MOMENT FLOER HOMOLOGY AND ARNOLD-GIVENTAL CONJECTURE

Contents

1. Outline

First I will recall the Morse inequality and Morse-Smale-Witten complex, and introduce the Morse-Bott case. Then I recall Arnold conjecture and Floer's Lagrangian intersection homology, which generalize the Morse homology. Then I start to talk about Frauenfelder's moment Floer homology. This homology can be used to prove Arnold conjecture for certain symplectic quotients.

2. Morse inequality and Morse homology

Definition 2.1. We say that two submanifold $N_0, N_1 \subset M$ intersect **transversely** if for any $p \in N_0 \cap N_1$,

$$T_pN_0 + T_pN_1 = T_pM.$$

We say that N_0 and N_1 intersect **cleanly**, which is weaker than transverse, if $N_0 \cap N_1$ is a smooth submanifold of N_i, and for any $p \in N_0 \cap N_1$,

$$T_p(N_0 \cap N_1) = T_pN_0 \cap T_pN_1.$$

Definition 2.2. A smooth function $f : M \to \mathbb{R}$ is **Morse** if df intersects with the zero section of T^*M transversely. It is **Morse-Bott** if they intersect cleanly.

By Poincaré-Hopf, we know that the minimal number of zeroes of transverse sections is bounded below from the Euler characteristic. But we have a much stronger estimate, which is the **Morse inequality**: using $\mathbb{Z}_2$-coefficient, or any field coefficient, denote by $P_t(M)$ the Poincaré polynomial of M, i.e.

$$P_t(M) = \sum_i t^i \dim H^i(M, \mathbb{Z}_2);$$

and $M_t(f)$, the Morse polynomial, i.e.,

$$M_t(f) = \sum_i t^i m_i(f).$$

Then

$$M_t(f) - P_t(M) = (1 + t)R(t)$$

for some polynomial R with **nonnegative** integer coefficients. In particular, set $t = 1$, we see

$$\#\mathrm{Crit}_f \geq \sum_i \beta_i(M;\mathbb{Z}_2). \tag{2.1}$$

We can see this fact by looking at Morse homology. In the Morse case, we can assign an integer to each critical point, which is called the Morse index. Take a Riemannian metric, then we can talk about the gradient of f, ∇f, hence the negative gradient flow, which is a 1-parameter family of diffeomorphisms $\phi_t : M \to M$, $t \in \mathbb{R}$ defined by

$$\frac{d}{dt}\phi_t(x) + \nabla f(\phi_t(x)) = 0.$$

Then for each $p \in \mathrm{Crit}^i_f$, we define the unstable submanifold

$$W^u(p) = \left\{ x \in M \mid \lim_{t\to-\infty} \phi_t(x) = p \right\}.$$

$$W^s(p) = \left\{ x \in M \mid \lim_{t\to+\infty} \phi_t(x) = p \right\}.$$

They are submanifolds of dimension $\lambda(p)$ and $n - \lambda(p)$ respectively.

The **Morse-Smale** condition on the pair (f, g) means that for any $p, q \in \mathrm{Crit}_f$, $W^u(p)$ and $W^s(q)$ intersect transversely. Then in particular, if $\lambda(p) = \lambda(q) + 1$, then the space

$$\mathcal{M}(p, q)$$

of trajectories of negative gradient flows is a finite set.

We define the differential complex

$$CM(f)^i(M;\mathbb{Z}_2) := \bigoplus_{p\in\mathrm{Crit}^i_f} \mathbb{Z}_2 < p > \tag{2.2}$$

with differentials

$$\partial < p >= \sum_{\substack{q \in \mathrm{Crit}_f, \\ \lambda(q) = \lambda(p) - 1}} (-1)^{\#\mathcal{M}(p,q)} < q > .$$

We can prove that $\partial^2 = 0$ and the cohomology of the complex is called the **Morse** homology of (M, f). We can prove that this is independent of the Morse-Smale pair (f, g) and is isomorphic to the usual homology of M.

