DONALDSON'S RESULTS ON SYMPLECTIC HYPERSURFACES AND LEFSCHETZ PENCILS

GUANGBO XU

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1. Statement of the theorems

We consider compact symplectic manifolds. The symplectic form ω represents a de Rham cohomology class $[\omega] \in H^2(V, \mathbb{R})$.

Theorem 1.1. [1, Theorem 1] *Let (V, ω) be a compact symplectic manifold of dimension $2n$ and suppose the cohomology class of ω is integral. Then for sufficiently large integer k, the Poincaré dual of $k[\omega]$ can be realized as a symplectic submanifold $W \subset V$.*

If V is Kähler, then there exists a positive line bundle L with Chern class $[\omega]$. Then for k large, a generic holomorphic section $s \in H^0(L^k)$ has a smooth vanishing locus which is a submanifold and is Poincaré dual to $k[\omega]$.

In [2] this theorem was enhanced to the following form. To state the enhancement, we need to introduce the notion of *topological Lefschetz pencils*.

Definition 1.2. A topological Lefschetz pencil on V consists of the following data,

(1) a codimension-4 submanifold $A \subset V$,
(2) a finite set of points $\{b_\lambda\} \subset V \setminus V$,
(3) a smooth map $f : V \setminus A \to S^2$ whose restriction to $V \setminus A \cup \{b_\lambda\}$ is a submersion, and $f(b_\lambda) \neq f(b_\mu)$ for $\lambda \neq \mu$.

This data is required to conform to the following standard local models.

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At a point $a \in A$, there are compatible local complex coordinates such that A is given by $z_1 = z_2 = 0$ and on the complement of A in a neighborhood of a, f is given by

$$(z_1, \dots, z_n) \mapsto z_1/z_2 \in \mathbb{P}^1.$$

At a point b_λ there are compatible local complex coordinates in which f is represented by the nondegenerate quadratic form

$$(z_1, \dots, z_n) \mapsto f(b_\lambda) + z_1^2 + \cdots + z_n^2.$$

Theorem 1.3. [2, Theorem 2] *In the same situation, for sufficiently large integer k, there is a topological Lefschetz pencil on V whose fibres are symplectic subvarieties, homologous to k times the Poincaré dual of* $[\omega]$.

The integrable case is also easy to understand. Take $L \to V$ whose Chern class is $[\omega]$. Then for k large, take a generic pair of holomorphic sections $s_0, s_1 \in H^0(L^k)$. The meromorphic function $f = s_0/s_1 : V \to \mathbb{P}^1$ gives a Lefschetz fibration , where the b_λ's are critical points of f. Equivalently, view $V \subset \mathbb{P}^N$ via the Kodaira embedding, then a generic pencil of hyperplane intersections gives a Lefschetz fibration.

In this note, we give a brief account of Donaldson's proofs of these two theorems.

2. Proof of Theroem 1.1

2.1. **The idea.** Let $\mathbb{C}^n$ have the standard metric and symplectic form ω. Let G be the Grassmannian of oriented real $2n-2$-planes in $\mathbb{C}^n$ and $G^+ \subset G$ the open subset of symplectic $2n-2$-planes Π, i.e., those for which the restriction of ω^{n-1} is positive relative to the orientation. G^+ only depends on the symplectic form ω. We also have a volume form Ω_Π on each such subspaces. We can define the "Kähler angle" $\theta : G \to [0, \pi]$ by

$$\theta(\Pi) = \cos^{-1}\left(\frac{1}{(n-1)!}\frac{\omega^{n-1}|_\Pi}{\Omega_\Pi}\right). \tag{2.1}$$

The complex subspaces are those with $\theta(\Pi) = 0$ and θ measures the amount by which a subspace fails to be complex-linear, and $G^+ = \theta^{-1}[0, \pi/2)$.

Now suppose Π is given by the kernel of a real linear map $A : \mathbb{C}^n \to \mathbb{C}$, $A = a' + a''$ (which induces a natural orientation). We see

1. A has real rank 2 unless $\overline{a''} = e^{i\alpha}a'$ for some α;
2. If A has rank 2 and $\Pi = \ker A$, then

$$\tan\theta(\Pi) = \frac{2\sqrt{|a'|^2|a''|^2 - |\langle a', \overline{a''}\rangle|^2}}{|a'|^2 - |a''|^2}$$

Therefore we see

Lemma 2.1. *If $|a''| < |a'|$, then* $\ker(a' + a'') \subset \mathbb{C}^n$ *is symplectic.*

Then we will prove

Theorem 2.2. *Let $L \to V$ be a complex line bundle and $c_1(L) = [\omega]$. Then there exists $C > 0$ such that for all large k, there is a smooth section $s \in \Gamma(L^k)$ such that*

$$|\overline{\partial} s| < \frac{C}{\sqrt{k}} |\partial s| \tag{2.2}$$

on the vanishing locus of s.

