Lecture 25:

Previously on Math 120A:

- homomorphisms: $\varphi: G_1 \to G_2$ st. $\forall g,h \in G_1, \varphi(gh) = \varphi(g)\varphi(h)$.
- lots of examples
  - $\varphi_a: \mathbb{Z} \to \mathbb{Z}$
    $\varphi_a(k) = ak$
  - $\det: GL_n(\mathbb{R}) \to \mathbb{R}^\times$
  - trivial homomorphism
    $\varphi: G \to H$
    $\varphi(g) = e \quad \forall g$
- isomorphism = bijective homomorphism
- Every homomorphism is ~~biject~~ surjective onto its image.
- $\operatorname{Ker}\varphi = \varphi^{-1}(\{e\}) = \{g \in G \mid \varphi(g) = e\}$.

ie $\varphi: G \to H$ may not be surjective but $\varphi: G \to \varphi(G)$ is always surjective

$\varphi: G \to H$

$\operatorname{Ker}\varphi$ is a subgroup. (proved in previous notes too)

Lemma: A homomorphism $\varphi: G \to H$ is injective if and only if $\ker\varphi = \{e\}$.

proof: Assume φ is injective, then if $\varphi(a) = e = \varphi(e)$, then $a = e$. So $\ker\varphi = \{e\}$.
Assume $\ker\varphi = e$, then if $\varphi(a) = \varphi(b)$, then $\varphi(a)\varphi(b)^{-1} = e$ so $\varphi(ab^{-1}) = e$. Since $\operatorname{Ker}\varphi = \{e\}$, $ab^{-1} = e$ so $a = b$. Hence φ is injective. $\square$

Recall also our talk about cosets.

$H \leq G$ subgroup

$aH \leftarrow$ left Hcoset containing a.

the cosets partition G.

Recall also the problem from the homework:

Lemma: $aH = Ha$ iff $aHa^{-1} = H$.

(left coset and right coset are same iff $aHa^{-1} = H$)

Definition: A subgroup H is called normal if all its left cosets and right cosets are the same, ie $\forall a \in G$, $aH = Ha$. (equivalently $aHa^{-1} = H$ forall $a \in G$).

Definition I will use: A subgroup H is normal (in G) iff $\forall a \in G$, $aHa^{-1} \subset H$.

equivalent definition.

Example: • G abelian $\Rightarrow$ every subgroup is normal.

• $G = D_6$, $H = \{1, s\}$ is not normal

$$rHr^{-1} = \{rr^{-1}, rsr^{-1}\}$$
$$= \{1, rsr^2\} = \{1, sr\}.$$

• $G = D_6$, $H = \{1, r, r^2\}$.

$$rHr^{-1} = H$$
$$sHs^{-1} = sHs = \{\underbrace{ss}_{=1}, \underbrace{srs}_{=r^2}, \underbrace{sr^2s}_{=r}\} = H$$

similarly for others. So $\langle r \rangle \subset D_6$ is a normal subgroup.

<u>Lemma</u>: For any homomorphism $\varphi: G \to H$,
$\operatorname{Ker}\varphi \leq G$ is a normal subgroup of G.

<u>proof</u>: Let $a \in G$, $k \in \operatorname{Ker}\varphi$, ie $\varphi(k) = e$.

then $\varphi(aka^{-1}) = \varphi(a)\underbrace{\varphi(k)}_{1}\varphi(a^{-1}) = \varphi(a)\varphi(a)^{-1} = e$.

so $aka^{-1} \in \operatorname{Ker}\varphi$.

So $\operatorname{Ker}\varphi$ is normal. "Kernels are normal".

<u>Remark</u>: Definition of normal is weird but it will make more sense soon.

<u>Quotient groups</u> ("factor groups" in the book).

Previously, we saw that $\mathbb{Z}/n\mathbb{Z}$ is actually a set of cosets.

$G = \mathbb{Z}$

$H = n\mathbb{Z}$

elements of $\mathbb{Z}/n\mathbb{Z}$ are $\bar{k} = \underset{\text{coset}}{k + n\mathbb{Z}}$.

But what about the addition operation on $\mathbb{Z}/n\mathbb{Z}$. $\bar{k}_1 + \bar{k}_2 = \overline{k_1 + k_2}$.

We had: $(k_1 + n\mathbb{Z}) + (k_2 + n\mathbb{Z}) = k_1 + k_2 + n\mathbb{Z}$.

But is this well-defined? ie does the answer change if we replace k_1 by something else in the same coset?

Let $k_1 + n\mathbb{Z} = k_1' + n\mathbb{Z}$.

Then $+k_1' - k_1 + n\mathbb{Z} = 0 + n\mathbb{Z}$ → so $n\mathbb{Z} = k_1' - k_1 + n\mathbb{Z}$.

so $-k_1' + k_1 \in n\mathbb{Z}$.

Look at $k_1 + n\mathbb{Z} + k_2 + n\mathbb{Z} = k_1 + k_2 + n\mathbb{Z}$

$= \cancel{k_1} + k_2 + k_1' - \cancel{k_1} + n\mathbb{Z}$.

$= k_1' + k_2 + n\mathbb{Z}$.

Conclusion: the answer of $(k_1 + n\mathbb{Z}) + (k_2 + n\mathbb{Z})$ doesn't depend on the representative we chose for $k_1 + n\mathbb{Z}$.

Similarly for k_2.

We want to do the same for all groups and cosets:

$G \geq H$ ← subgroup. $G/H = \{1.H, a_1H, a_2H, \ldots\}$ ↙ set of cosets.

We want to define coset multiplication:

$$(aH).(bH) = abH$$

But this is not always well-defined.

<u>Theorem</u>: Let $H \leq G$ be a subgroup. The left coset multiplication $aH \cdot bH = abH$ is well-defined iff $H \trianglelefteq G$ is a normal subgroup!