3. Arnold conjecture and Lagrangian intersection Floer homology

Let me say more on Morse theory. We know that T^*M is a symplectic manifold, with the standard symplectic structure

$$\omega = \sum_i dp_i \wedge dq_i.$$

Then a 1-form α, which is a section of T^*M and hence a submanifold, is Lagrangian if and only if α is closed. If $\alpha = df$, then consider the pull-backed function $f : T^*M \to \mathbb{R}$ and its Hamiltonian vector field with respect to the standard, symplectic structure. It is

$$X_f = -\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}.$$

The time-1 map ϕ_f^1 of the flow of X_f maps the zero section L_0 to the Lagrangian L_f defined by df. And $p \in M$ is a critical point of f if and only if $(p, 0) \in L_0 \cap$. Then Morse inequality can be viewed as giving a lower bound of $\#\left(L_0 \cap \phi_f^1(L_1)\right)$.

Arnold conjectured that this should hold for general symplectic manifold. Suppose $L \subset M$ is a Lagrangian and $H_t : M \to \mathbb{R}$ be a time-dependent Hamiltonian. Then the flow ϕ_H^t is defined by

$$\frac{d}{dt}\phi_H^t(x) = X_{H_t}(\phi_H^t(x)).$$

Arnold conjecture says that $\#(L \cap \phi_H^1(L)) \geq \sum_i \beta_i(L)$ with some coefficient.

The main breakthrough is due to Floer. Consider the space of arcs joining L_0 and L_1, i.e.,

$$\Omega = \{z : [0, 1] \to M \mid z(0) \in L_0, z(1) \in L_1\}.$$

This can be viewd as an infinite dimensional manifold, whose tangent space at $z \in \Omega$ is

$$T_z\Omega = \Gamma([0, 1], z^*TM).$$

We can define a 1-form on Ω by

$$\alpha(\xi) = \int_0^1 \omega(\dot{z}(t), \xi(t))dt.$$

It is easy to check that α is closed. Hence we may find a primitive function a, at least on the universal cover of Ω, such that $\alpha = da$.

Then the critical point of a, or the zero of α, corresponds to constant arcs, i.e., points in $L_1 \cap L_2$. Similar to the Morse-Smale-Witten complex, $L_0 \cap L_1$ generates the chain complex over some field $\mathbb{K}$

$$CF(L_0, L_1; \mathbb{K}) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{K}\{p\}.$$

The differential is defined using the trajectories of the Morse flow of the function a.

3.1. **Holomorphic strips.** Choose an ω-compatible almost complex structure J, i.e., $\omega(\cdot, J\cdot)$ defines a Riemannian metric, then this gives a metric on Ω. Then the dual of α is the gradient of a

$$\nabla a = J(z(t))\dot{z}(t).$$

The equation of the negative gradient flow line is

$$\partial_s u(s, t) + J\partial_t u(s, t) = 0, \ (s, t) \in \mathbb{R} \times [0, 1], \ u(s, 0) \in L_0, \ u(s, 1) \in L_1. \tag{3.1}$$

If the energy is finite, then we can show that

$$\lim_{s \to \pm\infty} u(s, t) = x_\pm \in L_0 \cap L_1$$

and we say that u is a holomorphic strip connecting x_- and x_+. Moreover, we can assign an integer, called the Conley-Zehnder index of $x_\pm$, which plays the role as Morse indices. With certain assumptions, we can show that the moduli space of holomorphic strips connecting $x_\pm$,

$$\widetilde{\mathcal{M}}(x_-, x_+)$$

is locally a smooth manifold of dimension $\lambda(x_+) - \lambda(x_-)$, where λ is the Conley-Zehnder index. This moduli space is usually non-compact, but in the simplest case which Floer assumed, we can rule out all bubbling phenomenon by topological restrictions, and the only noncompactness comes from the breaking of strips, as in the case of Morse-Smale-Witten complex. Moreover, if we use $\mathbb{Z}_2$-coefficients, then we don't care about orientations. Hence the space of holomorphic strips of relative index 1,