Together with Lemma 2.1, this theorem implies Theorem 1.1.

We briefly describe Donaldson's construction of the section s. For large k, chooing finitely many points $p_1, \dots, p_{M_k}$ on V such that balls centered at p_i of radius $O(k^{-1/2})$ cover V. For each i there is a section σ_i of L^k supported in B_i, which is constructed using the concrete coordinates around p_i. Then by some very complicated analytical argument, Donaldson proved that a delicate choice of a linear combination $s_{\underline{w}} = \sum w_i \sigma_i$ satisfies the transversality condition of Theorem 2.2.

2.2. **Local theory.** We first construct the sections σ_i, which is purely local.

2.2.1. *Symplectic structure.* Consider the standard symplectic form

$$\omega_0 = \frac{\boldsymbol{i}}{2} \sum_{\alpha=1}^{n} dz_\alpha d\overline{z}_\alpha \tag{2.3}$$

which is equal to $\boldsymbol{i} dA$ with

$$A = \frac{1}{4} \sum_{\alpha=1}^{n} \left(z_\alpha d\overline{z}_\alpha - \overline{z}_\alpha dz_\alpha \right) \tag{2.4}$$

Therefore, ω_0 is the curvature form of a $U(1)$-connection on the trivial line bundle with connection matrix A. This gives a $\overline{\partial}$-operator which reads

$$\overline{\partial}_A f = \overline{\partial} f + A^{0,1} f. \tag{2.5}$$

We see

$$\overline{\partial}_A e^{-|z|^2/4} = 0,\ \partial_A e^{-|z|^2/4} = \frac{1}{2} \Big(\sum_\alpha \overline{z}_\alpha dz_\alpha \Big) e^{-|z|^2/4}. \tag{2.6}$$

We see the trivial bundle ξ has a holomorphic section $e^{-|z|^2/4}$ which decays exponentially. For $k \geq 1$, the bundle ξ^k has connection form kA and holomorphic section $e^{-k|z|^2/4}$.

Now consider (V, ω) with

- a fixed compatible almost complex structure J;
- a line bundle $L \to V$ with $U(1)$-connection having curvature $-\boldsymbol{i}\omega$;

Let g be the metric $\omega(\cdot, J\cdot)$ and $g_k = kg$.

For any $p \in V$, there is a Darboux chart $\chi = \chi_p : B^{2n} \to V$ such that $\chi_p(0) = p$ and $\chi_p^*\omega = \omega_0$. Since V is compact, we may assume that all derivatives of χ_p are uniformly bounded. We also assume that χ_p is complex linear at 0, so there is $C > 0$ such taht

$$|\chi^* J - J_0| \leq C|z|, \ |\nabla(\chi^* J - J_0)| \leq C. \tag{2.7}$$

Given k, we compose χ with the dilation map

$$\widetilde{\chi} = \chi \circ \delta_{1/\sqrt{k}} : \sqrt{k} B^{2n} \to V. \tag{2.8}$$

Therefore we can assume

$$|\widetilde{\chi}^* J - J_0| \leq \frac{C}{\sqrt{k}}|z|, \ |\nabla(\widetilde{\chi}^* J - J_0)| \leq \frac{C}{\sqrt{k}} \tag{2.9}$$

with C taken uniformly.

On the other hand, since $\widetilde{\chi}^*(-\boldsymbol{i}k\omega)$, the pull-back of the curvature of L^k is the standard form $-\boldsymbol{i}\omega_0$ on $\mathbb{C}^n$. So we may take lift $\widetilde{\chi}$ to a map

$$\widetilde{\chi} : \sqrt{k} B^{2n} \times \mathbb{C} \to L \tag{2.10}$$

which preserves the connection. We have a locally defined section σ.

