Lecture 26:

Last time,

<u>Def</u>: A subgroup $H \leq G$ is <u>normal</u> if $\forall g \in G$, $gHg^{-1} \subset H$.

(equivalent definitions:

- $\forall g \in G$, $gHg^{-1} = H$
- left and right cosets coincide. ie $\forall g \in G$, $gH = Hg$.)

<u>Ex</u>: • If G is abelian, then every subgroup is normal.

- $\{1, r, r^2\} \leq D_6$ is normal
- $\{1, s\} \leq D_6$ is not normal
  $rsr^{-1} = sr \notin \{1, s\}$.
- For any $\varphi: G \to K$, group homomorphism, $\ker\varphi$ is normal in G.

<u>Quotient groups</u>: ("factor groups" in book)

We want to make $G/H = \{gH \mid g \in G\} =$ "set of H-cosets in G" into a group.

<u>Case study</u>:

$G = \mathbb{Z}$ $\quad H = n\mathbb{Z}$.

cosets are: $\mathbb{Z}/n\mathbb{Z} = \{\underset{\substack{\| \\ n+n\mathbb{Z}}}{0+n\mathbb{Z}}, 1+n\mathbb{Z}, 2+n\mathbb{Z}, \dots, n-1+n\mathbb{Z}\}$.

We have the addition <u>operation</u> in $\mathbb{Z}/n\mathbb{Z}$,

$$\bar{k}_1 + \bar{k}_2 = \overline{k_1 + k_2}$$

ie $(k_1 + n\mathbb{Z}) + (k_2 + n\mathbb{Z}) = k_1 + k_2 + n\mathbb{Z}$.

but how do we know that the result of the operation doesn't depend on choice of k_1 for the coset $k_1 + n\mathbb{Z}$. i.e. if $k_1 + n\mathbb{Z} = k_1' + n\mathbb{Z}$, then do we have $k_1 + k_2 + n\mathbb{Z} \overset{?}{=} k_1' + k_2 + n\mathbb{Z}$

<u>Yes</u>: If $k_1 + n\mathbb{Z} = k_1' + n\mathbb{Z}$, then $k_1' - k_1 \in n\mathbb{Z}$

so $k_1' - k_1 + n\mathbb{Z} = n\mathbb{Z}$.

so $\cancel{k_1} + k_2 + \underbrace{k_1' - \cancel{k_1} + n\mathbb{Z}}_{n\mathbb{Z}} = k_1' + k_2 + n\mathbb{Z}$

so the + operation is well-defined.

Def: Coset multiplication in G/H:

$$(g_1H)\cdot(g_2H) = g_1g_2H.$$

But is it well-defined? ie if $g_1H = g_1'H$, then is it true that

$$g_1H \cdot g_2H = \underbrace{g_1g_2H \overset{?}{=} g_1'g_2H} = g_1'Hg_2H.$$

similarly if $g_2H = g_2'H$, is it true that:

$$g_1g_2H \overset{?}{=} g_1g_2'H$$

Assume coset multiplication is well-defined; then, we should have:

$\forall g \in G$, $\forall h \in H$

$$\underbrace{hH}_{\overset{\parallel}{eH}} \cdot gH = hgH = gH = eH \cdot gH.$$

So $hgH = gH$

so $g^{-1}hgH = H$ so $\boxed{\forall g \in G, h \in H, \; g^{-1}hg \in H.}$

We just realized:

"Coset multiplication is well-defined"

$$\Longrightarrow H \leq G \text{ is normal}.$$

Theorem: Let $H \leq G$ be a subgroup. Then, coset multiplication $g_1H \cdot g_2H = g_1g_2H$ if and only if H is normal.

proof: We already proved "$\Rightarrow$".

"$\Leftarrow$" Assume H is normal.

Let $g_1H = g_1'H$, then $g_1' = g_1h$ for some h.

Then $g_1H \cdot g_2H = g_1g_2H$

equal!

$$g_1'Hg_2H = g_1'g_2H = g_1hg_2H$$
$$= g_1g_2\underbrace{g_2^{-1}hg_2}_{\in H}H$$
$$= g_1g_2H.$$

similarly for $g_2H = g_2'H$.

then: $g_2h = g_2'$ for some H.

$$g_1g_2H = g_1g_2hH = g_1g_2'H.$$

$\square$

Theorem: If $H \trianglelefteq G$ is normal subgroup. Then $G/H = \{gH \mid g \in G\}$ with coset multiplication is a group.

proof: Associativity follows from associativity in G.

$e_{G/H} = e_GH$. Inverses: $(gH)^{-1} = g^{-1}H$.