$$\mathcal{M}_1(x_-, x_+) := \widetilde{\mathcal{M}}_1(x_-, x_+)\Big/\mathbb{R}$$

consists of finitely many isolated strips. And we define the differential to be

$$\partial x_- := \sum_{x_+ \in L_0 \cap L_1} (-1)^{\# \mathcal{M}_1(x_-, x_+)} x_+. \tag{3.2}$$

There is no notorious bubbling of holomorphic disks, under Floer's assumption. Hence we can prove that $\partial^2 = 0$, which relies on a gluing theorem. Then $(CF(L_0, L_1; \mathbb{Z}_2), \partial)$ is a differential complex whose cohomology group is called the Floer homology group

$$HF(L_0, L_1; \mathbb{Z}_2).$$

Floer originally did this for $(L, \phi_H^1(L))$. He show that for two different choice of Hamiltonians, this homology groups are canonically isomorphic. Moreover, a neighborhood of L is symplectomorphic to T^*L, and a small time-dependent Hamiltonian will give $\phi_H^1(L)$ the same as a section of T^*L given by a Morse function on L. Then the Floer homology is isomorphic to the Morse homology of L. Then the Arnold conjecture follows immediately.

3.2. **General case.** In general Lagrangian intersection cannot be defined due to the so-called obstruction. See Fukaya-Oh-Ohta-Ono's book.

4. Moment Floer homology I: basic setting

The motivation is that, the original symplectic manifold usually will be topologically simpler than the symplectic quotient. Hence we can define homology group on the larger space which reduces to that in the symplectic quotient.

4.1. **Assumptions.** Let (M, ω) be a symplectic manifold. An effective action

$$\psi : G \hookrightarrow \mathrm{Diff}(M)$$

induces Lie algebra (anti)homomorphism

$$L : \mathfrak{g} \to \Gamma(TM).$$

This action is called **Hamiltonian**, if there exists a **moment map**

$$\mu : M \to \mathfrak{g}^*$$

which is G-invariant, i.e.

$$\langle \mu(gx), \xi \rangle = \langle \mu(x), \mathrm{Ad}_g^{-1} \xi \rangle$$

and

$$d\langle \mu, \xi \rangle = \iota_{X_\xi} \omega.$$

Hypothesis 4.1. The moment map $\mu : M \to \mathfrak{g}^*$ is proper, $0 \in \mathfrak{g}^*$ is a regular value of μ and G acts on $\mu^{-1}(0)$ freely.

This is a usual assumption on symplectic quotient, which implies that the Marsden-Weinstein quotient $\overline{M} := \mu^{-1}(0)/G$ is a smooth manifold of dimension

$$\dim_{\mathbb{R}} M - 2\dim_{\mathbb{R}} \mathbb{G}.$$

Let L_0, L_1 be two Lagrangian submanifolds.

Hypothesis 4.2. there exists $R_j : M \to M$, $j = 0, 1$ such that

$$R_j^*\omega = -\omega,\ R_j^2 = \mathrm{id},\ L_j = \mathrm{Fix} R_j.$$

We assume that the G-action is compatible with R_j, in the sense that

$$R_j \psi(g) R_j \in \psi(G).$$

Then this gives an involution $S_j : G \to G$. Then the induces involution on $\mathfrak{g}$ has ± 1 eigenspace decomposition. Denote

$$g_{L_j} = \left\{ \xi \in \mathfrak{g} \mid S_j \xi = \xi \right\},\ \mathfrak{g}_{L_j}^{\perp} = \left\{ \xi \in \mathfrak{g} \mid S_j \xi = -\xi \right\}.$$

Also we have the subgroup

$$G_{L_j} := \left\{ g \in G \mid gL_j = L_j \right\}.$$

Proposition 4.3. *With the above assumptions, if $\mu^{-1}(0) \cap L_j \neq \emptyset$, then $\mu^{-1}(0)$ and L_j intersect cleanly and*

$$\overline{L}_j := \mu^{-1}(0) \cap L_j / G_{L_j}$$

is a Lagrangian submanifold of $\overline{M}$.

