Math 222B, Complex Variables and Geometry

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Contents

1 Lecture 1

1.1 Example of $\mathbb{R}^{2n} = \mathbb{C}^n$

Remark 1.1. For now we will denote $\sqrt{-1}$ by i. However, later we will not do this, because the letter i is sometimes used as an index.

We consider $\mathbb{R}^{2n}$ and denote the coordinates as $x^1, y^1, \ldots, x^n, y^n$. Letting $z^j = x^j + iy^j$ and $\bar{z}^j = x^j - iy^j$, define complex one-forms

$$dz^j = dx^j + idy^j,$$
$$d\bar{z}^j = dx^j - idy^j,$$

and complex tangent vectors

$$\partial/\partial z^j = (1/2)\left(\partial/\partial x^j - i\partial/\partial y^j\right),$$
$$\partial/\partial \bar{z}^j = (1/2)\left(\partial/\partial x^j + i\partial/\partial y^j\right).$$

Note that

$$dz^j(\partial/\partial z^k) = d\bar{z}^j(\partial/\partial \bar{z}^k) = \delta^{jk},$$
$$dz^j(\partial/\partial \bar{z}^k) = d\bar{z}^j(\partial/\partial z^k) = 0.$$

The standard complex structure $J_0 : T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$ on $\mathbb{R}^{2n}$ is given by

$$J_0(\partial/\partial x^j) = \partial/\partial y^j, \quad J_0(\partial/\partial y^j) = -\partial/\partial x^j,$$

which in matrix form is written

$$J_0 = \mathrm{diag}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}. \tag{1.1}$$

Next, we complexify the tangent space $T \otimes \mathbb{C}$, and let

$$T^{(1,0)}(J_0) = \mathrm{span}\{\partial/\partial z^j, j = 1 \ldots n\} = \{X - iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{1.2}$$

be the i-eigenspace and

$$T^{(0,1)}(J_0) = \mathrm{span}\{\partial/\partial \bar{z}^j, j = 1 \ldots n\} = \{X + iJ_0X, X \in T_p\mathbb{R}^{2n}\} \tag{1.3}$$

be the $-i$-eigenspace of J_0, so that

$$T \otimes \mathbb{C} = T^{(1,0)}(J_0) \oplus T^{(0,1)}(J_0). \tag{1.4}$$

The map J_0 also induces an endomorphism of 1-forms by

$$J_0(\omega)(v_1) = \omega(J_0 v_1).$$

Since the components of this map in a dual basis are given by the transpose, we have

$$J_0(dx_j) = -dy_j, \quad J_0(dy_j) = +dx_j.$$

Then complexifying the cotangent space $T^* \otimes \mathbb{C}$, we have

$$\Lambda^{1,0}(J_0) = \mathrm{span}\{dz^j, j = 1 \ldots n\} = \{\alpha - iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.5}$$

is the i-eigenspace, and

$$\Lambda^{0,1}(J_0) = \mathrm{span}\{d\bar{z}^j, j = 1 \ldots n\} = \{\alpha + iJ_0\alpha, \alpha \in T_p^*\mathbb{R}^{2n}\} \tag{1.6}$$

is the $-i$-eigenspace of J_0, and

$$T^* \otimes \mathbb{C} = \Lambda^{1,0}(J_0) \oplus \Lambda^{0,1}(J_0). \tag{1.7}$$

We note that

$$\Lambda^{1,0} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(0,1)}\}, \tag{1.8}$$

and similarly

$$\Lambda^{0,1} = \{\alpha \in T^* \otimes \mathbb{C} : \alpha(X) = 0 \text{ for all } X \in T^{(1,0)}\}. \tag{1.9}$$

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}, \tag{1.10}$$

and note that

$$\dim_{\mathbb{C}}(\Lambda^{p,q}) = \binom{n}{p} \cdot \binom{n}{q}. \tag{1.11}$$

Note that we can characterize $\Lambda^{p,q}$ as those forms satisfying

$$\alpha(v_1, \ldots, v_{p+q}) = 0, \tag{1.12}$$

if more than p if the v_j-s are in $T^{(1,0)}$ or if more than q of the v_j-s are in $T^{(0,1)}$.

Finally, we can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{1.13}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$.

In general, J is not a complex structure on the space $\Lambda^k_{\mathbb{C}}$ for $k > 1$. Also, note that if $\alpha \in \Lambda^{p,p}$, then α is J-invariant.

1.2 Cauchy-Riemann equations

Let $f : \mathbb{C}^n \to \mathbb{C}^m$. Let the coordinates on $\mathbb{C}^n$ be given by

$$\{z^1, \ldots z^n\} = \{x^1 + iy^1, \ldots, x^n + iy^n\}, \tag{1.14}$$

and coordinates on $\mathbb{C}^m$ given by

$$\{w^1, \ldots w^m\} = \{u^1 + iv^1, \ldots, u^m + iv^m\} \tag{1.15}$$

Write

$$T_{\mathbb{R}}(\mathbb{C}^n) = \mathrm{span}\{\partial/\partial x^1, \ldots \partial/\partial x^n, \partial/\partial y^1, \ldots \partial/\partial y^n\}, \tag{1.16}$$
$$T_{\mathbb{R}}(\mathbb{C}^m) = \mathrm{span}\{\partial/\partial u^1, \ldots \partial/\partial u^m, \partial/\partial v^1, \ldots \partial/\partial v^m\}. \tag{1.17}$$

