Lecture 27:

Last time:

- $H \subseteq G$ is normal if $\forall g \in G$, $gHg^{-1} \subset H$.
  (equivalently, $gHg^{-1} = H$
  equivalently, $gH = Hg$)
- Quotient groups:
  coset-multiplication $(g_1H)\cdot(g_2H) = g_1g_2H$
  is well-defined <u>if and only if</u> H is <u>normal</u>

(key part of proof: assume H is normal,
and $g_1H = g_1'H$ (which implies $g_1' = g_1 h$ for some $h \in H$)
then $g_1'Hg_2H = g_1'g_2H = g_1hg_2H = \underbrace{g_1g_2g_2^{-1}}_{1}\overbrace{hg_2}^{\in H}H$

$= g_1g_2h_3H$

So choice of $g_1 \in g_1H$ doesn't change the answer.

$= g_1g_2H = g_1H \cdot g_2H$.)

- G/H = set of cosets $= \{gH \mid g \in G\}$
  with coset multiplication is a group if H is normal.

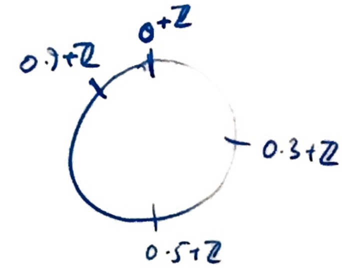
<u>Examples:</u> (1) $G = \mathbb{Z}$, $H = 5\mathbb{Z}$

$G/H = \{a + 5\mathbb{Z} \mid a \in \mathbb{Z}\} = \{0+5\mathbb{Z}, 1+5\mathbb{Z}, \ldots, 4+5\mathbb{Z}\}$.

$a + 5\mathbb{Z} + b + 5\mathbb{Z} = a + b + 5\mathbb{Z}$ (this is automatically well-defined because $5\mathbb{Z} \subset \mathbb{Z}$ is normal).

(2) $G = \mathbb{R}$, $H = \mathbb{Z} \subset \mathbb{R}$.

Cosets are $\alpha + \mathbb{Z}$, $\alpha \in \mathbb{R}$.



$$\mathbb{R}/\mathbb{Z} = \{ \alpha + \mathbb{Z} \mid \alpha \in \mathbb{R} \}$$

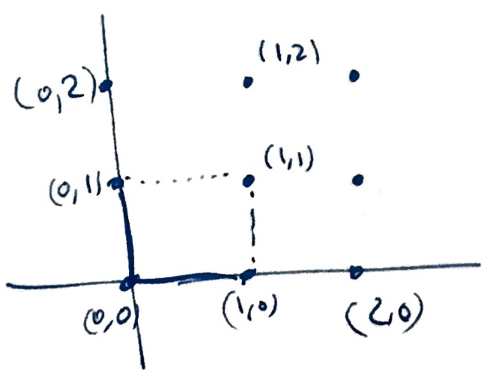
$$\alpha + \mathbb{Z} = \beta + \mathbb{Z} \iff \alpha - \beta \in \mathbb{Z}$$

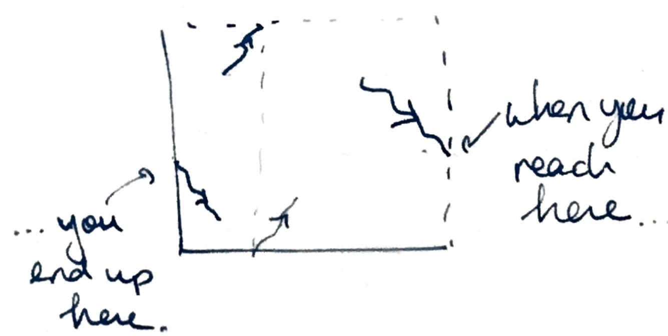
So each coset can be represented by a unique element $\alpha \in [0,1)$.

(it's like real numbers modulo 1).

(3) $\mathbb{R} \times \mathbb{R} / \mathbb{Z} \times \mathbb{Z}$ cosets are $(\alpha, \beta) + \mathbb{Z} \times \mathbb{Z}$

every coset has a unique representative in $[0,1) \times [0,1)$

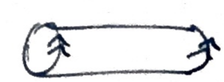




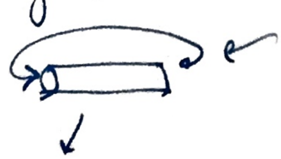
It's like you took

and glued together

and glued



the and together.



torus.

