Gauged linear σ-model and gauged Witten equation (joint work with Gang Tian)

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Motivation

Let $Q: \mathbb{C}^N \to \mathbb{C}$ be a homogeneous polynomial of degree r. Associated to Q there are two interesting geometric theories:

(i) The nonlinear σ-model of $X_Q \subseteq \mathbb{P}^{N-1}$ (Gromov-Witten)

based on the analysis of $\mathcal{M}(\Sigma) = \{u : \Sigma \to X_Q \mid \bar\partial u = 0\}$

Mathematical fundations: Ruan, Ruan-Tian, Fukaya-Ono, Li-Tian

(ii) Landau-Ginzburg model of the singularity $(\mathbb{C}^N, Q)$

based on the analysis of $\mathcal{W}(\Sigma) = \{u : \Sigma \to \mathbb{C}^N \mid \bar\partial u + \nabla Q(u) = 0\}$

Mathematical fundation: Fan-Jarvis-Ruan (2013),

following Witten's idea (1993')

In superstring theory, physicists care about the NLSM of Calabi-Yau manifolds. When $\deg Q = N$, $\overline{X}_Q$ is CY. Physicists discovered the so-called Landau-Ginzburg / Calabi-Yau correspondence between the two above theories, long before the mathematical theories rigorously constructed. The LG/CY correspondence remains mysterious to mathematicians.

A mathematically more accessible approach towards the LG/CY correspondence was introduced by Witten (1993). His theory is now refereed to as the gauged linear σ-model.

<u>The idea of GLSM</u>: Consider $\mathbb{C}^{N+1} = \{(x_1, \cdots, x_N, p)\}$ and a new potential $W(x,p) = p\,Q(x)$. W is invariant under a $\mathbb{C}^*$-action on $\mathbb{C}^{N+1}$ given by

$$\xi(x_1, \cdots, x_N, p) = (\xi x_1, \cdots, \xi x_N, \xi^{-r} p) \quad (r = \deg Q).$$

This action is Hamiltonian, with moment map $\mu = |x_1|^2 + \cdots + |x_n|^2 - r|p|^2 - \tau$

"GLSM" is a gauge theory with a superpotential W.

Classical vacuum: $(\mu^{-1}(0) \cap \mathrm{Crit}\, W) / S^1$

$$\partial_i W = p\, \partial_i Q, \quad \frac{\partial W}{\partial p} = Q \quad \Rightarrow \quad \tau > 0,\ |x| \neq 0, \text{ so } p = 0$$
$$Q = 0 \ : \ \overline{X}_Q$$

$\tau < 0 \Rightarrow p \neq 0$, $x_1 = \cdots = x_n = 0$. (Q is nondegenerate)

The variation of τ should relate the Landau-Ginzburg theory with the Calabi-Yau theory.

Our project is to give a mathematical construction of the gauged linear σ-model, then study its dependence on certain ~~var~~ parameters (including τ); we hope eventually the LG/CY correspondence can be understood.

<u>Gauged Witten Equation</u>

The first step is to set up a good elliptic PDE over a Riemann surface. Indeed we can work under a more general situation.

Let (X, ω, J) be a noncompact Kähler manifold and $W: X \to \mathbb{C}$ be a holomorphic function. Suppose we have a reductive Lie group $G = K^{\mathbb{C}}$ acting on X such that W is equivariant with respect to a character $\rho: G \to \mathbb{C}^*$, i.e. $W(gx) = \rho(g) W(x)$.

Let Σ be a Riemann surface. A "W-structure" over Σ is a pair (P, ϕ), where $P \to \Sigma$ is a holomorphic principal G-bundle and ϕ is an isomorphism : $\phi: P \times_\rho \mathbb{C} \xrightarrow{\sim} K_\Sigma$.

This allows us to lift W to the fibre bundle $Y = P \times_G X$. Choose local coordinate z on Σ and a frame e of P such that

$$\phi([e,1]) = dz. \qquad \left([e,1] \in P \times_\rho \mathbb{C} \right)$$

Then define $W_Y([p,x]) = W(x)\,dz \quad ([p,x] \in P \times_G X = Y)$.

Easy to check that $W_Y \in \Gamma(Y, \pi_Y^* K_\Sigma)$ is well-defined and holomorphic.

