agricultural land rent. In terms of w, we have

$$(4.12)\qquad v_r[w(R)]^{\alpha+1} = v_a, \qquad v_a = \frac{r_A b_i \theta R_i^2}{\sigma y N_0}.$$

The unknown R depends on ε and should be expanded as

$$R = \sum_{k=0}^{\infty} \overline{R}_k \varepsilon^k. \tag{4.13}$$

The value of w at $x = R$ depends on ε parametrically because of its appearance in the ODE and boundary conditions and also indirectly through $R(\varepsilon) : w(x = R; \varepsilon) = w(R(\varepsilon); \varepsilon)$. The perturbation series for $w(R(\varepsilon); \varepsilon)$ is

$$w(R(\varepsilon); \varepsilon) = w|_{\varepsilon=0} + \varepsilon \left.\frac{dw}{d\varepsilon}\right|_{\varepsilon=0} + \frac{1}{2}\varepsilon^2 \left.\frac{d^2 w}{d\varepsilon^2}\right|_{\varepsilon=0} + \cdots, \tag{4.14}$$

with
(4.15)

$$w|_{\varepsilon=0} = w_0(\overline{R}_0),$$

$$\left.\frac{dw}{d\varepsilon}\right|_{\varepsilon=0} = \left[\frac{\partial w}{\partial \varepsilon} + \frac{\partial w}{\partial R}\frac{dR}{d\varepsilon}\right]_{\varepsilon=0} = w_1(R_0) + \overline{R}_1 w'_0(\overline{R}_0),$$

$$\left.\frac{d^2 w}{d\varepsilon^2}\right|_{\varepsilon=0} = \left[\frac{\partial^2 w}{\partial \varepsilon^2} + 2\frac{\partial^2 w}{\partial R \partial \varepsilon}\frac{dR}{d\varepsilon} + \frac{\partial^2 w}{\partial R^2}\left(\frac{dR}{d\varepsilon}\right)^2 + \frac{\partial w}{\partial R}\frac{d^2 R}{d\varepsilon^2}\right]_{\varepsilon=0}$$

$$= 2\left[w_2(\overline{R}_0) + \overline{R}_1 w'_1(\overline{R}_0) + \frac{1}{2}\overline{R}_1^2 w''_0(\overline{R}_0) + \overline{R}_2 w'_0(\overline{R}_0)\right],$$

etc. The actual use of these series for the solution of the free boundary problem can be found in [**10**] (though there are several typographical errors in the expressions in (4.15) listed there).

In most investigations, σ is taken to be 1/4 which is roughly the fraction of income that households spend on housing. If σ is interpreted strictly as the fraction of income after travel cost spent on ground rent, then $\alpha + 1$ is typically larger than 10. For $\alpha + 1 \gg 1$, εv_r may not be small even if ε is and perturbation solution of the type (4.8) may no longer be appropriate. On the other hand, with $\alpha \gg 1$, a matched asymptotic expansion solution with $1/\alpha$ as the small parameter is possible and has been obtained in [**10**].

5. Maximum common household utility. Instead of prescribing the road width distribution, $1 - b$, as was done in [**7, 8, 10**], we consider here, as in [**5**, **6**], the problem of using $b(X)$ as a policy (or control) variable to maximize the common household utility $\overline{U} = U(\sigma^\sigma(1-\sigma)^{(1-\sigma)}y/\xi_0 r_i^\sigma)$. The uniform annual income of each household is known for the absentee landlord case; therefore, the dependence of $\overline{U}$ on $b(X)$ is only through r_i in this case. Since this dependence is specified by the conditions (3.2)–(3.6) for a city with a fixed outer boundary, these conditions assume the role of equality constraints in the optimization problem[1]. (Of course, there are also the inequality constraints of (3.7) on w, N and b, which are automatically satisfied for all cases analyzed in this article.) The first order necessary conditions for an *interior* maximum U subject to these constraints are [**4**]

$$\frac{d\phi}{dX} + \frac{\alpha(\alpha+1)\theta r_i}{y} X b w^{\alpha-1}\Psi = 0, \tag{5.1}$$

$$\phi(R_0) = 0, \tag{5.2}$$

$$\frac{d\Psi}{dX} + \frac{ak}{y}\frac{\phi}{N}\left[\frac{N}{\theta X(1-b)}\right]^k = 0, \tag{5.3}$$

$$\frac{(\alpha+1)\theta r_i}{y} X\Psi w^\alpha - \frac{ak}{y}\frac{\phi}{1-b}\left[\frac{N}{\theta X(1-b)}\right]^k = 0, \tag{5.4}$$

(5.5)
$$\frac{\sigma^\sigma(1-\sigma)^{1-\sigma}y}{\xi_0 r_i^\sigma}\overline{U}^{\boldsymbol{\cdot}}\left(\frac{\sigma^\sigma(1-\sigma)^{1-\sigma}y}{\xi_0 r_i^\sigma}\right) = \int_{R_i}^{R_0} \frac{r_i(\alpha+1)\theta}{y} X b w^\alpha \Psi \, dx,$$

where $\overline{U}^{\boldsymbol{\cdot}}(\beta) = d\overline{U}(\beta)/d\beta$ and where $\sigma\phi$ and $-\sigma\Psi$ are the Lagrange multiplier (or costate variables) associated with the constraints (3.2) and (3.3). The economic content of these necessary conditions (and of those for a more general "second-best allocation" problem which allows for more policy variables and inequality constraints) has already been discussed in [**6**].