Proof. We first prove that $\mu^{-1}(0) \cap L_j$ cleanly. First, we prove that for any $p \in L_j \cap \mu^{-1}(0)$, $d\mu(T_p L_j) = \mathfrak{g}_{L_j}^{\perp}$. Actualy, for any $\xi \in \mathfrak{g}_{L_j}$, $\mathcal{X}_\xi$ is tangent to L_j, hence for $v \in T_p L_J$,

$$\langle d\mu(v), \xi \rangle = \omega(\mathcal{X}_\xi, v) = 0$$

since L_j is Lagrangian. This show that $d\mu(T_p L_j) \subset \mathfrak{g}_{L_j}^{\perp}$. On the other hand, if $\xi \in \mathfrak{g}_{L_j}^{\perp}$ such that $\langle \xi, d\mu(v) \rangle = 0$ for all $v \in T_p L_j$, then we prove that $\xi = 0$. Indeed, for $w \in T_p M$, we write $w = w_+ + w_-$, with $(R_j)_* w_\pm = \pm w_\pm$. Hence

$$\langle \xi, d\mu(w) \rangle = \omega(\mathcal{X}_\xi, w_-) = -\omega(\mathcal{X}_\xi, -(R_j)_* w_-) = -\omega(\mathcal{X}_\xi, w_-) = -\langle \xi, d\mu(w) \rangle.$$

Since 0 is regular value of μ, this implies that $\xi = 0$. It is acturally holds for all moment value close to 0. Hence it is easy to see that $\mu(p) \subset \mathfrak{g}_{L_j}^{\perp}$ for $p \in L_j$ close to $L_j \cap \mu^{-1}(0)$.

Then consider a small neighborhood $U \subset \mathfrak{g}_{L_j}^{\perp}$. Then $\mu^{-1}(U)$ is a smooth submanifold. Then $\mu^{-1}(0)$ and $L_j \cap \mu^{-1}(U)$ intersect transversely in $\mu^{-1}(U)$. Hence they intersect cleanly in M.

Then we have the natural surjective map

$$(L_j \cap \mu^{-1}(0))/G_{L_j} \to G(L_j \cap \mu^{-1}(0))/G.$$

It is injective basically because G acts on $\mu^{-1}(0)$ freely. □

Hypothesis 4.4 (Morse-Bott). $\overline{L}_0$ and $\overline{L}_1$ intersect cleanly in $\mu^{-1}(0)/G$.

Hypothesis 4.5. We assume $\pi_2(M) = 0$, $\pi_1(M) = 0$, $\pi_1(L_j) = 0$, $\pi_0(L_j) = 0$.

The first two imply that M couldn't be compact. Hence we assume the convexity at infinity of M, which is a usual assumption when dealing with noncompact symplectic manifolds. As the author pointed out, this condition can be replaced by (M, L_j) being symplectic aspherical.

Hypothesis 4.6. Convexity at infinity.

5. Moment Floer homology II: moduli space of connecting orbits

Define the path space

$$\mathcal{P} = \left\{(x, \eta) \in C^\infty([0,1], M \times \mathfrak{g}) \mid x(j) \in L_j,\ \eta(j) \in \mathfrak{g}_{L_j}^\perp, j \in \{0,1\}\right\}.$$

Proposition 5.1. *The topological hypothesis implies that, $\mathcal{P}$ is connected and simply-connected.*

Define the gauge group

$$\mathcal{H} = \left\{g \in C^\infty([0,1], G) \mid g(j) \in G_{L_j},\ \left(g^{-1}\partial_t g\right)(j) \in \mathfrak{g}_{L_j}^\perp,\ j \in \{0,1\}\right\}. \tag{5.1}$$

It may be disconnected and let $\mathcal{H}_0$ be its identity component.