Suppose $G = K^{\mathbb{C}}$ and K is a compact Lie group. Choose a K-reduction of P. It induces a Hermitian metric on $T^\perp Y$. (Depending on the Kähler metric on X and the K-action is Hamiltonian.)

The vertical differential of W_Y is $dW_Y \in \Gamma(Y, (T^\perp Y)^* \otimes \pi_Y^* K_\Sigma)$.

Dualize it using the Hermitian metric of $T^\perp Y$, we obtain the vertical gradient

$$\nabla W_Y \in \Gamma(Y, T^\perp Y \otimes \pi_Y^* \Omega_\Sigma^{0,1}).$$

The "Witten equation" for sections $u \in \Gamma(Y)$ is

$$\bar\partial u + \nabla W_Y(u) = 0.$$

Here since Y is holomorphic, $\bar\partial u \in \Gamma(\Sigma, \mathrm{Hom}^{0,1}(T\Sigma, u^* T^\perp Y))$

$$\cong \Omega^{0,1}(u^* T^\perp Y);$$

$$\nabla W_Y(u) = u^* \nabla W_Y \in \Omega^{0,1}(u^* T^\perp Y).$$

Note that in writing down the Witten equation, we need to choose a W-structure and a K-reduction of P.

W-structures exists in a moduli and two W-structures (P, ϕ) and (P', ϕ') are equivalent if there is an isomorphism

$\psi : P \to P'$ such that the following diagram commutes.

$$\begin{tikzcd} P \times_\rho \mathbb{C} \arrow[r, "\phi"] \arrow[d, "\psi"'] & K_\Sigma \\ P' \times_\rho \mathbb{C} \arrow[ur, "\phi'"'] & \end{tikzcd}$$

In a simpler situation, if $G = G_1 \times \mathbb{C}^*$ and $\rho : G \to \mathbb{C}^*$ is induced from a character $\rho : \mathbb{C}^* \to \mathbb{C}^*$ (characterized by an integer r), then a W-structure on Σ consists of an arbitrary G_1-bundle $P_1 \to \Sigma$ together with an r-spin structure (L, ϕ). That is $L \to \Sigma$ is a holomorphic line bundle and ϕ is an isomorphism $\phi : L^{\otimes r} \xrightarrow{\sim} K_\Sigma$.

In particular, when G_1 is a finite group Γ, $W = Q : \mathbb{C}^N \to \mathbb{C}$ is homogeneous, then the Witten equation is the one considered by Fan-Jarvis-Ruan.

What is the gauged Witten equation? From now on we restrict to the splitting case, i.e. $G = \mathbb{C}^* \times G_1$ and $G_1 = K_1^{\mathbb{C}}$.

Instead of considering holomorphic G_1-bundle, we consider smooth K_1-bundles with connections. The connection A_1 is allowed to vary; we also allow the reduction on the $\mathbb{C}^*$-part, i.e. a Hermitian metric H_0 on L, to vary. Now for any smooth K_1-bundle $Q_1 \to \Sigma$ and an r-th root $L \to \Sigma$, for any K_1-connection $A_1 \in \mathcal{A}(Q_1)$ and Hermitian metric H_0 on L, we can form $\nabla^{H_0} W_Y \in \Gamma(Y, T^\perp Y \otimes \pi_Y^* \Omega_\Sigma^{0,1})$, and the Witten equation for $u \in \Gamma(Y)$ (here $Y = (Q_1 \times_\Sigma L) \times_{K_1 \times \mathbb{C}^*} X$).

$$\bar\partial_A u + \nabla^{H_0} W_Y(u) = 0.$$

To control the behavior of the variables A_1 and H_0, we borrow the idea of symplectic vortex equations. Choose a volume form $\nu \in \Omega^2(\Sigma)$. We can write down the "vortex equation"

$$* F_{A_1, H_0} + \mu_{H_0}(u) = 0.$$

Here $F_{A_1,H_0} \in \Omega^2(\Sigma, \mathrm{Ad}(\mathrm{Lie}\, K))$ is the curvature form of the K_1-connection A_1 and the Chern connection of H_0; $*$ is the Hodge-star induced from the volume form ν. On the other hand, $\mu : X \to \mathrm{Lie}(K)^* \cong \mathrm{Lie}\, K$ and $\mu_{H_0}(u) \in \Omega^0(\Sigma, \mathrm{Ad}(\mathrm{Lie} K))$ is well-defined. The gauged Witten equation is