For a fixed size city with a prescribed outer boundary R_0, the coupled fourth order system of four differential equations (3.2), (3.3), (5.1) and (5.3) and the algebraic equation (5.4), supplemented by the four boundary conditions (3.4), (3.5), (3.6) and (5.2), define a two-point boundary value problem for the five unknown functions w, N, Ψ, ϕ and b with r_i as an unknown parameter. The integral condition (5.3) then determines r_i. In the subsequent developments (particularly for the purpose of a perturbation series solution of the problem), we need a dimensionless form of this nonlinear boundary value problem. For this dimensionless form, we introduce the following dimensionless variables in addition to (4.2) and (4.3):

$$\tau(X) = \tau_0 T(x), \qquad u = \frac{N}{N_0}, \qquad \psi = N_0 \Psi, \tag{5.6}$$

$$\varepsilon_t^{k+1}\eta = \frac{\tau_0 R_i}{y} \equiv \frac{\bar{t}_d}{y}, \qquad \varepsilon_t^{k+1}(1-\eta) = \frac{aR_i}{y}\left(\frac{N_0}{\theta R_i}\right)^k \equiv \frac{\bar{t}_c}{y}, \tag{5.7}$$

$$\bar{v} = \frac{r_i \theta R_i^2}{\sigma y N_0}, \tag{5.8}$$

with $\varepsilon_t^{k+1} = (\bar{t}_d + \bar{t}_c)/y$, giving the order of magnitude of the fraction of income for travel costs. We write the fourth order boundary value problem and the integrated condition in terms of these new variables:

$$w' = -\varepsilon_t^{k+1}\left\{\eta T(x) + (1-\eta)\left[\frac{u}{x(1-b)}\right]^k\right\}, \tag{5.9}$$

$$w(1) = 1, \tag{5.10}$$

$$u' = -\bar{v}xbw^\alpha, \tag{5.11}$$

$$u(1) = 1, \tag{5.12}$$

$$u(R) = 1, \tag{5.13}$$

$$\phi' = -\alpha\bar{v}xb\psi w^{\alpha-1}, \tag{5.14}$$

$$\phi(R) = 0, \tag{5.15}$$

$$\psi' = -\varepsilon_t^{k+1}(1-\eta)\frac{k\phi}{u}\left[\frac{u}{x(1-b)}\right]^k, \tag{5.16}$$

$$\bar{v}x(1-b)w^\alpha\psi - \varepsilon_t^{k+1}(1-\eta)k\phi\left[\frac{u}{x(1-b)}\right]^k = 0, \tag{5.17}$$

$$\frac{\mu(\bar{v})}{\bar{v}} = \int_1^R xb(x)w^\alpha\psi\,dx, \tag{5.18}$$

where $(\quad)' = d(\quad)/dx$ and

$$\mu(\bar{v}) = \frac{\sigma^\sigma(1-\sigma)^{1-\sigma}y}{\xi_0 r_i^\sigma}\bar{U}'\left(\frac{\sigma^\sigma(1-\sigma)^{1-\sigma}y}{\xi_0 r_i^\sigma}\right). \tag{5.19}$$

The following three expressions for $\mu(\bar{v})$ correspond to utility functions used in existing literature (see [**6**]):

(5.20) $$\mu(\bar{v}) = 1 \quad \text{if} \quad U(z) = \ln(z),$$

(5.21) $$\mu(\bar{v}) = \frac{\theta R_i^2 y}{N_0 \xi_0^2 \bar{v}} \quad \text{if} \quad U(z) = z^2 \quad \text{and} \quad \sigma = 1/2,$$

(5.22) $$\mu(\bar{v}) = \sigma^\sigma (1-\sigma)^{1-\sigma} \left(\frac{y}{\xi_0}\right) \left[\frac{(\alpha+1)\theta R_i^2 y}{y N_0 \bar{v}}\right]^\sigma \quad \text{if} \quad U(z) = z.$$

In all cases, μ is a dimensionless quantity for the definition of ξ_0 previously given in (2.4).

It should be observe that, for a fixed $\bar{v}$, the form of the two-point boundary value problem (5.9)–(5.17) does not change for different utility functions of the form (2.4). However, this observation is of little practical value unless we can obtain the solution of the boundary value problem explicitly in terms of elementary or special functions as the solution depends on $\bar{v}$ which varies with U according to (5.18) and (5.19). In general, it is not possible to obtain such an explicit solution, and a numerical solution of the problem is necessary. While an efficient numerical solution scheme will be developed for the problem in a later section of this paper, it is of considerable interest to note that a perturbation solution of the problem suggests itself once a certain small parameter of the problem is identified. It will be seen in the next section that the first few terms of the perturbation solution for the various unknowns are (essentially) polynomials in the dimensionless distance variable x and that they provide an adequate approximation of the exact solution. A suitably truncated perturbation solution clearly delineates the structure of the solution of the optimization problem and provides a good initial guess for the BVP solver COLSYS used in our numerical solution scheme developed in section (7).

6. Perturbation solution. The solution of the nonlinear boundary value problem (4.9)–(4.18) depends on the value of ε_t parametrically. For an interpretation of ε_t, we see from (3.1) that the total transportation cost per annum t for each household consists of a distance cost component and a congestion cost component. With $R_0 - R_i = 0(R_i)$ for cases of interest, these two cost components are of the order of magnitude of $\bar{t}_d \equiv \tau_0 R_i$ and $\bar{t}_c \equiv a R_i (N_0/\theta R_i)^k$, respectively. On the other hand, the two equations in (5.7) defining ε_t and η may be combined to yield

(6.1) $$\varepsilon_t^{k+1} = \varepsilon_t^{k+1}\eta + \varepsilon_t^{k+1}(1-\eta) = \bar{t}_d/y + \bar{t}_c/y = 0(t/y).$$

The quantity ε_t^{k+1} is therefore of the order of magnitude of the fraction of the household income allocated for transportation costs; this fraction is typically very small compared to unity, $0 < \varepsilon_t^{k+1} \ll 1$. We will take advantage of this observation for a simple but accurate approximate solution for our problem.

