Morse homology of Lagrange multipliers and adiabatic limits

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0. Outline

1. Preliminaries on Morse-Smale-Witten complex

2. Apply the construction to Lagrange multipliers

3. The adiabatic limit $\lambda \to \infty$

4. $\lambda \to 0$

5. Wall-crossing

1. Morse-Smale-Witten chain complex

- Let M be a smooth manifold, $f : M \to \mathbb{R}$ a Morse function, g a complete Riemannian metric.

- For $p \in \mathrm{Crit} f$, $\mathrm{index}(p, f) \in \mathbb{Z}_{\geq 0}$.

- Negative gradient flow equation

$$\dot{x}(t) + \nabla f(x(t)) = 0.$$

- The Morse-Smale condition:

$$\forall p, q \in \mathrm{Crit} f,\ W^u(p) \pitchfork W^s(q)$$

  holds for generic pair (f, g).

We work with $\mathbb{Z}_2$-coefficients. The Morse-Smale-Witten complex $CM(f,g)$ is defined as:

- Generators and grading:

$$CM_i := \bigoplus_{\text{index}(p)=i} \mathbb{Z}_2 < p > .$$

- Differential is defined by counting trajectories:

$$\langle \partial p, q \rangle = \#\left[(W^u(p) \cap W^s(q))/\mathbb{R}\right] \text{ mod } 2.$$

- Palais-Smale condition on $f \Longrightarrow \#W^u(p) \cap W^s(q) < \infty$, and $\partial^2 = 0$.

- The Morse homology

$$HM(f,g) := H\left(CM_*(f,g), \partial\right).$$

- If M is compact, (f', g') is another pair, then there is a quasi-isomorphism

$$CM_*(f,g) \sim CM_*(f', g').$$

And $HM_*(f,g) \simeq H_*(M; \mathbb{Z}_2)$.

2. Lagrange multipliers

Let M be a compact manifold, $f, \mu : M \to \mathbb{R}$ be two Morse functions. Assume

- 0 is a regular value of μ.
- $f|_{\mu^{-1}(0)}$ is Morse.
- $\mathrm{Crit} f \cap \mathrm{Crit} \mu = \emptyset$.

We define the Lagrange multiplier

$$\begin{array}{rccc} \mathcal{F}: & M \times \mathbb{R} & \to & \mathbb{R} \\ & (x, \eta) & \mapsto & f(x) + \eta\mu(x). \end{array}$$

The critical point set of $\mathcal{F}$

$$\mathrm{Crit}\mathcal{F} = \{(x, \eta) \mid \mu(x) = 0,\ df(x) + \eta d\mu(x) = 0\}.$$

- There is a one-to-one correspondence

$$\begin{array}{rcl} \mathsf{Crit}\mathcal{F} & \simeq & \mathsf{Crit} f|_{\mu^{-1}(0)} \\ p := (x_p, \eta_p) & \mapsto & x_p \end{array}$$

- Indices shifted by one:

$$\mathsf{index}(p, \mathcal{F}) = \mathsf{index}\left(x_p, f|_{\mu^{-1}(0)}\right) + 1.$$

- Now choose a metric g on M, and e the Euclidean metric on $\mathbb{R}$. Denote

$$g_\lambda := g \oplus \lambda^{-2} e.$$

- The negative gradient flow equation

$$\begin{cases} \dot{x}(t) + \nabla f(x) + \eta \nabla \mu(x) &= 0, \\ \dot{\eta}(t) + \lambda^2 \mu(x) &= 0. \end{cases}$$

We call the solutions with finite energy "λ-trajectories". And the solution space $\mathcal{M}^\lambda(p, q)$.

- The chain complex obtained from the pair $(\mathcal{F}, g_\lambda)$ on $M \times \mathbb{R}$ is denoted by

$$C^\lambda(f, \mu, g) = \left(C^\lambda, \partial^\lambda\right).$$

- The homology H^λ of the chain complex is defined for and independent of generic $\lambda \in \mathbb{R}_{>0}$.

- When varying λ, the generators of C^λ don't change, while the trajectories change.

- We push λ to ∞ and 0, to see if there are "limit chain complexes". Then we need to know the behavior of λ-trajectories connecting p to q when λ is very large and very small.

3. $\lambda \to \infty$.

- Let's fix $p, q \in \mathrm{Crit}\mathcal{F}$. Then for any λ-trajectory $\gamma \in \mathcal{M}^{\lambda}(p,q)$, the energy of γ

$$E(\gamma) := \int_{-\infty}^{+\infty} \left(\|\dot{x}(t)\|^2 + \lambda^2 \, |\mu(x)|^2 \right) dt = \mathcal{F}(p) - \mathcal{F}(q) = f(x_p) - f(x_q)$$

doesn't depend on λ. So if $\lambda \to \infty$,

$$\|\mu(x)\|_{L^2} \to 0.$$

- **Compactness** Indeed, for any sequence $\lambda_i \to \infty$, $\gamma_i \in \mathcal{M}^{\lambda_i}(p,q)$, there is a subsequence converging to a broken trajectory of the negative gradient flow of $(f,g)|_{\mu^{-1}(0)}$ connecting x_p and x_q.

