Mathai-Quillen's Thom form and Atiyah-Hirzebruch's Riemann-Roch theorem

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1 Historic remarks

The classical Riemann-Roch theorem states that for a compact Riemann surface S and a holomorphic line bundle L over S, the difference of the dimensions of the space of holomorphic sections of L and the space of holomorphic differentials with coefficients in L is a topological invariant, i.e.

$$h^0(S, L) - h^1(S, L) = 1 - g(S) + \deg L. \tag{1}$$

In 1954, Hirzebruch generalized this classical results to higher dimension. If M is a compact complex manifold and $E \to M$ is a holomorphic vector bundle over M, then Hirzebruch shows that

$$\chi(M, E) = \langle \mathrm{Td}(TM) \cup \mathrm{ch}(E), [M] \rangle. \tag{2}$$

After that, Grothendieck extended this formula into relative case, which states a relation for a "proper morphism between varieties".

Atiyah and Hirzebruch then found a differentiable analogue of Grothendieck's formula. Their original approach was purely topological, using K-theory. I will give another approach which is in a differential-geometric flavor, and uses mainly de Rham theory. Before state it we need some preparation.

2 Chern-Weil theory

From now on everything is smooth. If $E \to M$ is a real or complex vector bundle, a connection on E is a linear differential operator $\nabla^E : \Gamma(E) \to \Omega^1(M, E)$ such that the Lebniz rule holds:

$$\nabla^E(fs) = df \otimes s + f\nabla^E s. \tag{3}$$

It extends to an operator $\nabla^E : \Omega^*(M, E) \to \Omega^{*+1}(M, E)$ such that

$$\nabla^E(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge \nabla^E\eta, \tag{4}$$

where $\omega \in \Omega(M)$ and $\eta \in \Omega(M, E)$. The **curvature form** of the connection ∇^E, is

$$R^E := \nabla^E \circ \nabla^E \in \Omega^2(M, E). \tag{5}$$

The Chern-Weil theory, roughly speaking, is to express characteristic classes of the bundle E in terms of the connection form. More precisely, if we have a polynomial f in k variables in $so(n)$ which is invariant under the adjoint action of $SO(n)$, then we define the characteristic form associated to f to be

$$f(R^E, \dots, R^E) \in \Omega^*(M). \tag{6}$$

Chern-Weil theory asserts that:

1. $f(R^E, \dots, R^E)$ is a closed form;
2. the cohomology class $[f(R^E, \dots, R^E)]$ in de Rham cohomology is independent of the choice of the connection.

Historically, the first is due to Chern and the second is due to Weil.

For example, we have

- For a complex vector bundle E, the **Chern character form** is

$$\mathrm{ch}(E, \nabla^E) = \mathrm{Tr}\left(\exp\left(-\frac{R^E}{2\pi i}\right)\right); \tag{7}$$

- for a complex vector bundle E, the **Todd form** is

$$\mathrm{Td}(E, \nabla^E) = \det\left(\frac{R^E/2\pi i}{e^{R^E/2\pi i} - 1}\right); \tag{8}$$

- for a real vector bundle E, the **$\hat{A}$-form** is

$$\hat{A}(E, \nabla^E) = \det{}^{1/2}\left(\frac{R^E/4\pi i}{\sinh(R^E/4\pi i)}\right). \tag{9}$$

Note that in the last example,

$$\det\left(\frac{X/2}{\sinh(X/2)}\right) \tag{10}$$

is an analytic function of entries of X for $X \in so(n)$, and the zero order term of its Taylor expansion is 1. So it has a unique analytic square root for X small such that the zero order term is also 1. Since entries of R^E is nilpotent, the $\hat{A}$-form we defined is a polynomial of entries of R^E.

2.1 Superconnections

It will be convenient if we go to the "super" category. A super vector space is just a bigraded vector space $V = V^+ \oplus V^-$. Equivalently, there is a endomorphism $\tau : V \to V$ such that $\tau^2 = 1$ and the decomposition is the decomposition with respect to τ. And End$V$ is also bigraded, where $(\mathrm{End}E)^\pm$ is the subspace of endomorphisms which commute (resp. anticommute) with τ. The supertrace of $A \in \mathrm{End}(E)$ is defined to be the trace of τA. Example: the Grassman algebra $\Lambda V = \Lambda^{\mathrm{even}} V \oplus \Lambda^{\mathrm{odd}} V$. If V, W are both bigraded, then $V \otimes W$ is also bigraded.