The algebraic equation (5.17), rearranged in the form

(6.2) $$x(1-b) = \varepsilon_t \left[\frac{(1-\eta)k\phi u^k}{\bar{v}\psi w^\alpha}\right]^{1/(k+1)},$$

or

(6.3) $$b = 1 - \frac{\varepsilon_t}{x}\left[\frac{(1-\eta)k\phi u^k}{\bar{v}\psi w^\alpha}\right]^{1/(k+1)},$$

may be used to eliminate $b(x;\varepsilon_t)$ from (5.9), (5.11), (5.14) and (5.16). The form of the resulting four coupled first order differential equations for w, u, ϕ and ψ together with the boundary conditions (5.10), (5.12), (5.13) and (5.15) indicates that parametric expansions for the four unknown functions of x and the unknown constant $\bar{v}$ in powers of ε_t are appropriate, i.e.,

(6.4) $$\{w, u, \bar{\phi}, \psi, \bar{v}\} = \sum_{m=0}^{\infty}\{\bar{w}_m(x), \bar{u}_m(x), \phi_m(x), \psi_m(x), \bar{v}_m\}\varepsilon_t^m.$$

Correspondingly, equation (6.3) suggests that we take an expansion for $b(x;\varepsilon_t)$ in the form

(6.5) $$b(x;\varepsilon_t) = 1 - \left(\frac{\varepsilon_t}{x}\right)[\beta_1(x) + \beta_2(x)\varepsilon_t + \beta_3(x)\varepsilon_t^2 + \ldots].$$

Upon substituting (6.4) and (6.5) into (5.9)–(5.18) and requiring the resulting equations to be satisfied identically in ε_t, we get a sequence of linear boundary value problems for the determination of the coefficients of the various expansions. For illustration, we list here the first two problems of this sequence and their solution for $k = 1$ and a logarithmic-additive utility function $U(\cdot) = \ln(\cdot)$:

The 0(1) problem.

(6.6)
$$\begin{aligned} &\bar{w}_0' = 0, \qquad \bar{w}_0(1) = 1, \\ &\bar{u}_0' + \bar{v}_0 x \bar{w}_0^\alpha = 0, \qquad \bar{u}_0(1) = 1, \qquad \bar{u}_0(R) = 0, \\ &\psi_0' = 0, \\ &\phi_0' + \alpha \bar{v}_0 x \psi_0 \bar{w}_0^{\alpha-1} = 0, \qquad \phi_0(R) = 0, \end{aligned}$$

and

(6.7)
$$\bar{v}_0 \int_1^R x \psi_0 \bar{w}_0^\alpha \, dx = 1.$$

The solution of this problem is

(6.8)
$$\begin{aligned} \bar{w}_0(x) = \psi_0(x) = 1, \qquad \bar{v}_0 = 2/(R^2 - 1), \\ \bar{u}_0(x) = (R^2 - x^2)/(R^2 - 1), \qquad \phi_0(x) = \alpha(R^2 - x^2)/(R^2 - 1). \end{aligned}$$

The 0(ε_t) problem.

$$\beta_1 = \left[\frac{(1-\eta)\phi_0 \bar{u}_0}{\bar{v}_0 \psi_0 \bar{w}_0^\alpha}\right]^{1/2},$$

$$\bar{w}_1' + (1-\eta)\frac{\bar{u}_0}{\beta_1} = 0, \qquad \bar{w}_1(1) = 0,$$

$$\bar{u}_1' + \bar{v}_0 \bar{w}_0^\alpha \left[x\left(\frac{\bar{v}_1}{\bar{v}_0} + \alpha\frac{\bar{w}_1}{\bar{w}_0}\right) - \beta_1\right] = 0, \qquad \bar{u}_1(1) = \bar{u}_1(R) = 0,$$

(6.9)
$$\bar{u}_1' + (1-\eta)\frac{\phi_0}{\beta_1} = 0,$$

$$\phi_1' + \alpha \bar{v}_0 x \psi_0 \bar{w}_0^{\alpha-1}\left[\frac{\bar{v}_1}{\bar{v}_0} + (\alpha-1)\frac{\bar{w}_1}{w_0} + \frac{\psi_1}{\psi_0} - \frac{\beta_1}{x}\right] = 0, \qquad \phi_1(R) = 0,$$

and

$$\bar{v}_1 \int_1^R x\psi_0 \bar{w}_0^\alpha \, dx = \bar{v}_0 \int_1^R x\psi_0 \bar{w}_0^\alpha \left[\frac{\beta_1}{x} - \frac{\psi_1}{\psi_0} - \alpha\frac{\bar{w}_1}{\bar{w}_0}\right] dx. \tag{6.10}$$

With $\bar{w}_0, \bar{u}_0, \psi_0, \phi_0$ and $\bar{v}_0$ known from (5.8), the solution of this problem is found to be
(6.11)

$$\beta_1(x) = \frac{\alpha_0}{\bar{v}_0}\frac{R^2 - x^2}{R^2 - 1}, \qquad \frac{\bar{v}_1}{\bar{v}_0^2} = \frac{\alpha_0}{3}(2R^3 - 3R^2 + 1),$$

$$\bar{w}_1(x) = -\frac{\alpha_0}{\alpha}(x - 1), \qquad \psi_1 = \frac{2\alpha_0}{3}\left\{\frac{R^3 - 1}{R^2 - 1} - \frac{3}{2}x\right\},$$

$$\bar{u}_1(x) = \frac{\alpha_0\bar{v}_0}{6}\left\{(x^3 - 1) - (4R^3 - 3R^2 - 1)\frac{x^2 - 1}{R^2 - 1} + 3R^2(x - 1)\right\},$$

$$\phi_1(x) = \alpha\bar{u}_1(x) - \bar{v}_0 a_0\left\{\left[\frac{1}{2}(R^2 - 1) - \frac{\alpha}{3}(R^3 - 1)\right]\frac{R^2 - x^2}{R^2 - 1} + \frac{\alpha - 1}{3}(R^3 - x^3)\right\},$$

where $\alpha_0 = [\bar{v}_0\alpha(1 - \eta)]^{1/2}$.