(4) $G = D_6$ $\quad H = \{1, r, r^2\}$

(note: we can't take $H = \{1, s\}$ because it's not normal)

$$G/H = D_6 / \{1, r, r^2\} = \{1.H, sH\}$$

$$sH \cdot sH = s^2H = 1.H$$

it's isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(we knew that already because every group with two elements is isomorphic to $\mathbb{Z}/2\mathbb{Z}$).

First isomorphism theorem:

recall: that for $\varphi: G \to K$ a homomorphism

$\mathrm{Ker}\varphi \subset G$ is a subgroup $\quad \mathrm{Ker}\varphi = \varphi^{-1}(\{e_K\})$

- recall: $\mathrm{Im}\varphi = \varphi(G)$ is a subgroup. $\quad = \{g \in G \mid \varphi(g) = e_K\}$

Book calls it "fundamental homomorphism theorem")

Theorem: (1st iso theorem) Let $\varphi: G \to K$ be a group homomorphism. The map:

$$\tilde{\varphi}: G/_{\mathrm{Ker}\varphi} \longrightarrow \mathrm{Im}\varphi$$

given by $\tilde{\varphi}(g\,\mathrm{Ker}\varphi) = \varphi(g)$ is an isomorphism.

proof: We need to check that $\tilde{\varphi}$ is well-defined.

Let $g_1 \mathrm{Ker}\varphi = g_2 \mathrm{Ker}\varphi$, then $\exists k \in \mathrm{Ker}\varphi$ st $g_1 = g_2 k$

Then $\tilde{\varphi}(g_1 \mathrm{Ker}\varphi) = \varphi(g_1) = \varphi(g_2 k) = \varphi(g_2)\underbrace{\varphi(k)}_{e}$

$\tilde{\varphi}(g_2 \mathrm{Ker}\varphi) = \varphi(g_2)$ equal

So $\tilde{\varphi}$ is well-defined.

- To see that $\tilde{\varphi}$ is injective (recall that a homomorphism is injective iff its kernel is trivial) observe that $\operatorname{Ker}\tilde{\varphi} = \{ g\operatorname{Ker}\varphi \mid \tilde{\varphi}(g\operatorname{Ker}\varphi) = \varphi(g) = e \}$
$= \{ 1 \cdot \operatorname{Ker}\varphi \}$.

g must be already in Kernel so $g\operatorname{Ker}\varphi = 1.\operatorname{Ker}\varphi$.

- $\tilde{\varphi}$ is surjective by definition (since we made it map to $\operatorname{Im}\varphi$..

- $\tilde{\varphi}$ is a homomorphism since

$$\tilde{\varphi}(g_1\operatorname{Ker}\varphi \cdot g_2\operatorname{Ker}\varphi) = \tilde{\varphi}(g_1g_2\operatorname{Ker}\varphi) = \varphi(g_1g_2)$$
$$= \varphi(g_1)\varphi(g_2)$$
$$= \tilde{\varphi}(g_1\operatorname{Ker}\varphi)\,\tilde{\varphi}(g_2\operatorname{Ker}\varphi)$$

$\square$

Ex: • $\det_n : GL_n(\mathbb{R}) \to \mathbb{R}^\times$

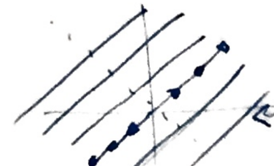
Then: $\operatorname{Ker}(\det_n) = SL_n(\mathbb{R})$

$$GL_n(\mathbb{R}) / SL_n(\mathbb{R}) \cong \mathbb{R}^\times$$

✓ · $\det_n$ is surjective so $\operatorname{Im}(\det_n) = \mathbb{R}^\times$

- Let $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $\varphi(a,b) = a-b$
φ is a homomorphism ($+$ is the operation), (check it).
φ is surjective. $\operatorname{Ker}\varphi = \{(a,b) \mid a-b=0\}$
$= \{(a,b) \mid a=b\} = \langle (1,1) \rangle$

So $\mathbb{Z}\times\mathbb{Z} / \langle (1,1) \rangle \cong \mathbb{Z}$



cosets.

Cool remark: A subgroup H in G is normal if and only if it is the kernel of some homomorphism $\varphi: G \to K$ (for some group K).

proof: we already know that kernels are normal.

If H is normal, it is the kernel of $\varphi: G \longrightarrow G/H$

$$\varphi(g) = gH$$