Then it acts on $\mathcal{P}$ by

$$g_*(x, \eta) = \left(g(t)x(t),\ \mathrm{Ad}_g\eta - \partial_t g g^{-1}\right). \tag{5.2}$$

Choose a base path $x_0 : [0,1] \to M$ with $x_0(j) \in L_j$ we define the **action functional** $\mathcal{A}_\mu : \mathcal{P} \to \mathbb{R}$ by

$$\mathcal{A}_\mu(x, \eta) = -\int_{[0,1]\times[0,1]} \overline{x}^*\omega + \int_0^1 \langle \mu(x(t)), \eta(t) \rangle dt. \tag{5.3}$$

The critical points of $\mathcal{A}_\mu$ are all paths (x, η) such that

$$\dot{x}(t) + \mathcal{X}_\eta(x(t)) = 0,\ \mu(x) \equiv 0. \tag{5.4}$$

Proposition 5.2. *We have natural identification*

$$\mathrm{Crit}\mathcal{A}_\mu/\mathcal{H} \simeq \overline{L}_0 \cap \overline{L}_1.$$

Proof. The map is given by $[x, \eta] \mapsto \pi(x(0))$. □

Take an almost complex structures J compatible with ω, which is also G-invariant, and a biinvariant metric on the Lie algebra $\mathfrak{g}$, we can write down the equation for negative gradient flow lines

$$\begin{cases} \partial_s u + J(\partial_t u + \mathcal{X}_\Psi) = 0, \\ \partial_s \Psi + \mu(x) = 0 \end{cases} \tag{5.5}$$

with boundary condition

$$u(s, j) \in L_j,\ \Psi(s, j) \in \mathfrak{g}_{L_j}^\perp. \tag{5.6}$$

The second boundary condition is natural becaues $\mu(L_j) \subset \mathfrak{g}_{L_j}^\perp$.

5.1. **Energy.** The energy for solutions (u, Ψ) is defined to be

$$E(u, \Phi, \Psi) := \frac{1}{2}\int_{\Theta} |\partial_s u|^2 + |\partial_t u + \mathcal{X}_{\Psi}(u)|^2 + |F_A|^2 + \left|\mu(u)\right|^2 dsdt \tag{5.7}$$

where

$$F_A = F_{\Psi dt} = (\partial_s \Psi) dsdt \in \Omega^2(\Theta, \mathfrak{g}).$$

For a connecting orbit with finite energy, we can prove that as $s \to \pm\infty$, $(u(s,t), \Psi(s,t))$ approaches to $(x_\pm, \eta_\pm) \in \mathrm{Crit}\mathcal{A}_\mu$. Then we define the evaluation map

$$ev_\pm : (u, \Psi) \mapsto [x_\pm, \eta_\pm] \in \pi_0(\mathrm{Crit}\mathcal{A}_\mu).$$

In fact, $\pi_0(\mathrm{Crit}\mathcal{A}_\mu)$ has a action by $\mathcal{H}/\mathcal{H}_0$, whose quotient is $\pi_0\left(\overline{L}_0 \cap \overline{L}_1\right)$.

For $c_\pm \in \pi_0(Crit\mathcal{A}_\mu)$, define $\widetilde{\mathcal{M}}(c_-, c_+)$ the space of connecting orbits with $ev_\pm(u, \Psi) = c_\pm$, and

$$\mathcal{M}(c_-, c_+) := \widetilde{\mathcal{M}}(c_-, c_+)/\mathcal{H}_0.$$

Modulo the transversality arguments, we can show that the above moduli is a smooth manifold. It is standard to prove the compactness theorem, which is similar to Morse trajectories and has no bubbling by our topological assumption. And a gluing theorem.