We see from (6.8) and (6.11) that w, u, ϕ and ψ are all polynomials in x, at least for the first two terms in their respective perturbation expansion. On the other hand, we have

$$\begin{aligned} b(x;\varepsilon_t) &= 1 - \varepsilon_t\frac{\beta_1(x)}{x} - \varepsilon_t^2\frac{\beta_2(x)}{x} + 0(\varepsilon_t^3) \\ &= 1 - \varepsilon_t\left[\left(\frac{R^2 - x^2}{x}\right)\sqrt{\frac{\alpha(1 - \eta)}{2(R^2 - 1)}}\right] + 0(\varepsilon_t^2), \end{aligned} \tag{6.12}$$

so that to order ε_t, the road width fraction $(1 - b)$ is a simple rational function of x. The corresponding two-term perturbation solution for the market allocation, denoted by $b_m(x;\varepsilon_t)$, is obtained in the Appendix of this paper (see equation (A.15)). The ratio

$$\frac{1 - b}{1 - b_m} = \sqrt{\frac{\alpha}{\alpha + 1}}[1 + 0(\varepsilon_t)] \tag{6.13}$$

To order ε_t, the cost-benefit criterion based on market land price (A.1) *allocates more land for roads than the second-best allocation.*

Note that with (6.11), we can obtain $\beta_2(x)$ without solving another boundary value problem since

$$\beta_2(x) = \frac{1}{2}\beta_1(x)\left[\frac{\bar{u}_1}{\bar{u}_0} + \frac{\phi_1}{\phi_0} - \frac{\bar{v}_1}{\bar{v}_0} - \frac{\psi_1}{\psi_0} - \alpha\frac{\bar{w}_1}{\bar{w}_0}\right]. \tag{6.14}$$

The two-term perturbation solution for the net income fraction after transportation cost is

$$w(x;\varepsilon_t) = 1 - \varepsilon_t\left[(x-1)\sqrt{\frac{2(1-\eta)}{\alpha(R^2-1)}}\right] + 0(\varepsilon_t^2). \tag{6.15}$$

The corresponding solution for the market allocation, denoted by $w_m(x;\varepsilon_t)$, is given by (A.13). The ratio

$$\frac{1-w}{1-w_m} = \sqrt{\frac{\alpha+1}{\alpha}}[1+0(\varepsilon_t)] \tag{6.16}$$

indicates that

The transportation cost for a household at a given location is smaller in a market city (since there is more land for roads) giving it a larger after transportation income for housing and consumption goods.

The second-best unit land rent at the edge of the CBD is

$$r_i = \frac{yN_0\bar{v}}{(\alpha+1)\theta R_i^2},$$

$$\frac{r_i}{r_0} = 1 + \varepsilon_t\left[\frac{1}{3}(2R^3 - 3R^2 + 1)\sqrt{\left(\frac{2}{R^2-1}\right)^3\alpha(1-\eta)}\right] + 0(\varepsilon_t^2) \tag{6.17}$$

with

$$r_0 = \frac{2yN_0}{(\alpha+1)\theta(R_0^2 - R_i^2)}.$$

From (A.14), we have

$$(6.18) \qquad \frac{(r_i/r_0)-1}{(r_m/r_0)-1} = \sqrt{\frac{\alpha^2+\alpha}{\alpha^2+\alpha+1/4}}\,[1+0(\varepsilon_t)],$$

where r_m is the value for r_i of the market allocation. To the extent that the common household utility increases as r_i decreases, we have:

The second-best land allocation gives a larger individual utility.

Evidently, better and cheaper housing more than makes up for the higher transportation cost of the second-best allocation.

The comparison between a second-best allocation and the corresponding market allocation for a close city with fixed boundaries may not be an appropriate comparison. But whatever the appropriate comparison may be, a two-term perturbation solution (in ε_t) more readily provides significant information similar to what has been listed above. Even when the exact solution is available in terms of elementary functions (as in the case of the market allocation), the perturbation solution is still more useful and informative. While we have limited ourselves here to a perturbation solution for the case $k=1$ in the transportation cost function and algorithmic-additive utility function, it is clear that the same technique can be applied to $k>1$ and other utility functions as well.

When applicable, the perturbation solutions show a significant qualitative difference between second-best solutions and equilibrium solutions on the effect of the two different components of the transportation. When $\bar{t}_d$ and $\bar{t}_c$ are of comparable magnitude, their contributions to the equilibrium solution for a prescribed road width are equally important. On the other hand, the contribution of the congestion cost component is at least an order of magnitude more significant than the contribution of the distance cost component in the second-best solution. For $k=1$, we see from (6.9) that the distance cost still has no effect on the first order correction term of the perturbation solution. In fact, upon using (6.3) to eliminate b from (5.9), we get

$$(6.19) \qquad w' = -\varepsilon_t^{k+1}\eta - \varepsilon_t(1-\eta)\left[\frac{\bar{v}xw^{\alpha}\psi}{k\phi(1-\eta)}\right]^{k/(k+1)}.$$

The following conclusion is evident from (6.19):

The contribution of the distance travel cost term in the second-best allocation is smaller by $O(\varepsilon_t^k)$ *than the contribution of the congestion cost of transportation.*

In the course of obtaining the perturbation solution for the second-best allocation, we have ignored the inequality constraints (3.7). When a truncated perturbation solution violates one or more of these constraints, an accurate numerical solution by the method of section (7) can be obtained to see whether the inequality constraints are in fact binding. It turns out that the constraints (3.7) are automatically satisfied by the two-term perturbation solution (as well as the accurate numerical solution) in all cases considered in this paper.