Let d_k be the distance function of g_k and define

$$e_k(p,q) = \begin{cases} e^{-d_k(p,q)^2/5}, & d_k(p,q) \leq k^{1/4}; \\ 0, & d_k(p,q) > k^{1/4}. \end{cases} \tag{2.11}$$

(e_k is supported in a $O(k^{-1/2})$-neighborhood of the diagonal).

Proposition 2.3. *For each $p \in V$ and sufficiently large k, there is a smooth section σ_p of L^k over V such that for*

1. *For fixed R, $|\sigma_p(q)| \geq C^{-1}$ if $d_k(p,q) \leq R$.*
2. *$|\sigma_p(q)| \leq e_k(p,q)$.*
3. *$|\nabla_V \sigma_p| \leq C(1 + d_k(p,q))e_k(p,q)$.*
4. *$|\overline{\partial}_L \sigma_p(q)| \leq \frac{C}{\sqrt{k}} d_k(p,q)^2 e_k(p,q)$.*
5. *$|\nabla_V \overline{\partial}_L \sigma_p(q)| \leq \frac{C}{\sqrt{k}}(d_k(p,q) + d_k(p,q)^3)e_k(p,q)$.*

Here $\overline{\partial}_L$ is the $\overline{\partial}$-operator on L (and L^k), ∇_V is the covariant derivative induced from the Levi-Civita connection on V and the connection on L^k.

Proof. Choose a cut-off function $\beta_k : \mathbb{C}^n \to [0,1]$ by rescaling a standard cut-off function β with

$$\beta_k(z) = \beta\Big(\frac{z}{k^{1/6}}\Big). \tag{2.12}$$

Then we define

$$\sigma_p = \widetilde{\chi}(\beta_k \sigma) \in \Gamma(L^k). \tag{2.13}$$

Since $\widetilde{\chi}$ shrinks with a factor $k^{-1/2}$, σ_p is supported in a ball of radius $O(k^{-1/3})$ centered at p. Then we can prove everything by explicit calculation. □

2.3. **A very dense open cover.**

Lemma 2.4. *There is a constant $C > 0$ such that for all k, we can cover V by g_k-unit balls with centers $p_1, \dots, p_{M_k}$ such that*

$$\sum_{i=1}^{M_k} d_k(p_i, q)^r e_k(p_i, q) \le C, \ \forall q \in V, \ r = 0, 1, 2, 3. \tag{2.14}$$

Proof. If $\Lambda \subset \mathbb{C}^n$ is a lattice in $\mathbb{C}^n$, then for any $a, r > 0$ and $w \in \mathbb{C}^n$,

$$\sup_w \sum_{\mu \in \Lambda} |\mu - w|^r e^{-a|\mu - w|^2} < \infty. \tag{2.15}$$

Choose a finite cover $\phi_s : O_s \to V$, $s = 1, \dots, S$ with O_s bounded in $\mathbb{C}^n$ such that

$$\frac{1}{2}|x - y| \le d(\phi_s(x), \phi_s(y)) \le 2|x - y|. \tag{2.16}$$

Choose slightly smaller $O_s' \subset\subset O_s$ such that O_s' also cover V.

Let Λ_k be the lattice $\alpha(\mathbb{Z}^n \oplus \boldsymbol{i}\mathbb{Z}^n)$ with

$$\alpha = \frac{\sqrt{\frac{n}{2}}}{2\sqrt{k}}. \tag{2.17}$$

Let Λ_s be the image under ϕ_s of $O_s \cap \Lambda_k$. Then when k is large, the ball of radius $k^{-1/2}$ centered at points of Λ_s cover $\phi_s(O_s')$. Take p_i be the union of those lattice points. Then to bound the quantity, we need to bound the individual ones

$$R_s(q) = \sum_{p \in \Lambda_s} d_k(p, q)^r e_k(p, q). \tag{2.18}$$

Since e_k vanishes if $d_k(p, q) > k^{1/4}$, so we only need to consider the case when q lies in $\phi_s(O_s)$. Then

$$R_s(q) \le \sum_{\lambda \in \Lambda_k} 2^r k^{r/2} |z - \lambda|^r e^{-k|z-\lambda|^2/20} = \sum_{\mu \in \Lambda_0} 2^r |w - \mu|^r e^{-|\mu - w|^2/20}. \tag{2.19}$$

□

So for each k, we fix the choice of $p_1, \dots, p_{M_k}$ and denote $\sigma_i = \sigma_{p_i}$. Let $B_i \subset V$ be the g_k-unit ball centered at p_i.