A super vector bundle is just a bigraded vector bundle $E = E^+ \oplus E^-$. Then the space of sections are also bigraded. Two important examples are $\Lambda(T^*M) \otimes E$ and $\Lambda(T^*M) \otimes \mathrm{End}E$ and the supertrace over a supervector space can be naturally generalized to

$$\mathrm{Str} : \Omega(M) \otimes \mathrm{End}(E) = \Omega(M, E) \to \Omega(M). \tag{11}$$

To introduce the notion of superconnections, we first look at connections, if E is ungraded, which means $E^- = 0$, then a connection on E is an operator mapping $\Omega(M, E)^\pm$ to $\Omega(M, E)^\mp$. To generalized, a superconnection of a super vector bundle E is a first order differential operator

$$\mathbb{A} : \Omega(M, E)^\pm \to \Omega(M, E)^\mp \tag{12}$$

which satisfies the Leibniz rule:

$$\mathbb{A}(\omega \wedge s) = d\omega \wedge s + (-1)^{\deg\omega} \mathbb{A}s. \tag{13}$$

The curvature form is defined to be

$$\mathbb{F}_{\mathbb{A}} = \mathbb{A}^2 \in \Omega^+(M, \mathrm{End}E).$$

A connection ∇^E can be locally written as $d+\omega$ where $\omega \in \Omega^1(M, \mathrm{End}E)$. Similarly, a superconnection $\mathbb{A}$ can be locally written as $d+\omega$ where $\omega \in \Omega(M, \mathrm{End}E)^-$, which might not be homogeneous of degree 1.

We have a generalization of the Chern character for superconnections. If E is a complex supervector bundle and $\mathbb{A}$ is superconnection, $\mathbb{F} = \mathbb{A}^2$, then the Chern character form is defined to be

$$ch(E, \mathbb{A}) = \mathrm{Str}\left(\exp\left(-\frac{\mathbb{F}}{2\pi i}\right)\right). \tag{14}$$

Also, the cohomology class of this form does not depend on the superconnection we choose. In particular, its class coincides with that defined by a usual connection. And we see that actually

$$ch(E) = ch(E^+) - ch(E^-) \in H^*_{dR}(M). \tag{15}$$

If you know K-theory, you know that Chern character is a homomorphism from $K(M)$ to $H^*(M)$.

It is easy to show that

$$ch(E_1 \otimes E_2, \nabla^{E_1} \otimes 1 + 1 \otimes \nabla^{E_2}) = ch(E_1, \nabla^{E_1}) \wedge ch(E_2, \nabla^{E_2}), \tag{16}$$

$$\hat{A}(E_1 \oplus E_2, \nabla^{E_1} \oplus \nabla^{E_2}) = \hat{A}(E_1, \nabla^{E_1}) \wedge \hat{A}(E_2, \nabla^{E_2}), \tag{17}$$

on the level of differential forms!

2.2 Statement of Atiyah-Hirzebruch's theorem

If $i : Y \hookrightarrow X$ is an embedding between two closed compact oriented differentiable manifolds, such that $\dim X - \dim Y = n = 2l$, and the normal bundle N_Y over Y is spin (i.e. $w_2(N_Y) = 0 \in H^2(Y, \mathbb{Z}_2)$). Then for any complex vector bundle μ over Y, there is a "direct image" $i_!\mu$ which is a complex super vector bundle over X, such that

$$\int_Y \hat{A}(TY) \wedge ch(\mu) = (-1)^{n/2} \int_X \hat{A}(TX) \wedge ch(i_!\mu). \tag{18}$$

Note that both sides are topological invariants.