7. Numerical solutions by COLSYS. We can validate the accuracy of our two-term perturbation solution for the range of parameter values of interest by obtaining an accurate numerical solution by the general BVP solver COLSYS [**2**]. For this purpose, we undertake a somewhat unorthodox reformulation of the boundary value problem with an integral constraint (5.9)–(5.18). In this new formulation, we think of $\bar{v}$ as a function of x and introduce a new differential equation

$$\bar{v}' = 0, \tag{7.1}$$

to recover the fact that $\bar{v}$ is really independent of x. Next, we introduce another differential equation

$$\Lambda' = xb\psi w^{\alpha}, \tag{7.2}$$

for a new auxiliary function $\Lambda(x)$. In terms of $\Lambda(x)$, the integral constraint (5.18) is equivalent to the boundary conditions

$$\Lambda(1) = 0 \quad \text{and} \tag{7.3}$$

$$\bar{v}(R)\Lambda(R) - \mu(\bar{v}(R)) = 0. \tag{7.4}$$

The six first order differential equations (5.9), (5.11), (5.14), (5.16), (7.1) and (7.2), the algebraic equation (5.17), and the six boundary conditions (5.10), (5.12), (5.13), (5.15), (7.3) and (7.4) define a two-point boundary value problem. This boundary value problem is equivalent to the original fourth order problem with an integral constraint.

The new boundary value problem for the sixth order system determines the seven unknowns $w, u, b, \bar{v}, \phi, \psi$ and Λ and may be discretized and solved by a number of unknown methods. A computer code COLSYS, developed by U. Ascher et al. [**2**] for solving general boundary value problems involving ordinary differential equations has been used to obtain solutions of this problem for different utility functions, congestion cost functions and input parameters. In all cases, solution components obtained meet a prescribed relative error tolerance of 10^{-5}; the actual estimated errors are usually much smaller. Approximate perturbation (and matched asymptotic solution not included here) have been used as initial guesses to speed up the convergence of the iterative solution scheme whenever appropriate. The method of continuation has also been used to achieve the same results outside the range of validity of the perturbation and asymptotic solutions. While COLSYS itself allows for automatic mesh selections with more mesh points in regions of abrupt changes, the knowledge gained from the matched asymptotic solution concerning the locations of these abrupt changes often reduces the number of iterations needed to meet the prescribed error tolerance and thereby the computing cost.

TABLE 1. Accuracy of two-term perturbation solutions.
($k = 1$, $y = 10^4$, $N_0 = 10^5$, $R_i = 1$, $R_0 = 5$, $\theta = 2\pi$, $\tau_0 = 0$)

	Case	$w(R_0)$	$b(R_i)$	$r_i \times 10^{-6}$	$\overline{U}$
(I)	Exact	0.78885	0.46318	8.7897	2.8442
$\sigma = 0.5$	Perturbation	0.79399	0.45977	8.3012	3.0116
$a_0 = 0.02$	% Error	0.65%	−0.74%	−5.5%	5.9%
(II)	Exact	0.86281	0.23415	5.4969	4.7681
$\sigma = 0.25$	Perturbation	0.88106	0.24076	4.7618	4.8040
$a_0 = 0.02$	% Error	2.1%	2.8%	−13.4%	0.75%
(III)	Exact	0.96074	0.69245	3.8295	4.8585
$\sigma = 0.25$	Perturbation	0.96239	0.69261	3.7730	4.8622
$a_0 = 0.002$	% Error	0.17%	0.02%	−1.5%	0.08%

In Table 1, we show some typical results obtained by the perturbation method and by COLSYS to illustrate the accuracy of the perturbation solution for small ε_t. Altogether, results for three different cases are presented with $k = 1$, $y = 10^4$, $N_0 = 10^5$, $R_i = 1$, $R_0 = 5$, $\theta = 2\pi$ and $\tau_0 = 0$ (so that $\bar{\tau}_d = 0$) in all cases. In Case (I), we have $\overline{U}(\cdot) = (\cdot)^2$, $\sigma = 1/2$ and $a_0 = 0.02$ so that $\varepsilon_t^2 = 1/(10\pi)$. Case (II) differs from Case (I) only in $U(\cdot) = \ln(\cdot)$ and $\sigma = 1/4$ so that we have again $\varepsilon_t^2 = 1/(10\pi)$; it is the first problem analyzed in [**8**, **10**]. Case (III) is the same as Case (II) except $a_0 = 0.002$ so that $\varepsilon_t^2 = 1/(100\pi)$; the smaller value of a is chosen to demonstrate how the accuracy of the perturbation solution improves with decreasing ε_t. The table gives for each case the exact values to five significant figures obtained by COLSYS (and confirmed by the conventional shooting method) and the approximate values from the two-term perturbation solutions for (i) the fraction of income after transportation cost for households at the edge of the city $w(R_0)$, (ii) the fraction of land allocated for housing at the edge of the CBD[2], $b(R_i) = b_i$, (iii) the land rent per unit area at the edge of the CBD, r_i and (iv) the common utility for all households in the second-best world.

The substantial difference among the percentage errors in the perturbation solutions for $w(R_i)$, $b(R_i)$ and r_i in each case is not unexpected given the different order of magnitude of the terms neglected in the expansions for these quantities as observed in [**10**]. It was also shown in [**10**] that a matched asymptotic expansion solution should be used for $\sigma \ll 1$. On the other hand, the percentage error for $\overline{U}$ is significantly affected by the functional $\overline{U}(\cdot)$. For example, the surprisingly low percentage error for $\overline{U}$ in Case II (given the high percentage error for r_i) is due to the fact that $U(\cdot) = \ln(\cdot)$.

The results for $w(R_i)$, $b(R_i)$ and r_i are typical for the perturbation solutions of all other quantities at various locations. In general, the terms neglected in the two-term perturbation solutions for u, ψ and ϕ are $0(\varepsilon_t^2/\sigma)$ relative to the terms retained. This order of magnitude error estimates are usful guides to the appropriate use of the perturbation solutions. For example, the perturbation method is not useful for Case (IV): $\sigma = 0.2$, $y = 1$, $N_0 = 10^5$, $R_i = 50$, $R_0 = 118.2$, $\theta = 2$ and $a_0 = 10^{-5}$ with $\varepsilon_t^2 = 0.5$ and $\varepsilon_t^2/\sigma = 1.0$. (The market allocation for this case was investigated in [**5**].) To see whether the conclusion on second-best versus market allocation also holds for this

TABLE 2. Second-best versus market allocation.