So far we can only assume the clean intersection between $\overline{L}_0$ and $\overline{L}_1$.

6. Moment Floer homology III: definition of the homology group

6.1. **Novikov ring.** Because of our topological restriction, the path space $\mathcal{P}$ is already simply-connected. Hence the actional functional and the Conley-Zehnder index can be globally defined for $(x, \eta) \in \mathrm{Crit}\mathcal{A}_\mu$, by choosing a base (x_0, η_0). For any $h \in \mathcal{H}$, define

$$E(h) = E(h(x_0, \eta_0), (x_0, \eta_0)),\ \lambda_{CZ}(h) = \dim\mathcal{M}(h(x_0, \eta_0), (x_0, \eta_0)).$$

Define

$$\Gamma = \frac{\mathcal{H}}{\ker\lambda_{CZ} \cap \ker E} \tag{6.1}$$

and the **Novikov ring** over $\mathbb{Z}_2$,

$$\Lambda_\gamma := \left\{ \sum_\gamma a_\gamma \gamma \mid \gamma \in \Gamma, a_\gamma \in \mathbb{Z}_2,\ \forall \kappa \in \mathbb{R}, \#\{\gamma : a_\gamma \neq 0, E(\gamma) \leq \kappa\} < \infty \right\}. \tag{6.2}$$

This ring is graded by the function λ_{CZ}.

6.2. **Moment Floer homology in the transverse case.** Assume that $\overline{L}_0$ and $\overline{L}_1$ intersect transversely. Then define

$$\mathscr{C}_F := \frac{\mathrm{crit}\mathcal{A}_\mu}{\ker\lambda_{CZ} \cap \ker E}. \tag{6.3}$$

Γ acts on $\mathscr{C}_F$ freely with quotient isomorphic to $\overline{L}_0 \cap \overline{L}_1$.

Define

$$CF_k(L_0, L_1, \mu) = \left\{ \xi = \sum_{c \in \mathscr{C}_F,\ \lambda_{CZ}(c) = k} \xi_c c \,\middle|\, \xi_c \in \mathbb{Z}_2,\ \#\{c : \xi_c \neq 0,\ E(c) \leq \kappa\} < \infty, \forall \kappa \in \mathbb{R} \right\}. \tag{6.4}$$

Then CF_* is a graded Λ_Γ-module: we see

$$\left(\sum_\gamma a_\gamma \gamma\right) * \left(\sum_c \xi_c c\right) = \sum_c \sum_{\gamma c' = c} (a_\gamma \xi_{c'}) c. \tag{6.5}$$

Then, define the boundary operator $\partial_F : CF_k(L_0, L_1, \mu) \to CF_{k-1}(L_0, L_1, \mu)$, by

$$\partial_F(c) = \sum_{c' \in \mathscr{C}_{k-1}} (-1)^{\#\mathcal{M}(c,c')} c'. \tag{6.6}$$

This satisfies the finiteness property basically by compactness theorem.

The momen Floer homology group is just defined as the cohomology:

$$HF_k(L_0, L_1, \mu) := \frac{\ker(\partial_F|_{CF_k})}{\operatorname{im}(\partial_F|_{CF_{k+1}})}. \tag{6.7}$$

6.3. **An example.** Consider $M = \mathbb{C}$, $L_0 = L_1 = \mathbb{R}$, $G = S^1 = U(1)$. Let the moment map be

$$\mu(z) = \frac{1}{2i}\left(|z|^2 - 1\right). \tag{6.8}$$

Then $\mu^{-1}(0)$ is the unit circle and the quotient is a single point. Hence everything is transverse. Now $\pi_0(\mathcal{H})$ is represented by $g_k = e^{k\pi i t}$, $t \in [0,1]$ and elements in $\mathscr{C}_F$ are represented by $(g_k(t), -k\pi i t)$. And it is easy to compute

$$\lambda_{CZ}(g_k) = k, E_{\mathcal{H}}(g_k) = \pi k. \tag{6.9}$$

The Novikov ring is just the ring of $\mathbb{Z}_2$-Laurent series.