$$\begin{cases} \bar\partial_{A_1} u + \nabla^{H_0} W_Y(u) = 0 \\ * F_{A_1, H_0} + \mu_{H_0}(u) = 0 \end{cases} \quad \text{(gauge invariant under } \mathcal{G}(Q_1) \times C^\infty(\Sigma, S^1)\text{)}$$

Special case: When $W=0$, we don't need W-structures and the setting can be extended to Hamiltonian K-manifolds (X, ω, μ) with K-invariant almost complex structure J. This is the case of symplectic vortex equation (introduced by Cieliebak-Gaio-Salamon and I. Mundet).

Energy functional. The symplectic vortex equation is a natural equation in the sense that solutions are minimizers of the Yang-Mills-Higgs functional on (A,u), given by

$$YMH(A,u) = \frac{1}{2}\left(\|d_A u\|_{L^2}^2 + \|\mu(u)\|_{L^2}^2 + \|F_A\|_{L^2}^2 \right).$$

For gauged Witten equation, solutions are minimizers of the following energy functional:

$$E(A,u) = \frac{1}{2}\left(\| d_{A_1, H_0} u\|_{L^2}^2 + \| \mu_{H_0}(u) \|_{L^2}^2 + \| F_{A_1,H_0}\|_{L^2}^2 \right) + \|\nabla^{H_0} W(u)\|_{L^2}^2$$

We could continue working on the general situation but let us restrict to the linear case.

Let $Q: \mathbb{C}^N \to \mathbb{C}$ be homogeneous of degree r. Let $X = \mathbb{C}^{N+1}$

Let $W(x_1,\cdots x_N,p) = p\,Q(x_1,\cdots x_N)$ be as previously discussed.

Now $G = \mathbb{C}^* \times \mathbb{C}^*$, which acts on $\mathbb{C}^{N+1}$ as

$$(\xi_0,\xi_1)(x_1,\cdots x_N,p) = (\xi_0\xi_1 x_1,\cdots,\xi_0\xi_1 x_N,\ \xi_1^{-r}p)$$

This action is Hamiltonian, with moment map

$$\mu(x_1,\cdots x_N,p) = \begin{pmatrix} \sum_{i=1}^N |x_i|^2 - \tau_0 \\ \sum_{i=1}^N |x_i|^2 - r|p|^2 - \tau_1 \end{pmatrix} \qquad \tau_0,\tau_1 \in \mathbb{R}.$$

The parameter τ_1 will be responsible for the wall-crossing.

W is invariant under the second $\mathbb{C}^*$-action but homogeneous under the first $\mathbb{C}^*$-action. $\rho(\xi_0,\xi_1) = \xi_0^r \in \mathbb{C}^*$

$$W(gx) = \rho(g)\,W(x).$$

A W-structure is an r-spin structure (L_0,ϕ), $L_0^{\otimes r} \xrightarrow[\sim]{\phi} K_\Sigma$ together with an arbitrary holomorphic line bundle L_1.

We consider H_0 : Hermitian metrics on L_0 and consider L_1 as a Hermitian line bundle with arbitrary unitary connections: (u, H_0, A_1)

$$u \in \Gamma(\underbrace{L_0\otimes L_1 \oplus \cdots \oplus L_0\otimes L_1}_{n} \oplus L_1^{-r}) \qquad \begin{cases} \bar\partial_{A_1} u + \nabla^{H_0} W(u) = 0 \\ *F_{H_0,A_1} + \mu_{H_0}(u) = 0 \end{cases}$$

All the above are working with a general Riemann surface. For compact Riemann surfaces, the above setting is not enough. Motivated from Fan-Jarvis-Ruan's work, we consider compact Riemann surfaces Σ, with marked points $p_1, \cdots p_k$. Then the r-th root L_0 of K_Σ is allowed to have orbifold structures at $p_1, \cdots p_k$, and K_Σ should be replaced by

$$K_{log} = K_\Sigma \otimes \mathcal{O}(p_1) \otimes \cdots \otimes \mathcal{O}(p_k).$$