Case		$\overline{U}$	$r(R_i)$	$r(R_o)$	Area for Roads	Total Rent	Total Transportation Cost
I	Market	2.7897	8.9617×10^6	6.3870×10^6	13.070	4.4836×10^8	1.0327×10^8
	Second-Best	2.8442	8.7898×10^6	5.4697×10^6	9.2569	4.3162×10^8	1.3675×10^8
II	Market	4.7652	5.5588×10^6	3.3695×10^6	17.391	2.3005×10^8	7.9792×10^7
	Second-Best	4.7681	5.4969×10^6	3.0463×10^6	14.267	2.2712×10^8	9.1503×10^7
III	Market	4.8580	3.8367×10^6	3.3430×10^6	5.8505	2.4467×10^8	2.1333×10^7
	Second-Best	4.8585	3.8295×10^6	3.2625×10^6	4.9867	2.4383×10^8	2.4680×10^7
IV	Market	.35225	15.106	1.0012	5.6374	1.4595×10^4	2.7025×10^4
	Second-Best	.36143	13.281	.61954	4.2278	1.4623×10^4	2.6887×10^4

(1) Cases (I), (II) and (III) all have a total land area $= 73,398$.

(2) Case (IV): $\sigma = 0.2$, $y = 1$, $N_o = 10^5$, $R_i = 50$, $R_o = 118.2$, $\theta = 2$, $a = 10^{-5}$, $\tau_o = 0$, $k = 1$ with a total land area $= 1.1471\times10^4$.

TABLE 2. Continued.

Case		$b(R_i)$	$n(R_i)\times10^{-3}$	$n(R_o)\times10^{-3}$	$s(R_i)\times10^{3}$	$s(R_o)\times10^{3}$	$c(R_i)$	$c(R_o)$
I	Market	0.24813	2.7944	47.536	0.55793	0.66089	5×10^3	4.2211×10^3
	Second-Best	0.46318	5.1160	43.566	0.56885	0.72111	5×10^3	3.9443×10^3
II	Market	0.045447	0.63507	47.990	0.44965	0.65464	7.5×10^3	6.6173×10^3
	Second-Best	0.23415	3.2348	44.368	0.45480	0.70808	7.5×10^3	6.4711×10^3
III	Market	0.63663	6.1388	43.481	0.65161	0.72252	7.5×10^3	7.2461×10^3
	Second-Best	0.69245	6.6644	42.676	0.65283	0.73619	7.5×10^3	7.2056×10^3
IV	Market	0.18638	1.4078	2.0363	13.240	116.09	0.8	0.46490
	Second-Best	0.39844	2.6459	1.3518	15.059	174.88	0.8	0.43337

(1) Cases (I), (II) and (III) all have a total land area $= 73,398$.

(2) Case (IV): $\sigma = 0.2$, $y = 1$, $N_o = 10^5$, $R_i = 50$, $R_o = 118.2$, $\theta = 2$, $a = 10^{-5}$, $\tau_o = 0$, $k = 1$ with a total land area $= 1.1471\times10^4$.

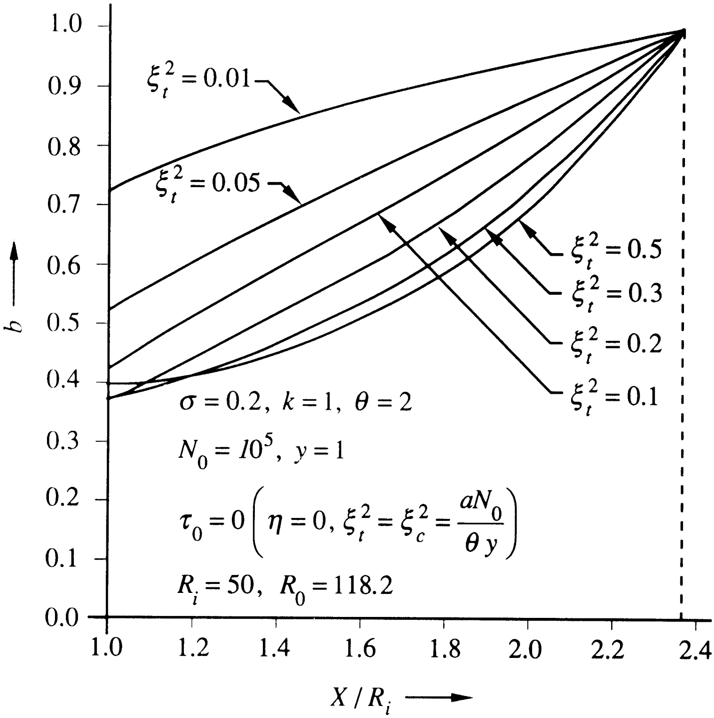

FIGURE 1. Distributions of fraction of land for housing.

case, we give in Table 2 both types of results for this case as well as for Cases (I)–(III) obtained by COLSYS accurate to at least five significant figures.

