How to write solutions for "Solving Linear System" problems

We are learning the basic strategy and technique of solving systems of linear equations, i.e., using augmented matrix and row reductions. Let's give a typical example of solving equations, which satisfies the minimal requirement for a complete solution.

Problem. Solve the following system of linear equations.

$$\begin{cases} x_1 & -2x_2 & -x_3 & +3x_4 & = & 0 \\ -2x_1 & +4x_2 & +5x_3 & -3x_4 & = & 3 \\ 3x_1 & -5x_2 & -6x_3 & +8x_4 & = & 2 \end{cases}$$

Solution. The augmented matrix of this system is

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -3 & 3 \\ 3 & -5 & -6 & 8 & 2 \end{array}\right] \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

We apply the row reduction algorithm to transform it to RREF as follows.

$$\xrightarrow[③ - 3\,①]{② + 2\,①} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 1 & -3 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{\text{interchange ② \& ③}} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{③\times(1/3)} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 1 & -3 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[② + 3\,③]{① + ③} \left[\begin{array}{rrrr|r} 1 & -2 & 0 & 4 & 1 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{① + 2\,②} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 8 & 11 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

The system corresponds to the RREF (the last matrix) is

$$\begin{cases} x_1 & & & +8x_4 & = & 11 \\ & x_2 & & +2x_4 & = & 5 \\ & & x_3 & +x_4 & = & 1 \end{cases}$$

So the general solution to the original system is

$$\begin{cases} x_1 = 11 - 8x_4 \\ x_2 = 5 - 2x_4 \\ x_3 = 1 - x_4 \\ x_4 \text{ is free} \end{cases}$$

(1) You have to first say **True**, or **False**, at the beginning or your solution.
(2) If your assertion is wrong then you get zero point.
(3) You have to explain your assertion. The ideal form of your explanation should be a rigorous mathematical proof or disproof. However, we don't require that much. The minimal requirement is to give some reasoning so that the grader can see that your reasoning indeed leads to your assertion.

How to determine True or False?

It depends. A typical statement in a True of False problem describes a situation and then claim something. Basically there are two types of True or False problems. One type is that it describes a **concrete** situation and claim something. For example,

- The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form.

When considering it, you have to know the basic terms (concepts) appeared in the statement, which is "reduced row echelon form". You have to know its definition and see if the given matrix satisfies or violates the required conditions.

A perfect answer can be

False. The entry 2 *on the second row and second column is the leading (pivot) entry of that row. If it is in RREF, then it will contradicts with the requirement that the leading entry of each row must be* 1.

Another type describes an abstract situation. Usually it reads like

- If **P**, then **Q**. (**P** could be, "a linear system has no free variable", and **Q** could be "it has only one solution".)

For such a case, you have to understand the two concepts "having no free variable" and "having only one solution". Then you have to think about if there is such an implication or not. The assertion of true of false is not for the correctness of either **P** or **Q**, but for the **correctness of the implication** $\mathbf{P} \implies \mathbf{Q}$. Therefore, if there is a conter-example (which is a concrete case satisfying **P** but not **Q**), then this statement is false.

The division of concrete and abstract situations are not sharp. In any case, the most important thing is to know the definitions of the concepts, and knowing the relations among the concepts (theorems).

What if I have no idea?

This happens most of the time for abstract statements. For example

- If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^n$ so that no one of them is a multiple of another, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

Someone may think it is true because it is true if there are only two vectors. But there is no such a theorem. Indeed one can **consider a concrete example**, like the three vectors are in $\mathbb{R}^2$. But we know that if the number of vectors (which is 3) is greater than the dimensin (which is 2), then they must be linearly dependent. So in this situation, you see if cannot be true that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.

So a perfect answer to this can be

False. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *are vectors in* $\mathbb{R}^2$*, then* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *must be linearly dependent (because there is a theorem which says that if* $\mathbf{v}_1, \ldots, \mathbf{v}_p$ *are vectors in* $\mathbb{R}^n$ *and* $p > n$*, then they are linearly dependent). This gives a counter-example to the statement.*

An equally perfect answer can also be

False. Consider $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. *In this case, none of* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is a multiple of another. But* $\mathbf{u} - \mathbf{v} + \mathbf{w} = 0$, *which means they are linearly dependent. This gives a counter-example of the statement.*

An example for a true statement.

- If A is an $m \times n$ matrix and $\mathbf{y}$ is a linear combination of the columns of A, then the linear system $A\mathbf{x} = \mathbf{y}$ is consistent.

This is some rephrasing of the relation between linear combination of columns and solutions to a linear system. But if you forget why it is true, at least you have to know the definition of linear combinations and the meaning of being consistent. Then you may try to think of a concrete matrix A (for example, 2×2), then probably you realize that it is true for this case and there is nothing preventing it from being true if it is a larger matrix.

A perfect answer can be

True. Suppose the columns of A *are* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *and suppose* $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, *then it means the vector* $\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}$ *is a solution to the linear system* $A\mathbf{x} = \mathbf{y}$. *So it is consistent.*

Some conclusion.

So you see that the most important part lying in true or false problems is your understanding of concepts and theorems. If you don't have any idea, then try to write down the definition of all terms appeared in the statement and try to compare and connect them. Then you probably can find a reason for being true or being false.