2.4. **A nice linear combination.** Our desired section will be a linear combination of those σ_i. For complex numbers $\underline{w} = (w_1, \dots, w_{M_k})$, take

$$s = s_{\underline{w}} = \sum_{i=1}^{M_k} w_i \sigma_i. \tag{2.20}$$

We always consider coefficients w_i with $|w_i| \leq 1$.

Lemma 2.5. *For any $\underline{w}$, $s = s_{\underline{w}}$ satisfies*

$$|s| \leq C,\ |\overline{\partial}_L s| \leq \frac{C}{\sqrt{k}},\ |\nabla_V \overline{\partial}_L s| \leq \frac{C}{\sqrt{k}}. \tag{2.21}$$

Proof. By Proposition 2.3 and Lemma 2.4, we have

$$|s(q)| \leq \sum_{i=1}^{M_k} |\sigma_i(q)| \leq \sum_{i=1}^{M_k} e_k(p_i, q) \leq C. \tag{2.22}$$

Other items of Proposition 2.3 together with Lemma 2.4 imply the other two estimates. □

Proposition 2.6. *There is an $\epsilon > 0$ such that for all large k, we can choose $\underline{w}$ such that s satisfies the* **transversality condition**

$$|\partial_L s| > \epsilon \tag{2.23}$$

on the zero locus of s.

Lemma 2.5 and Proposition 2.6 imply Theorem 2.2.

2.5. **Proof of Proposition 2.6.** Since B_i is of radius $O(k^{-1/2})$, $\widetilde{\chi}_i^{-1}(B_i)$ is contained in a bounded region, say $\Delta = \frac{11}{10}B^{2n}$. Take Δ^+ be the polydisk of radius $\frac{22}{10}$. Over Δ^+ we have the section σ_i trivializting L^k. The following lemma, which can be proved by straightforward calculation, shows that we only need to check the transversality condition on each chart.

Lemma 2.7. *Let $s = s_{\underline{w}}$ is a section of L^k with $|w_i| \leq 1$ and $f_i : \Delta^+ \to \mathbb{C}$ is the corresponding function. Then*

(1) $\|f_i\|_{C^1(\Delta^+)} \leq C$;
(2) $\|\overline{\partial} f_i\|_{C^1(\Delta^+)} \leq \frac{C}{\sqrt{k}}$;
(3) *If $|\partial f_i| > \epsilon$ on $f_i^{-1}(0) \cap \Delta$, then for k large, $|\partial_L s| \geq C^{-1}\epsilon$ on $s^{-1}(0) \cap B_i$.*

Since the number M_k grows with k, we subdivide the points p_i into finitely many groups.

Lemma 2.8. *Given $D > 0$, there is $N(D) > 0$ independent of k such that for all k with $p_1, \ldots, p_{M_k}$ given by ..., there is a partition*

$$I = \{1, \ldots, M_k\} = I_1 \cup I_2 \cup \cdots \cup I_N \tag{2.24}$$

such that for $\alpha \in \{1, \ldots, N\}$,

$$d(p_i, p_j) \geq D, \ \forall p_i, p_j \in I_\alpha. \tag{2.25}$$

Fix $D > 0$. Denote

$$V_\alpha = \bigcup_{i \in I_\beta, \ \beta \leq \alpha} B_i, \ \emptyset = V_0 \subset V_1 \subset \cdots \subset V_N = V. \tag{2.26}$$

The construction of the coefficient vector $\underline{w}$ (and hence the section s) is done inductively. We start with an arbitrary $\underline{w}$. Then we modify those w_i with $i \in I_1$ such that the controlled transversality condition holds on V_1. Suppose we have chosen $\underline{w}_\alpha$ such that the controlled transversality holds on V_α, then we modify those w_i with $i \in I_{\alpha+1}$ to obtain $\underline{w}_{\alpha+1}$, such that $s_{\alpha+1}$ satisfies the controlled transversality over $V_{\alpha+1}$. The induction finishes in finite steps.

Definition 2.9. Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \to \mathbb{C}$ be a smooth function. For $\eta > 0$ and $w \in \mathbb{C}$ we say that f is η-transverse to w over U if

$$|f(z) - w| \leq \eta \implies |\partial f(z)| \geq \eta. \tag{2.27}$$

We say a section $s \in \Gamma(L^k)$ is η-transverse over B_i if $f_i = s/\sigma_i$ is η-transverse to 0 over the corresponding set Δ.