In all four cases, we see from the numerical results that, compared to the market allocation, the second-best allocation leads to

(i) a lower rent per unit land area at all locations and therefore a larger common household utility (for the class of utility functions considered),

(ii) less land for roads, giving more housing space for each household,

(iii) a lower (commodity) consumption for each household, except those at the edge of the CBD,

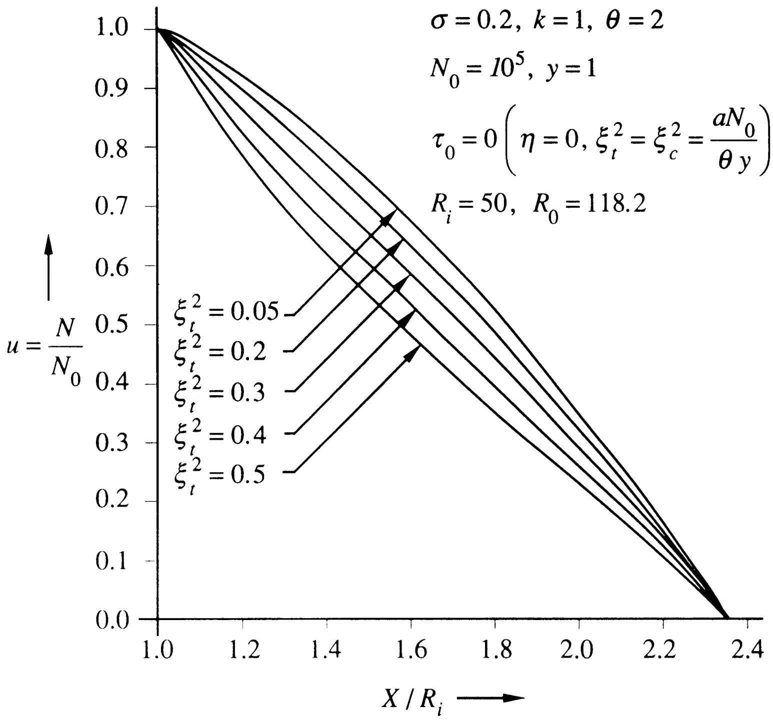

FIGURE 2. Total traffic through an arc of distance X from the city center.

(iv) a higher residential density near the CBD and a lower density near the city limit,

(v) less total rent and a higher total transportation cost in the first three cases and conversely for Case (IV).

In addition to the information given in Tables (1) and (2), the actual numerical solution of the second-best allocation problem also provides us with the distribution of various quantities of interest, such as the road width fraction, etc., throughout the city. As expected, the unit land rent and the after transportation cost income fraction decrease more rapidly with distance from the CBD for higher unit congestion costs (or more precisely, for higher values of ε_t) while its profile becomes more concave.

What is not expected is that the distribution of the fraction of land for housing $b(x)$ changes from concave downward in Cases I and II to concave upward (or convex to the origin) in Case IV which has a relatively higher congestion cost. The same is true for $N(x)$, the distribution of households living outside the ring of radius X. We show in Figures 1 and 2 the profiles of $b(x)$ and $u(x) \equiv N/N_0$ for Case IV and the corresponding profiles for lower unit congestion costs to demonstrate this unexpected qualitative feature (with $\xi_t \equiv \varepsilon_t$ in these figures). However, the effect of distance cost of transportation is likely to be significant and should be included for the true picture in this case since $\varepsilon_t^2/\sigma = 1.0$.

Figure 1 also shows that the road width fraction at the edge of the CBD, $1 - b(x = 1)$, first increases with ε_t, reaching a maximum between $\varepsilon_t = 0.25$ and $\varepsilon_t = 0.30$, and then decreases (very slowly) with ε_t after that. It seems that more roads are needed at and near the edge of the CBD to offset the increase in unit congestion cost for the (smaller number of) travellers living away from the CBD when the transportation cost is still relatively low. On the other hand, more housing space is needed to slow down the increase in unit land rent near the CBD due to housing demand of the large population there when the unit congestion cost is sufficiently high.

APPENDIX

Land allocation by a cost-benefit criterion. Suppose instead of the maximum common utility policy of Section (4), land allocation for roads and housing is made by a cost-benefit criterion based on the market land price. According to this criterion, the land policy variable b is determined by the condition that the marginal benefit from more land for roads (leading to a reduction of the congestion cost of transportation) equals the marginal cost of land rent lost from residential use priced at the market rent [**5**, **6**]:

$$ak\left[\frac{N}{\theta X(1-b)}\right]^{k+1} = r = r_i w^{\alpha+1}. \tag{A1}$$

In terms of the dimensionless quantities introduced in (5.6)–(5.8), the condition (A1) becomes

$$\frac{u}{x(1-b)} = \frac{\zeta}{\varepsilon_t} w^{(\alpha+1)/(k+1)}, \tag{A2}$$

where

$$\zeta = \left[\frac{\bar{v}}{k(\alpha+1)(1-\eta)}\right]^{1/(k+1)}. \tag{A3}$$

Upon using (A2) to eliminate u from (5.9), we obtain

$$w' = -\varepsilon_t(1-\eta)\zeta^k w^{k(\alpha+1)/(k+1)} - \varepsilon_t^{k+1}\eta. \tag{A4}$$

Rather remarkably, the housing land fraction b does not appear in this first order equation. It has been eliminated along with u because u and b appear in (5.9) only in that combination which appears on the left side of (A2). Hence, the separable first order ODE (A4) along with the initial condition $w(1) = 1$ determines $w(x; \varepsilon_t)$.

Next, we use (A2) to express u in terms of $\ell_T \equiv x(1-b)$ and the known function $w(x; \varepsilon_t)$:

$$u = \frac{\zeta}{\varepsilon_t} \ell_T w^{(\alpha+1)/(k+1)}, \tag{A5}$$

which is in turn used to eliminate u from (5.11), taken in the form

$$u' = \bar{v}\ell_T w^\alpha - \bar{v} x w^\alpha = -k\zeta^{k+1}(\alpha+1)(1-\eta)(x-\ell_T)w^\alpha, \tag{A6}$$

to get

$$[\ell_T w^{(\alpha+1)/(k+1)}]' = -\varepsilon_t^k \zeta^k (\alpha+1)(1-\eta)(x-\ell_T)w^\alpha. \tag{A7}$$

The boundary condition $u(R) = 0$ (see (5.13)) and (A5) imply

$$\ell_T(R; \varepsilon_t) = 0; \tag{A8}$$

it serves as an auxiliary condition for (A7). An exact solution of the "terminal value" problem (A7) and (A8) for ℓ_T is also immediate since (A7) is a first order linear differential equation.