3 Thom form

How to prove this? Since X is closed, the integral on the right side does not depend on the superconnection on $i_!\mu$. We will construct a family of superconnections $\mathbb{A}_T$ such that the Chern character form has exponential decay outside a tubular neighborhood of Y in X, and the integral can be localized into this neighborhood. Since N_Y is diffeomorphic to such a neighborhood, we first integrate along the fibre of the normal bundle. Namely,

$$\int_X \hat{A}(TX, \nabla^{TX}) \wedge ch(i_!\mu, \mathbb{A}_T) = \left(\int_{N_\varepsilon} + \int_{X-N_\varepsilon}\right) \hat{A}(TX, \nabla^{TX}) \wedge ch(i_!\mu, \mathbb{A}_T). \tag{19}$$

Because this value is independent of $T > 0$, we let $T \to \infty$, and we will see that the second term goes to zero while the first converges to a current $\Psi \wedge \delta_Y$. The spin condition of the relative bundle N_Y is essential in constructing the family of Chern characters.

Let's recall the notion of Thom isomorphism in de Rham cohomology.

If $E \to M$ is an oriented vector bundle, then we have the integration along the fibre

$$\int_{E/M} : \Omega_{cv}(E) \to \Omega(M), \tag{20}$$

which is an isomorphism in the level of cohomology. Its inverse is to pull back a class on M to E then wedge the Thom form.

A Thom form $\Phi \in \Omega_{cv}(E)$ is a closed differential form such that $\int_{E/M} \Phi = 1$.

In application, we will not restrict ourselves to forms with vertical compact support, but also consider those rapidly decreasing along the fibre, which are also integrable along the fibre.

3.1 Clifford algebra

To define the direct image $i_!\mu$ and construct the superconnection we want, I need to talk about Clifford algebra. If V is a Euclidean space of dimension $n = 2l$, the Clifford algebra $C(V)$ is the algebra generated by elements in V with the relations $vw + wv = -2 < v, w >$. If we choose an orthonormal basis e_i, then $C(V)$ is generated by e_i with the relations $e_ie_j + e_je_i = -2\delta_{ij}$. Clifford algebra is what Dirac needed to find the square root of the Laplacian $\Delta = -\sum \partial^2/\partial x_i^2$. So if $D = \sum a_i\partial/\partial x_i$ and $D^2 = \Delta$, we see that exactly $a_ia_j + a_ja_i = -2\delta_{ij}$.

In fact $C(V)$ is isomorphic to the Grassman algebra ΛV as vector spaces, which is given by

$$e_{i_1}e_{i_2}\cdots e_{i_k} \mapsto e_{i_1}\wedge e_{i_2}\wedge\ldots\wedge e_{i_k}. \tag{21}$$

but not as algebras: the product is "quantized". And $C(V) = C^+(V)\oplus C^-(V)$. The subspace of $C(V)$ corresponding to $\Lambda^2 V$, $C^2(V)$ is isomorphic to the Lie algebra $so(V)$. And on the level of Lie groups, it gives double cover $\mathrm{Spin}(V)\to SO(V)$, where $\mathrm{Spin}(V)\subset C(V)$ and the multiplication for the group is the Clifford multiplication. For dimension 3, this is the universal cover of $SO(V)$.

There is a representation of $\mathrm{Spin}(V)$ which does not come from a representation of $SO(V)$, $S = S^+\oplus S^-$, which is called the spinor representation. Also $C(V)$ acts on S and $c(v)S^\pm = S^\mp$. Indeed, $\mathrm{End}S = C(V)\otimes\mathbb{C}$ as super algebras. And since S is a super vector space, we can take the supertrace on it, it is actually

$$\mathrm{Str}(c(e_{i_1})\ldots c(e_{i_k})) = 0, k < n, \tag{22}$$

$$\mathrm{Str}(c(e_1)\ldots c(e_n)) = (-2i)^{n/2}. \tag{23}$$

3.2 Spinor bundle

If $E\to M$ is a real vector bundle which is spin, of rank $n = 2l$, then the principal bundle of oriented orthogonal frames $SO(E)$ has a nontrivial double cover $\mathrm{Spin}(E)$ which is a $\mathrm{Spin}(n)$ principal bundle. Then we have the associated spinor bundle $S(E)$, by the spinor representation. If ∇^E is a connection on E, which is equivalent to a connection 1-form $\omega\in\Omega^1(SO(E), so(n))$, since the Lie algebra of $SO(n)$ and $\mathrm{Spin}(n)$ are isomorphic, we have the pull back of ω onto $\mathrm{Spin}(E)$, and the associated connection $\nabla^{S(E)}$ on $S(E)$. If E has an Euclidean metric, then $S(E)$ has the associated Hermitian metric and if ∇^E is compatible with the metric, then $\nabla^{S(E)}$ is also compatible with the associated metric. And $\nabla^{S(E)}$ satisfies another important condition:

$$[\nabla^{S(E)}_X, c(s)] = c(\nabla^E_X s), \tag{24}$$

where X is any vector field and s is a section of E.