3. The transversality theorem

For $\delta \in (0, 1)$, $p > 0$, introduce

$$Q_p(\delta) = \frac{1}{\big(-\log \delta\big)^p}. \tag{3.1}$$

The burden of proving Proposition 2.6 is given to the following theorem.

Theorem 3.1. *For $\sigma > 0$, let $\mathcal{H}_\sigma$ denote the set of functions f on Δ^+ such that*

(1) $\|f\|_{C^0(\Delta^+)} \leq 1$,
(2) $\|\bar{\partial} f\|_{C^1(\Delta^+)} \leq \sigma$.

Then there is an integer p depending only on n such that for any $\delta \in (0, 1/2)$, if $\sigma < Q_p(\delta)\delta$, then for any $f \in \mathcal{H}_\sigma$, there is $w \in \mathbb{C}$ with $|w| \leq \delta$ such that f is $Q_p(\delta)\delta$-transverse to w over Δ. Moreover, w can be chosen in any preferred half-plane in $\mathbb{C}$.

Now we start the induction argument.

Lemma 3.2. *Let $s = s_{\underline{w}}$, and for any α, let $\underline{w}'$ be another coefficient vector which agrees with $\underline{w}$ except for coefficients belonging to I_α. Suppose $|w'_j - w_j| \leq \delta$ for some δ. Then*

(1) *For any i, $\|f'_i - f_i\|_{C^1(\Delta^+)} \leq C\delta$.*
(2) *For $i \in I_\alpha$, $\|f'_i - f_i - (w'_i - w_i)\|_{C^1(\Delta^+)} \leq C\delta \exp(-D^2/5)$.*

Proposition 3.3. *There are constants $\rho < 1$ and p such that if*

(1) *$s^{\alpha-1} = s_{\underline{w}^{\alpha-1}} \in \Gamma(L^k)$ is $\eta_{\alpha-1}$-transverse over $V_{\alpha-1}$ with $\eta_{\alpha-1} \leq \rho$;*
(2) *$\frac{1}{\sqrt{k}} \leq Q_p(\eta_{\alpha-1})\eta_{\alpha-1}$;*
(3) *$\exp(-D^2/5) \leq Q_p(\eta_{\alpha-1})$,*

then there is another section $s^\alpha = s_{\underline{w}^\alpha}$ which is η_α-transverse over V_α where $\eta_\alpha = Q_p(\eta_{\alpha-1})\eta_{\alpha-1}$.

Proof. Consider some $i \in I_\alpha$. Suppose we choose w_i^α in such a way that

$$|w_i^\alpha - w_i^{\alpha-1}| \leq \delta_\alpha. \tag{3.2}$$

Then by the (1) of Lemma 3.2, s^α is $\eta_{\alpha-1} - C\delta_\alpha$-transverse over V_α. Therefore choosing δ_α properly, we have s^α still $\eta_{\alpha-1}/2$-transverse over $V_{\alpha-1}$. On the other hand, if $s^{\alpha-1} = f_i\sigma_i$ for $i \in I_\alpha$, then by Lemma 2.7,

$$\|\overline{\partial} f_i\|_{C^1(\Delta^+)} \leq \frac{C}{\sqrt{k}}. \tag{3.3}$$

Hence $f_i \in \mathcal{H}_\sigma$ for $\sigma = \frac{C}{\sqrt{k}}$. We can modify p properly to absorb the constant C. Then choose k big enough such that

$$\frac{1}{\sqrt{k}} \leq \delta_\alpha Q_p(\delta_\alpha), \tag{3.4}$$

then by Theorem 3.1, there exists v_i with $|v_i| \leq \delta_\alpha$ and f_i is $Q_p(\delta_\alpha)\delta_\alpha$-transverse to v_i over the unit ball. In other words, $f_i - v_i$ is $Q_p(\delta_\alpha)\delta_\alpha$-transverse to 0. We define

$$w_j^\alpha = w_j^{\alpha-1},\ \forall j \neq i,\ w_i^\alpha = w_i^{\alpha-1} - v_i. \tag{3.5}$$

By choosing an appropriate half-plane, we may still have $|w_i^\alpha| \leq 1$.