In what follows, we limit ourselves to the case $k = 1$. We have for this case

$$\text{(A9)} \quad \ell_T(x;\varepsilon_t) = x(1-b) = \frac{\varepsilon_t\zeta(\alpha+1)(1-\eta)}{g(x;\varepsilon_t,\zeta)w^{(\alpha+1)/2}(x;\varepsilon_t,\zeta)} \int_x^R \xi g(\xi;\varepsilon_t,\zeta)[w(\xi;\varepsilon_t,\zeta)]^\alpha \, d\xi$$

where

$$\text{(A10)} \quad g(x;\varepsilon_t,\zeta) = \exp\left\{ -\varepsilon_t\zeta(1-\eta)(\alpha+1)\int_1^x [w(\xi;\varepsilon_t,\zeta)]^\alpha \, d\xi\right\}.$$

The remaining unknown parameter $\bar{v}$ (or equivalently ζ) is determined by the boundary condition $u(1) = 1$. In view of (A5) and (A9) (as well as $w(1;\varepsilon_t) = 1$ and $g(1;\varepsilon_t) = 1$), this condition implies

$$\text{(A11)} \qquad (\alpha+1)(1-\eta)\zeta^2 \int_1^R xg(x;\varepsilon_t,\zeta)[w(x;\varepsilon_t,\zeta)]^\alpha \, dx = 1.$$

Finally, we have from (A9),

$$\text{(A12)} \quad b(x;\varepsilon_t) = 1 - \frac{\varepsilon_t\zeta(\alpha+1)(1-\eta)}{xgw^{(\alpha+1)/2}} \int_x^R \xi g(\xi;\varepsilon_t,\zeta)[w(\xi;\varepsilon_t,\zeta)]^\alpha \, d\xi.$$

For $\varepsilon_t \ll 1$, we see from (A12) that a fraction of land allocated to roads $(1-b)$ is to a first approximation $0(\varepsilon_t)$ of the total land available outside the CBD. This is consistent with the expectation that the amount of land allocated for roads is a decreasing function of the congestion cost. More quantitative results can be obtained for $\varepsilon_t \ll 1$ by discarding terms of order ε_t^2 or smaller in (A9), (A12), etc. However, they are more simply obtained by seeking a perturbation solution for the various initial (or terminal) value problems as in Section (6). We omit the calculation and simply give the following results for comparison with the second-best solution of Section (6):

$$\text{(A13)} \quad w(x;\varepsilon_t) = 1 - \frac{t}{y} = 1 - \varepsilon_t\left[\sqrt{\frac{2(1-\eta)}{(R^2-1)(\alpha+1)}}(x-1)\right] + 0(\varepsilon_t^2),$$

$$\text{(A14)} \qquad r_i = \frac{yN_0\bar{v}}{(\alpha+1)\theta R_i^2} \equiv r_0\tilde{v}(\varepsilon_t,\sigma,R,\eta),$$

(A15)
$$r_0 = \frac{2yN_0}{(\alpha+1)\theta(R_o^2 - R_i^2)},$$

(A16)
$$\tilde{v} = 1 + \varepsilon_t \left[\frac{1}{6}(2R^3 - 3R^2 + 1)\left(\frac{2\alpha+1}{\alpha+1}\right)\sqrt{\left(\frac{2}{R^2-1}\right)^3 (1-\eta)(\alpha+1)}\right] + 0(\varepsilon_t^2),$$

(A17)
$$b(x;\varepsilon_t) = 1 - \varepsilon_t \left[\frac{R^2 - x^2}{x}\sqrt{\frac{(\alpha+1)(1-\eta)}{2(R^2-1)}}\right] + 0(\varepsilon_t^2).$$

While we have limited ourselves to the case $k = 1$ and a fixed city boundary, it is evident that the same technique can be used to obtain approximate solutions for other values of k and/or an unknown outer city limit which is determined as part of the solution process by the agricultural rent beyond.

ENDNOTES

1. To keep our description of the solution procedures as simple as possible, we will use the more restrictive formulation of [**7**, **8**, **10**] instead of the more general one in [**6**]. Also, it is more convenient to treat r_i as the primary unknown parameter. In other words, we think of $\overline{U}$ as a function of r_i rather than r_i as a function of $\overline{U}$ as in [**5**, **6**].

2. The β_2 term is included in the perturbation solution for b since the perturbation expansion is effective for $x(1-b)$.

REFERENCES

1. R.J. Arnott and J.G. MacKinnon, *Market and shadow land rents with congestion*, Amer. Econ. Rev. **68** (1978), 588–600.

2. U. Ascher, J. Christiansen and R.D. Russell, *A collocation solver for mixed order systems of boundary value problems*, Math. Comp. **33** (1979), 659–679.

3. U. Ascher and F.Y.M. Wan, *Numerical solutions for maximum sustainable consumption growth with a multi-grade exhaustible resource*, SIAM J. Sci. Statist. Comput. **1** (1980), 160–172.

4. A.E. Bryson and Y.C. Ho, *Applied optimal control*, Ginn and Company, Waltham (Mass.) Toronto/London, 1969.

5. Y. Kanemoto, *Cost-benefit analysis and the second best land use for transportation*, J. Urban Econ. **4** (1977), 483–503.

6. ———, *Theories of urban externalities*, North-Holland Publishing Co., Amsterdam/New York/Oxford, 1980.

7. R.M. Solow, *Congestion, density and the use of land in transportation*, Swedish J. Econom. **74** (1972), 161–173.

8. ———, *Congestion cost and the use of land for streets*, Bell J. Econom. Management Sci. **4** (1973), 602–618.

9. R.M. Solow and W.S. Vickery, *Land use in a long narrow city*, J. Econom. Theory **3** (1971), 430–447.

10. F.Y.M. Wan, *Perturbation and asymptotic solutions for problems in the theory of urban land rent*, Stud. Appl. Math. **56** (1977), 219–239.

DEPARTMENT OF APPLIED MATHEMATICS, UNIVERSITY OF WASHINGTON, FS-20, SEATTLE, WA 98195