Also, if $R^E = \sum_{i<j} dx_i\wedge dx_j R^E_{ij}$, where $R^E_{ij}\in so(E_p)$, then

$$R^{S(E)} = \frac{1}{4}\sum_{i<j,k,l} dx_i\wedge dx_j (R^E_{ij}e_k, e_l)c(e_k)c(e_l)\in \mathrm{End}(S(E_p)). \tag{25}$$

Denote $\pi : E \to M$, we will consider the pull back bundle $\pi^*(S(E))$ over the total space E. We know that $C(V)$ acts on the spinors, so at each point $x \in E$, $c(x)$ is a linear transformation on $\pi^*(S(E))_x$. We consider the following family of superconnections

$$\mathbb{A}_T = \pi^*\nabla^{S(E)} + T\sqrt{-1}c(x) \tag{26}$$

and compute its Chern character forms.

First

$$\mathbb{F}_T = \pi^* R^{S(E)} + T\sqrt{-1}(dx^i + x^j\omega_{ji})c(e^i) + T^2\|x\|^2. \tag{27}$$

Its Chern character can be computed using Mathai-Quillen's methods.

3.3 Algebraic computations

Suppose $\Omega \in so(n)$ and $\gamma_1, \ldots, \gamma_n$ is the degree 1 generators of the Grassman algebra $\Lambda(\mathbb{R}^n)$ which is an oriented basis of $\mathbb{R}^n$, and write $\gamma = (\gamma_1, \ldots, \gamma_n)^t$. Then the coefficient of $\gamma_1 \wedge \ldots \wedge \gamma_n$ in

$$\exp\left(\frac{1}{2}\gamma^t\Omega\gamma\right)$$

is denoted by

$$T\left(\exp(\frac{1}{2}\gamma^t\Omega\gamma)\right) = \mathrm{Pf}(\Omega). \tag{28}$$

Note that the definition of Pfaffian depends on the orientation of $\mathbb{R}^n$.

Indeed, we have

$$\exp(\frac{1}{2}\gamma^t\Omega\gamma) = \sum_{I\subset\{1,\ldots,n\}} \mathrm{Pf}(\Omega_I)\gamma^I. \tag{29}$$

And since both sides are analytic in entries of Ω, it extends to Ω with entries in any commutative algebra.

We also want to compute the Berezin integral of

$$\exp\left(\frac{1}{2}\gamma^t\Omega\gamma + J^t\gamma\right)$$

, where J consists of degree 1 elements of a supercommutative algebra.

In application, the super algebra will be the algebra of differential forms.

Actually,

$$T(\exp(\frac{1}{2}\gamma^t\Omega\gamma + J^t\gamma)) = \sum_{I\subset\{1,\ldots,n\}} \mathrm{Pf}(\Omega_I)\varepsilon_{I,I'}(-1)^{\frac{|I'|}{2}}J^{I'}. \tag{30}$$

Now we want to compute an analogue of the above results. If we regard γ_i not in the Grassman algebra but in the Clifford algebra and the multiplication is the Clifford multiplication, then the coefficient of $\gamma_1 \dots \gamma_n$ is also well-defined.

$$T(\exp(\frac{1}{2}\gamma^t\Omega\gamma)) = \det{}^{1/2}\left(\frac{\sinh\Omega}{\Omega}\right)\mathrm{Pf}(\Omega). \tag{31}$$

And

$$T(\exp(\frac{1}{2}\gamma^t\Omega\gamma + J^t\gamma)) = \det{}^{1/2}\left(\frac{\sinh\Omega}{\Omega}\right)\sum_{I\subset\{1,\dots,n\}}\mathrm{Pf}(\Omega_I)\varepsilon_{I,I'}(-1)^{\frac{|I'|}{2}}J^{I'}. \tag{32}$$

