Algebra Lecture 1:

- Class setup:

  Homework: 30%
  In-discussion problem session: 10%
  Midterm: 20%
  Final: 40%

- Sources of the idea of a group:

  - Symmetry is the main source.

    For example look at the solutions of

    $$x^2+y^2+z^2 = 1 \text{ in } \mathbb{C}^3 \text{ (or } \mathbb{R}^3\text{)}$$

Remark: In lecture the equation was $x^3+y^3+z^3=0$ but that had more symmetries, so I changed it.

Do you notice any symmetry?

We can swap x, y, z around. How many ways can we do that?

$$(x,y,z) \xmapsto{\tau} (x,z,y) \qquad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(x,y,z) \longmapsto (z,y,x) \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \tau \circ \sigma \circ \sigma$$

$$(x,y,z) \longmapsto (y,x,z) \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tau \circ \sigma$$

$$(x,y,z) \xmapsto{\sigma} (y,z,x) \qquad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(x,y,z) \longmapsto (z,x,y) \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \sigma \circ \sigma$$

$$(x,y,z) \longmapsto (x,y,z) \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These are linear maps:

$\sigma : \mathbb{R}^3 \to \mathbb{R}^3$

$\sigma(x,y,z) = (y,z,x)$

or in matrix form: $\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\tau : \mathbb{R}^3 \to \mathbb{R}^3$

$\tau(x,y,z) = (x,z,y)$

$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

...

we can write the others as matrices too (page before)
But we can also write them as compositions of σ and τ.

- Note that, we can compose these maps.
(indeed, if something has two symmetries, applying both symmetries is also a symmetry).

- Also note that we can invert these maps.
(and the inverse of a symmetry is a symmetry)

general
<u>Conclusions</u>:
- Identity is always a symmetry
- Symmetries can be composed.
- Symmetries can be inverted.

Specific to symmetries of solution set of $x^2+y^2+z^2=0$.

Specific to our example (without proof)

There are six symmetries:

$$\{ \mathbb{1}, \sigma, \tau, \sigma^2, \tau\sigma, \tau\sigma^2 \}$$

· these can be composed with each other. and the composition is always among the six elements.

(by multiplying the corresponding matrices)

· They each have an inverse: (eg. $\sigma^{-1} = \sigma^2$)

Such a structure: i.e.

* A set
* with ~~composition~~ associative operation
* with identity element
* and inverses

is called a <u>group</u>. (official definition later)