Now we define $\underline{w}^\alpha$ by

$$w_j^\alpha = w_j^{\alpha-1},\ j \notin I_\alpha,\ w_j^\alpha = w_j^{\alpha-1} - v_j,\ j \in I_\alpha. \tag{3.6}$$

By (2) of Lemma 3.2, the extra term is given by $C\delta_\alpha \exp(-D^2/5)$. So if

$$C\delta_\alpha \exp(-D^2/5) \leq \frac{1}{2} Q_p(\delta_\alpha)\delta_\alpha, \tag{3.7}$$

then s^α will be

Eventually we have s^α is $\widetilde{\eta}_\alpha$-transverse with $\widetilde{\eta}_\alpha = \frac{1}{2}Q_p(\delta_\alpha)\delta_\alpha$. Lastly, if

$$\frac{1}{2}Q_p(\delta_\alpha)\delta_\alpha \geq Q_p(\eta_{\alpha-1})\eta_{\alpha-1} = \eta_\alpha \tag{3.8}$$

then s^α is η_α-transverse. Otherwise, modify the value of p and δ_α such that the above becomes an equality. □

One can prove that for suitably chosen D and hence the partition, an induction argument can be carried out to produce an ϵ-transverse section over V.

4. Asymptotic behavior of s_k

In fact, in the proof we can see the submanifold W_k becomes extremely complicated as k grows, which will eventually fill out all of V. Indeed as currents, $\frac{k}{W}k$ converges to the symplectic form.

Proposition 4.1. [1, Proposition 40] *There is a constant $C > 0$ such that for any test form $\psi \in \Omega^{2n-2}(V)$,*

$$\left| \int_{W_k} \psi - k \int_V \omega \wedge \psi \right| \leq C\sqrt{k} \| d\psi \|_{L^\infty(V)}. \tag{4.1}$$

Let $s \in \Gamma(L^k)$ cut out W_k. Consider the 1-form $A = s^{-1}\nabla s$ on the complement of W_k, which has integrable singularity. Hence A can be viewed as a current. Indeed, we have

$$dA = W_k - k\omega. \tag{4.2}$$

Therefore it suffices to prove that $\int_V |A| d\mu \leq C\sqrt{k}$.

If we consider it within a g_k-unit ball, i.e., radius $O(1/\sqrt{k})$ balls, identified with B^{2n} via $\widetilde{\chi}$, then the number of balls is roughly $O(k^n)$ while the pull-back to the unit ball will enlarge the volume form by k^n. The pull-back will also bring in a factor to ∇s. In a small scale we see that ∇s is uniformly bounded, so it suffices to control the integral of $|s|^{-1}$ in any g_k-unit ball.

Proposition 4.2. *Given $\rho > 0$ let $\mathcal{K}_\rho$ be the space of complex-valued functions f on $2B^{2n}$ with $\|\overline{\partial} f\|_{C^1} \leq \rho/2$, $\|f\|_{C^0} \leq 1$ and $|f| \leq \rho \implies |\partial f| \geq \rho$. Then there is $C(\rho) > 0$ such that for all $f \in \mathcal{K}_\rho$,*

$$\int_{B^{2n}} \frac{1}{|f|} d\mu \leq C(\rho). \tag{4.3}$$

Proof. Divide B^{2n} into two parts I_1 and I_2 by whether $|f| \leq \rho$ or $|f| \geq \rho$. I_2 is easy to control. On the other hand, we have the "co-area" formula:

$$I_1 = \int_{|f|\leq\rho} \frac{1}{|f|} d\mu = \int_{|\sigma|\leq\rho} |\sigma|^{-1} \Big(\int_{f^{-1}(\sigma)} |J_f|^{-1} d\nu \Big) d\sigma. \tag{4.4}$$

We have

$$J_f = \sqrt{\left(|\partial f|^2 + |\overline{\partial} f|^2\right)^2 - 4|\langle \overline{\partial f}, \overline{\partial} f \rangle|^2} \geq \frac{3\rho^2}{4}. \tag{4.5}$$

Therefore it suffices to give a uniform bound on the volume of each $f^{-1}(\sigma)$. If $f^{-1}(\sigma)$ doesn's have uniformly bounded volume, then choose a sequence f_i, σ_i. By elliptic regularity, f_i converges in C^1 to f and σ_i converges to σ with $|\sigma| \leq \rho$. However, $Z_{\sigma_i}(f_i)$ converges to $Z_\sigma(f)$ in the C^1-sense so the volume should converge.

□

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Department of Mathematics, University of California, Irvine, Irvine, CA 92697 USA

E-mail address: guangbox@math.uci.edu