Lecture 3:

Last time: binary operations

$$S \times S \xrightarrow{*} S$$

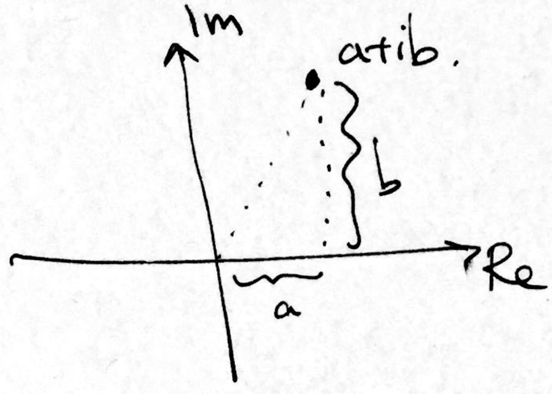
examples, commutativity, associativity.

Today: • Recap on complex numbers

Complex numbers:

Q: What are the solutions to $z^6 = 1$ for $z \in \mathbb{C}$.

$$\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}.$$



$$i = \sqrt{-1}$$

$$i^2 = -1.$$

$$(a+ib)(c+id)$$
$$= ac + iad + ibc - bd$$
$$= ac - bd + i(bc + ad).$$

- $z = a+ib$

  $|z| = \sqrt{a^2+b^2}$ ← length of z or "norm" of z.

- $z = a+ib$

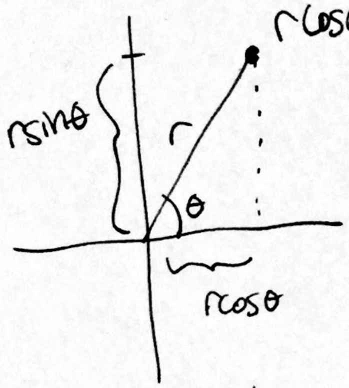
  $\bar{z} := a-ib$ so $z\bar{z} = |z|^2$.

  $z \cdot \bar{z} = a^2+b^2$

- $e^{i\theta} = \cos\theta + i\sin\theta$

  $r\cos\theta + ir\sin\theta = r e^{i\theta}$ ← θ is called the argument of z.

  (r ← $|z|$)



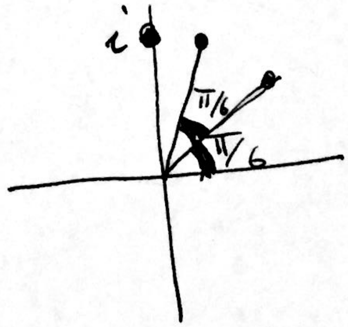
So every complex number can be written

as $z = r e^{i\theta}$, $r \geq 0$.

- Let $z_1 = r_1 e^{i\theta_1}$ then
  $z_2 = r_2 e^{i\theta_2}$

  $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

  the norms multiply   the arguments get added.



$$z_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$z_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$z_1 \cdot z_2 = i$$

because $z_1 = e^{i\pi/6}$

$$z_2 = e^{i\pi/3}$$

$$z_1 \cdot z_2 = e^{i\pi/6} \cdot e^{i\frac{\pi}{3}} \overset{+}{=} e^{i\pi/2} = i$$

Powers of z:

Let $z = re^{i\theta}$

$$z^2 = r^2 e^{i2\theta}$$

$$z^k = r^k e^{ik\theta}.$$

Solving $z^6 = 1$. we see that

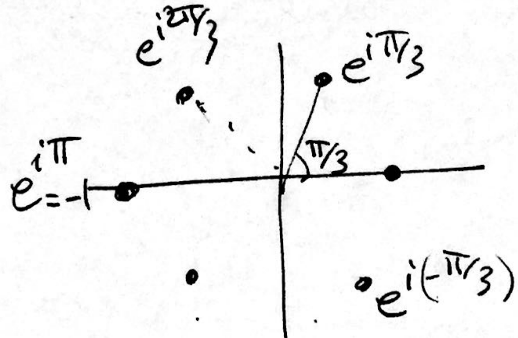
$r^6 e^{i6\theta} = 1$ $\qquad r = 1$ ← because $|r^6| \cancel{|e^{i6\theta}|}^{1} = |1|$

and $e^{i6\theta} = 1$.

but $1 = e^{i2\pi s}$ for $s = 0, 1, 2, \dots, -1, -2, \dots$

so $e^{i6\theta} = e^{i2\pi s}$.

So $\theta = s \cdot \frac{\pi}{3}$



these solutions

$$1, e^{i\pi/n}, e^{i2\pi/n}, \ldots, e^{i\frac{(n-1)}{n}\pi}$$

are called "nth roots of unity".

"Fundamental theorem of algebra".

<u>Theorem</u>: Every polynomial of degree d in one variable, with complex coefficients has d solutions (counting with multiplicity eg. $(x-2)^2$ has double solution)

(equivalently, the polynomial factors into linear factors)

<u>Theorem</u>: (but much less fancy) (conjugate root theorem)

If $p(x) \in \mathbb{R}[x]$ is a real polynomial, then its complex roots are in pairs $z, \bar{z}$ (ie if z is a root, so is $\bar{z}$)

<u>proof</u>: $p(z) = 0 \Rightarrow a_0 + a_1 z + \ldots + a_d z^d = 0$

$$\Rightarrow \overline{p(z)} = \overline{a_0} + \overline{a_1}\bar{z} + \ldots + \overline{a_d}\bar{z}^d = 0$$

$$= a_0 + a_1\bar{z} + \ldots + a_d\bar{z}^d = 0$$

$$= p(\bar{z}).$$