Then consider (u, Ψ) solves the equation, i.e.,

$$\partial_s u + i(\partial_t u + \mathcal{X}_\Psi) = 0,\ \partial_s \Psi + \mu(u) = 0$$

which is equivalent to

$$\partial_s u_1 - \partial_t u_2 + u_1 \Psi = 0,\ \partial_s u_2 + \partial_t u_1 + u_2 \Psi = 0,\ 2\partial_s \Psi + |u|^2 = 1.$$

By the complex conjugation, solutions to the above equation with boundary condition can be extended to the punctured disk, and the if the winding around the outside circle is k_1, around the origin is k_2, then the solution belongs to the moduli

$$\widetilde{\mathcal{M}}(g_{k_1}, g_{k_2}).$$

By residue formular(roughly), $k_2 - k_1$ comes from the poles inside the disk. Then we see poles away from $\partial\Theta$ contribute in pairs. And we can prove that for each distribution of poles, there exists a unique solution. Hence we see the index 1 moduli is identified with $\partial\Theta$. By $\mathbb{R}$-translation, this means the boundary operator is zero.

7. Moment Floer homology IV: computation

7.1. **Morse-Bott homology.** Let $C = \cup C_i$ be the critical submanifold of a Morse-Bott function f. Then each C_i has a Morse index. Now usually we have to use another (co)homology theory on C, maybe singular, de Rham, or Morse. The moduli of trajectories between C_i will give a double complex.

For example(Morse homology on C), choose Morse-Smale pair (h, g) on C. Then the double complex is generated by critical points of h, and the degree for $p \in \mathrm{Crit} h|_{C_i}$ is given by

$$\deg p := \mathrm{index}_f(C_i) + \mathrm{index}(h|_{C_i}).$$

Then the boundary operator is defined by counting flow lines with cascades. To prove that this homology coincides with the usual homology, one need to pass to a spectral sequence.

7.2. **Moment Morse homology.** In this case we let $L_0 = L_1$ which is the Morse-Bott case. Frauenfelder defined the **moment Morse homology**, by counting Morse flow on $\overline{L}$ with cascades, by which we mean connecting orbits connecting different component of $\mathcal{H}(\mu^{-1}(0) \cap L)$. This is in analog with the Morse-Bott homology approach.

To prove that this is isomorphic to the moment Floer homology, we choose a generic Hamiltonian such that $\overline{L}_0$ and $\phi^1_{\overline{H}}(\overline{L}_1)$ intersect transversely, and use Piunikhin-Salamon-Schwarz's spiked discs, to produce a chain isomorphism between the two chain complexes.

While, on the other hand, because we have the antisymplectic involution, which means cascades always appear in pairs. Since we are using $\mathbb{Z}_2$-coefficient, this means cascades don't contribute to the boundary operator of the moment Morse homology. Then this homology reduces to the Morse homology on $\overline{L}$.

Theorem 7.1. *We have natural isomorphism*

$$HF_*(L_0, L_1, \mu; \Lambda_\Gamma) \simeq H_*\left(\overline{L}; \mathbb{Z}_2\right) \otimes_{\mathbb{Z}_2} \Lambda_\Gamma.$$

Corollary 7.2. *If L is the fixed point set of an antisymplectic involution and H_t is a generic G-invariant Hamiltonian flow, the*

$$\#\left(\overline{L} \cap \phi^1_{\overline{H}}(\overline{L}_0)\right) \geq \sum_i \beta_i(\overline{L}_0; \mathbb{Z}_2).$$