3.4 Go back to the Chern character

If we set $\Omega_{kl} = \frac{1}{2}(R^E e_k, e_l)$, where e_k is a local oriented orthonormal basis of E, and $J_k = T\sqrt{-1}(dx_k + x_l\omega_{kl})$ where ω is the connection matrix with respect to the basis e_k, then by the above computation

(33)

$$\mathrm{Str}(\exp(-\mathbb{F})) = e^{-T^2\|x\|^2}\pi^*\hat{A}^{-1}(E,\nabla^E)\wedge\sum_I \mathrm{Pf}(R^E_I)\varepsilon_{I,I'}(-1)^{\frac{|I'|}{2}}(dx_k + x_l\omega_{lk})^{I'}(-2\sqrt{-1})^{n/2}.$$

We claim that besides the $\pi^*\hat{A}^{-1}(E,\nabla^E)$, the other terms is $(-1)^{n/2}$ of a Thom form.

So in summary, by computing the Chern character of $(\pi^*S(E), \mathbb{A}_T)$, we find a family of Thom forms.

4 Direct Image

We go back to the setting of Atiyah-Hirzebruch's theorem. $\mu \to Y$ is a Hermitian vector bundle with a connection ∇^μ compatible with the metric. We choose a small tubular neighborhood of Y, which is N_ε diffeomorphic to the set of normal vectors of length less than ε. So we have $\pi : N_\varepsilon \to Y$. Also we have the spinor bundle $S(N)$ over Y. Then set $E^\pm = \pi^*(S^\pm(N)\otimes\mu)$ over N_ε. We can find a complex vector bundle F over Y such that $S^-(N)\otimes\mu\oplus F$ is trivial. So set $\xi^\pm = \pi^*(S^\pm(N)\otimes\mu F)$ and for $v \in N_\varepsilon - Y$, $\sqrt{-1}c(v)\otimes\pi^*\mathrm{Id}_\mu\oplus\pi^*\mathrm{Id}_F$ is invertible from ξ_+ to ξ_-. Hence both ξ_+ and ξ_- are trivial on the boundary of N_ε. So we can extend them trivially to the whole X. And extend the morphism on N_ε trivially to the whole X as well, which is denoted by $V : \xi_\pm \to \xi_\mp$.

Since V is positive and bounded below outside N_ε, we see that the Chern character form of the superconnection $\nabla^\mu + TV$ has exponential decay outside N_ε. While inside N_ε, F doesn't contribute to the Chern character and we see by the above result,

$$(34)\quad ch(\xi, \nabla^\mu + TV) = \left(\frac{1}{2\pi i}\right)^{n/2} e^{-T^2\|x\|^2}(Ti)^n \pi^* \hat{A}^{-1}(N, \nabla^N) \wedge (-2i)^{n/2}(-1)^{n/2} dx_1 \ldots dx_n + \ldots$$

But along each fibre,

$$(35)\qquad \lim_{T\to\infty} \int_{B_\varepsilon} e^{-T^2\|x\|^2} T^n dx_1 \ldots dx_n = \lim_{T\to\infty} \int_{B_{T\varepsilon}} e^{-\|x\|^2} dx_1 \ldots dx_n = \pi^{n/2}.$$

5 Relation with Atiyah-Singer index theorem

There are several different proofs of Atiyah-Singer index theorem. The first one is the cobordism proof, which is to modify Hirzebruch's proof of Riemann-Roch. But the founders of index theorem thought that this proof is not natural. Then motivated by Grothendieck-Riemann-Roch, Atiyah and Singer gave the second proof, which is called the K-theory proof, in which Bott periodicity played a significant role. There are also the heat equation proof and physical proof of this theorem.

Actually, the characteristic number

$$\langle \hat{A}(TX), [X] \rangle$$

(the special case where μ is the trivial line bundle in above situation) is called the $\hat{A}$-genus of X. It used to be mysterious why this number is an integer when TX is spin. This was one of the starting point of the index theorem and this integer is equal to the index of the Dirac operator on X.

References.

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