Introductory talk on Hitchin systems

February 21, 2011

During this weekend our department held a retreat at some mysteriour place. The topic of the retreat is "Integrable system", which is divided into several diverse talks. The talk I contributed to is on the integrable system on the moduli space of vector bundles over Riemann surfaces, or the so-called "Hitchin system". And my talk is mainly based on Hitchin's beautiful paper [Hit87]. In this notes I only pointed out the basic ideas of Hitchin's approach.

1 Moduli space of stable vector bundles over Riemann surfaces

Let's first recall some classical stuff: moduli space of holomorphic vector bundles over Riemann surfaces, and we take the gauge-theoretic approach.

So let $V \to M$ be a smooth vector bundle over a Riemann surface M of certain genus g. The space of holomorphic structures on V is isomorphic to the space of Cauchy-Riemann operators

$$\mathcal{A} = \left\{ d''_A : \Omega^0(M, V) \to \Omega^{0,1}(M, V) \right\}. \tag{1.1}$$

Definition 1.1 *A holomorphic vector bundle* (V, d''_A) *is* **stable** *if any proper holomorphic subbundle* $W \subset V$ *satisfies*

$$\frac{\deg W}{\mathrm{rank} W} < \frac{\deg V}{\mathrm{rank} V}. \tag{1.2}$$

Denote by $\mathcal{A}_s \subset \mathcal{A}$ be the subspace of stable holomorphic structures. The stability can be generalized to G-principal bundles for semi-simple complex Lie group G and works for $G = GL(m, \mathbb{C})$. However, we only consider here the latter case, i.e., the case of vector bundles. The dimension of the moduli space $\mathcal{N}$ is given by Riemann-Roch:

$$\dim_{\mathbb{C}} \mathcal{N} = m^2(g-1) + 1. \tag{1.3}$$

The gauge group $\mathcal{G}$ is the space of complex automorphisms of V. $\mathcal{G}$ acts on $\mathcal{A}$ by conjugation, i.e.

$$g^* d''_A := g^{-1} \circ d''_A \circ g. \tag{1.4}$$

On the other hand, $\mathcal{A}$ is an affine space modelled on the vector space

$$\Omega^{0,1}(M, \mathrm{End} V). \tag{1.5}$$

Hence at each $A \in \mathcal{A}$, its tangent space at A is isomorphic to the above space. Its dual, the cotangent space is hence isomorphic to

$$\Omega^0(M, \mathrm{End} V \otimes K_M) \tag{1.6}$$

where $K_M \to M$ is the canonical bundle. The pairing is given by

$$\begin{array}{ccccc} \Omega^{0,1}(M, \mathrm{End} V) & \times & \Omega^0(M, \mathrm{End} V \otimes K_M) & \to & \mathbb{C} \\ (\alpha & , & \beta) & \mapsto & \int_M \alpha \wedge \beta \end{array} \tag{1.7}$$

The gauge transformations hence induces an action on $T^*\mathcal{A}$. This is a cotangent bundle hence have a canonical symplectic form $\omega_{\mathcal{A}}$.

Proposition 1.2 *The gauge transformation is Hamiltonian. A moment map is given by*

$$\mu(A,\Phi) = d''_A\Phi \in \Omega^{0,1}(M, \mathrm{ad}P \otimes K_M) \simeq (\mathfrak{g}(P))^* . \tag{1.8}$$

Proof. The symplectic structure is given by

$$\begin{array}{cccccc} \omega_{\mathcal{A}}: & \Omega^{0,1}(M,\mathrm{ad}P) & \times & \Omega^0(M,\mathrm{ad}P\otimes K_M) & \to & \mathbb{C} \\ & (\Psi & , & \Phi) & \mapsto & \int_M \langle \Psi, \Phi\rangle. \end{array} \tag{1.9}$$

Where $\langle,\rangle$ is given by (e.g.) the Killing form. Since the gauge transformation has no effect on Φ, for each $\psi \in \mathfrak{g}$, we have

$$\iota_{X_\psi}\omega_{\mathcal{A}}(\Phi) = \int_M \langle d''_A\psi, \Phi\rangle. \tag{1.10}$$

On the other hand, the derivative of the function $\mu = d''_A\Phi$ on the Φ along the Φ-direction is equal $d''_A\Phi$. Pairing with ψ gives the same(or up to a sign) quantitiy. qed.

We will see that the gauge group action is indeed Hamiltonian. A moment map is very simple, i.e.

$$\mu(A,\Phi) = d''_A\Phi \in \Omega^{0,1}(M, \mathrm{End}V \otimes K) \simeq \left(\Omega^0(M,\mathrm{End}V)\right)^* \simeq (\mathfrak{g}(V))^* . \tag{1.11}$$

If we restrict to the subspace of stable vector bundles $\mathcal{A}_s \subset \mathcal{A}$, then the quotient $\mathcal{A}_s/\mathcal{G}$ is a complex manifold $\mathcal{N}$, as well as $T^*\mathcal{N}$. Actually, $T^*\mathcal{N}$ is the Marsden-Weinstein quotient of the Hamiltonian action on $T^*\mathcal{A}_s$.

Hence $T^*\mathcal{N}$ inherits a symplectic structure.