- **Gluing** Conversely, for any $\gamma^\infty \in \mathcal{M}^\infty(x_p, x_q)$, using implicit function theorem, one shows that there is a homeomorphism

$$\mathcal{M}^\infty(x_p, x_q) \simeq \mathcal{M}^\lambda(p, q), \ \forall \lambda >> 0.$$

- Hence we actually proved

$$H_*^\lambda \simeq HM_{*-1}\left((f,g)|_{\mu^{-1}(0)}\right) \simeq H_{*-1}(\mu^{-1}(0); \mathbb{Z}_2).$$

- Compare with Gaio-Salamon on Hamiltonian Gromov-Witten invariants of $M \sim$ Gromov-Witten of $M//G$, or with Katrin's lectures on Donaldson invariants $\sim$ quilts. We are in the classical(not quantum) situation so the story is much simplified.

4. The other direction $\lambda \to 0$

- Look at the second equation:

$$\begin{cases} \dot{x}(t) + \nabla f(x) + \eta \nabla \mu(x) &= 0, \\ \dot{\eta}(t) + \lambda^2 \mu(x) &= 0. \end{cases}$$

  As $\lambda \to 0$, η will converge to constant on compact intervals.

- A **fast trajectory** is a trajectory of the flow of $-\nabla f - \eta \nabla \mu$ in M for some $\eta \in \mathbb{R}$. Its image can be viewed as a subset in $M \times \{\eta\} \subset M \times \mathbb{R}$.

- But near points where $\nabla f + \eta \nabla \mu = 0$ and for small λ, both x and η changes slowly along the flow.

- To describe "limit" trajectories, we introduce the **slow manifold**:

$$C_{\mathcal{F}} := \{(x, \eta) \mid \nabla f(x) + \eta \nabla \mu(x) = 0\} \subset M \times \mathbb{R}.$$

- In generic situation, $C_{\mathcal{F}}$ is a 1-dimensional submanifold of $M \times \mathbb{R}$, and

$$\mathrm{Crit}\mathcal{F} = C_{\mathcal{F}} \cap (\mu^{-1}(0) \times \mathbb{R}).$$

- A **slow trajectory** is an oriented arc $\gamma \subset C_{\mathcal{F}}$ such that $\mathcal{F}|_\gamma$ is decreasing.

- A **fast-slow trajectory** connecting p to q is a sequence

$$\left(\gamma_f^1, \gamma_s^1, \gamma_f^2, \gamma_s^2, \dots, \gamma_f^k, \gamma_s^k\right)$$

where γ_f^i are fast trajectories and γ_s^i are slow trajectories, such that the end of γ_f^i is the beginning of γ_s^{i+1}, etc., and the beginning of γ_f^1 is p and the end of γ_s^k is q.

- Note that a fast-slow trajectory doesn't necessarily start/end with a fast/slow trajectory, but we allow trivial trajectories appearing in the sequence.

Compactness For any sequence $\lambda_i \to 0$ and $\gamma_i \in \mathcal{M}^{\lambda_i}(p,q)$, there exists a subsequence "converging"(e.g., in Hausdorff topology) to a fast-slow trajectory γ_0.

Gluing Suppose $\text{index}(p) - \text{index}(q) = 1$. Then for any fast-slow trajectory γ_0 and small enough λ, there exists a unique $\gamma_\lambda \in \mathcal{M}^\lambda(p,q)$ which is "close" to γ_0.

The proof of the gluing theorem is not using the usual implicit function theorem approach. Instead, we are using the "generalized exchange lemma" of Schecter(2008) in the context of geometric singular perturbation theory with minor modifications.

- Let $\mathcal{M}^0(p,q)$ be the moduli space of fast-slow trajectories connecting p and q. Under various transversality assumptions

$$\text{index}(p) - \text{index}(q) = 1 \Longrightarrow \#\mathcal{M}^0(p,q) < \infty.$$

- This allows us to define a chain complex (C^0, ∂^0) by couting fast-slow trajectories. $(\partial^0)^2 = 0$ follows from the 1-1 correspondence between $\mathcal{M}^0$ and $\mathcal{M}^{\lambda<<1}$ and $(\partial^\lambda)^2 = 0$.

- Hence the complex (C^0, ∂^0) gives an equivalent definition of the Morse homology of $\mu^{-1}(0)$, by counting those singular objects.

We give two examples of the objects we are counting. Critical points of $\mathcal{F}$ are also critical points of $\mathcal{F}|_{C_\mathcal{F}}$. We assume that p is a local maximum and q a local minimum of $\mathcal{F}|_{C_\mathcal{F}}$ and assuming index(p) − index$(q) = 1$.

- It is possible that p and q lie in the same connected component of $C_\mathcal{F}$, with a slow trajectory γ_s connecting them. Then γ_s itself forms a fast-slow trajectory.

- A fast-slow trajectory connecting p and q may contain several "handle-slides", that is, fast trajectories with relative index zero; and "cusp trajectories", that is, fast trajectories with beginning or ending a degenerate critical point.

5. Wall-crossing

- The classical Morse theory studies the change of the level set $\mu^{-1}(c)$ when c crosses a critical value of μ.

- We replace μ by $\mu - c$, and look at the change of the chain complex C^0 as c varies.

- Note that $C_{\mathcal{F}} = \{(x, \eta) \mid \nabla f(x) + \eta \nabla \mu(x) = 0\}$ and the fast flow doesn't change, while the generators and the slow flows will change.

Two examples of wall-crossing:

- "Birth-death": The chain complexes differ by two redundant generators.

- Crossing a critical value of μ: The chain complexes differ by two generators of degree k and $n-1-k$, corresponding to the attaching of a k-handle(or $(n-k)$-handle).

Thank you!