Since $\phi: L_0^{\otimes r} \xrightarrow{\cong} K_{log}$, the local group of L_0 near p_i must be $\mathbb{Z}_r$ (or its subgroup). So locally, L_0 is identified with $\mathbb{D} \times \mathbb{C} / \mathbb{Z}_r$, with $\mathbb{Z}_r$-action given by $\xi(x, t) = (\xi x, \xi^{m_i} t)$

The above structures gives an "r-spin curve" $(\Sigma, p_1, \cdots p_k, L_0, \phi)$

<u>Perturbations</u>. When there is an orbifold point whose $m_i = 0$, the Witten equation looks like the Floer type equation:

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \nabla W(u) = 0.$$

In general W has degenerate critical points, so the linearization of the Witten equation or the gauged Witten equation is not naturally a Fredholm operator.

Therefore we need to perturb the equation, not to achieve transversality but to achieve Fredholmness.
(There are other reasons why we need to perturb.)

A perturbation is given as follows. near the cylindrical ends at p_i with cylindrical coordinates (s,t), consider

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t}\right) + \nabla W(u) + \beta(s)\, \nabla F(u) = 0.$$

Here $\beta(s)$ is a cut-off function supported near $s=+\infty$ and $F: X \to \mathbb{C}$ is a holomorphic function such that

$W+F: X \to \mathbb{C}$ is a holomorphic Morse function.

If $W = pQ$, then we choose $F = -ap + \sum_{i=1}^{N} b_i x_i$ and

$W+F = p(Q-a) + \sum_{i=1}^{N} b_i x_i$ (Lefschetz pencil)

<u>Thm.</u> (Tian-X., 2014) In the above setting for solutions to the perturbed gauged Witten equation, we have:

① Asymptotic behavior: $m \neq 0 \Rightarrow u(s,t) \to \{0\} \times \mathbb{C}$; $m=0 \Rightarrow$
$u(s,t) \to \mathrm{Crit}(W+F)$

② Fredholm theory ③ Compactness (fixing an r-spin curve).

<u>Invariants</u>. How to define (formally) an invariant out of the moduli space of solutions:

<u>The state space</u> $\mathcal{H}_Q = \bigoplus_{k=0}^{r-1} \mathcal{H}_Q^{(k)}$ primitive cohomology

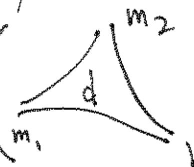
$k=1,\cdots r-1$, $\mathcal{H}_Q^{(k)} \cong \mathbb{Q}\{e^k\}$; $k=0$, $\mathcal{H}_Q^{(0)} \cong PH^{N-2}(\bar{X}_Q)$

If we fix an r-spin curve $C = (\Sigma, p_1, \cdots p_k; L_0, \phi)$ with monodromies $\gamma_i = \exp(2\pi i m_i / r)$, $m_i \in \{0, 1, \cdots r-1\}$ then the invariant is a multilinear function (for $d \in \mathbb{Z}$)

$$\langle , \cdots , \rangle^d_C : \mathcal{H}_Q^{(m_1)} \otimes \cdots \otimes \mathcal{H}_Q^{(m_k)} \to \mathbb{Q}.$$

<u>Example</u>. When $k=3$, suppose $m_1 \neq 0$, $m_2 \neq 0$, $m_3 = 0$. Choose $\theta \in \mathcal{H}_Q^{(0)} = PH^{N-2}(\bar{X}_Q) \cong H^{N-1}(Q^{-1}(a); \mathbb{Q})^{\mathbb{Z}_r}$. Then

$$\langle e^{m_1}, e^{m_2}, \theta \rangle^d_C = \sum_{\kappa \in \mathrm{Crit}(W+F)} A^\theta_\kappa \; \# \Big(\quad \Big).$$



Here # is the virtual counting of solutions to the perturbed gauged Witten equation whose asymptotic at p_3 is given by K. It is nonzero only when the virtual dimension is zero. A^θ_κ is a topological intersection number in $Q^{-1}(a)$ b.c K gives a cycle $H_{N-1}(Q^{-1}(a), \infty)$.