2 Integrable Hamiltonian system

Definition 2.1 *Let (M,ω) be a $2n$-dimensional holomorphic symplectic manifold. A completely integrable system is n holomorphic functions $f_1,\dots,f_n$ such that f_i Poisson commute and $df_1\wedge\cdots\wedge df_n$ is generically nonzero. It is called algebraically completely integrable, if a generic fibre of the map $p = (f_1,\dots,f_n)$ is an open subset of an n-torus and the vector fields X_{f_i} are linear.*

Recall that we have the natural symmetric functions $a_i, i = 1,\dots,m$ on $\mathfrak{g} = gl(m,\mathbb{C})$. By symmetric I mean *ad*-invariant. They are defined by

$$\det(x - A) = x^m + a_1(A)x^{m-1} + \ldots + a_m(A),\ A \in gl(m,\mathbb{C}). \tag{2.1}$$

In particular, $a_1(A) = -\mathrm{tr}(A)$ and $a_m(A) = (-1)^m \det A$. The function f_i will be defined on $T^*\mathcal{A}_s$ and then pass to the quotient $T^*\mathcal{N}$. In fact, at each $(d''_A,\Phi) \in T^*\mathcal{A}_s$, $\Phi \in \Omega^0(M,\mathrm{End}V\otimes K_M)$, and apply a_i on Φ, one obtains

$$a_i(\Phi) \in \Omega^0\left(M, K_M^i\right). \tag{2.2}$$

Hence we get a map

$$\widetilde{p} = \left(\widetilde{f_1},\dots,\widetilde{f_m}\right) : T^*\mathcal{A}_s \to \oplus_{i=1}^m \Omega^0(M, K_M^i). \tag{2.3}$$

Actually, $T^*\mathcal{A}_s = \mathcal{A}_s \times \Omega^0(M, \mathrm{End}V \otimes K_M)$ and $\widetilde{p}$ only depends on the fibre direction. This implies that all the components a_i actually Poisson commute in $T^*\mathcal{A}_s$. On the other hand, the gauge transformation only acts on the base direction hence $\widetilde{p}$ descends to the symplectic quotient $T^*\mathcal{N}$:

$$p = (f_1, \ldots, f_m) : T^*\mathcal{N} \to \oplus_{i=1}^m H^0(M, K_M^i). \tag{2.4}$$

So these functions actually Poisson commute in the quotient.

Also,

$$h^0(M, K_M) = g,\ h^0(M, K_M^i) = (2i-1)(g-1),\ i > 1. \tag{2.5}$$

So the right hand side of (2.4) is equal to $\dim\mathcal{N} = N$.

3 Generic fibre of the map p, the spectral curve

For a generic choice of $a_i \in H^0(M, K_M^i)$, we want to study the preimage of $(a_i)_{i=1}^m$ under p. Indeed, if we pull-back everying over M to the total space of K_M. We have a tautological section of π^*K_M, denoted by λ. Then a_i defines a section of $(\pi^*K_M)^n$ by

$$s = \lambda^m + \lambda^{m-1}a_1 + \cdots + a_m. \tag{3.1}$$

If we vary a_i, this gives a linear system inside $(\pi^*K_M)^n$. We can argu that this linear system is base-point-free, and hence by Bertini's theorem a generic divisor in this linear system is a smooth curve inside the surface K_M. Then for generic a_i, this curve is given by the vanishing locus of (3.1), which is denoted by S. We can compute the genus of S by adjunction formula:

Suppose $(d_A'', \Phi) \in p^{-1}\left((a_i)_{i=1}^m\right)$. Then we see that

$$\det(x - \Phi) = x^n + x^{n-1}a_1 + \cdots + a_n. \tag{3.2}$$

Hence $\det(\lambda - \Phi)|_S = 0$. Hitchin argus that λ is an eigenvalue of Φ and hence generically its associated eigenspace is a line bundle $L \subset \pi^*V$. To argu that this indeed gives us a line bundle over S(even that there exist points on S where the eigenvalue has higher dimensional eigenspace), one sees that locally the line gives an analytic function from S into $\mathbb{P}(V)$. Hence as (d_A'', Φ) varies inside the fibre $p^{-1}\left(\{a_i\}_{i=1}^m\right)$, the associated L varies in the Jacobian of S.

Conversely, if we fix the spectral curve S and give an arbitrary line bundle L, we can construct a vector bundle over M, basically by pushing forward L on to M. Also, Φ naturally arises. However, the pair (V, Φ) may not be necessarily stable. The stable pairs corresponds to an open subset of the Jacobian.

We want to say that the system is algebraically completely integrable. That means, the associated vector fields are linear inside the torus(the Jacobian). It is equivalent to say that the vector fields extends holomorphically to the Jacobian.

We can use Grothendieck-Riemann-Roch for the covering $S \to M$ to compute the degree of L. Indeed,

$$\deg L = -m(m-1)(g-1) - \deg V^*. \tag{3.3}$$

4 Other classical groups

We have similar integrable systems for $G = SO(2m;\mathbb{C}), SO(2m+1;\mathbb{C}), Sp(m;\mathbb{C})$.

References

[Hit87] N. Hitchin, *Stable bundles and integrable systems*, Duke mathematical journal **54** (1987), no. 1, 91–114.