Then the real Jacobian of

$$f = (f^1, \ldots f^{2m}) = (u^1 \circ f, u^2 \circ f, \ldots, v^{2m} \circ f). \tag{1.18}$$

in this basis is given by

$$\mathcal{J}_{\mathbb{R}} f = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \ldots & \frac{\partial f^1}{\partial y^n} \\ \vdots & \ldots & \vdots \\ \frac{\partial f^{2m}}{\partial x^1} & \ldots & \frac{\partial f^{2m}}{\partial y^n} \end{pmatrix} \tag{1.19}$$

Definition 1.2. A differentiable mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is pseudo-holomorphic if

$$f_* \circ J_{0,\mathbb{C}^n} = J_{0,\mathbb{C}^m} \circ f_*. \tag{1.20}$$

That is, the differential of f commutes with J_0.

We have the following characterization of pseudo-holomorphic maps.

Proposition 1.3. *A mapping $f : \mathbb{C}^m \to \mathbb{C}^n$ is pseudo-holomorphic if and only if the Cauchy-Riemann equations are satisfied, that is, writing*

$$f(z^1, \dots z^m) = (f_1, \dots, f_n) = (u_1 + iv_1, \dots u_n + iv_n), \tag{1.21}$$

and $z^j = x^j + iy^j$, for each $j = 1 \dots n$, we have

$$\frac{\partial u_j}{\partial x^k} = \frac{\partial v_j}{\partial y^k} \qquad \frac{\partial u_j}{\partial y^k} = -\frac{\partial v_j}{\partial x^k}, \tag{1.22}$$

for each $k = 1 \dots m$, and these equations are equivalent to

$$\frac{\partial}{\partial \bar{z}^k} f_j = 0, \tag{1.23}$$

for each $j = 1 \dots n$ and each $k = 1 \dots m$

Proof. First, we consider $m = n = 1$. We compute

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \\ \frac{\partial f_2}{\partial x^1} & \frac{\partial f_2}{\partial y^1} \end{pmatrix}, \tag{1.24}$$

says that

$$\begin{pmatrix} \frac{\partial f_1}{\partial y^1} & -\frac{\partial f_1}{\partial x^1} \\ \frac{\partial f_2}{\partial y^1} & -\frac{\partial f_2}{\partial x^1} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_2}{\partial x^1} & -\frac{\partial f_2}{\partial y^1} \\ \frac{\partial f_1}{\partial x^1} & \frac{\partial f_1}{\partial y^1} \end{pmatrix}, \tag{1.25}$$

which is exactly the Cauchy-Riemann equations. In the general case, rearrange the coordinates so that $(x^1, \dots, x^m, y^1, \dots, y^m)$ are the real coordinates on $\mathbb{R}^{2m}$ and $(u^1, \dots, u^n, v^1, \dots, v^n)$, such that the complex structure J_0 is given by

$$J_0(\mathbb{R}^{2m}) = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \tag{1.26}$$

and similarly for $J_0(\mathbb{R}^{2n})$. Then the computation in matrix form is entirely analogous to the case of $m = n = 1$.

Finally, we compute

$$\frac{\partial}{\partial \bar{z}^k} f_j = \frac{1}{2}\Big(\frac{\partial}{\partial x^k} + i \frac{\partial}{\partial y^k} \Big)(u_j + iv_j) \tag{1.27}$$

$$= \frac{1}{2}\Big\{ \frac{\partial}{\partial x^k} u_j - \frac{\partial}{\partial y^k} v_j + i\Big(\frac{\partial}{\partial x^k} v_j + \frac{\partial}{\partial y^k} u_j \Big) \Big\}, \tag{1.28}$$

the vanishing of which again yields the Cauchy-Riemann equations. $\square$

From now on, if f is a mapping satisfying the Cauchy-Riemann equations, we will just say that f is *holomorphic*.

For any differentiable f, the mapping $f_* : T_{\mathbb{R}}(\mathbb{C}^n) \to T_{\mathbb{R}}(\mathbb{C}^m)$ extends to a mapping

$$f_* : T_{\mathbb{C}}(\mathbb{C}^n) \to T_{\mathbb{C}}(\mathbb{C}^m). \tag{1.29}$$

Consider the bases

$$T_{\mathbb{C}}(\mathbb{C}^n) = \text{span}\{\partial/\partial z^1, \dots \partial/\partial z^n, \partial/\partial \overline{z}^1, \dots \partial/\partial \overline{z}^n\}, \tag{1.30}$$

$$T_{\mathbb{R}}(\mathbb{C}^m) = \text{span}\{\partial/\partial w^1, \dots \partial/\partial w^m, \partial/\partial \overline{w}^1, \dots \partial/\partial \overline{w}^m\}. \tag{1.31}$$

The matrix of f_* with respect to these bases is the complex Jacobian, and is given by

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} \frac{\partial f^1}{\partial z^1} & \cdots & \frac{\partial f^1}{\partial z^n} & \frac{\partial f^1}{\partial \overline{z}^1} & \cdots & \frac{\partial f^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f^m}{\partial z^1} & \cdots & \frac{\partial f^m}{\partial z^n} & \frac{\partial f^m}{\partial \overline{z}^1} & \cdots & \frac{\partial f^m}{\partial \overline{z}^n} \\ \frac{\partial \overline{f}^1}{\partial z^1} & \cdots & \frac{\partial \overline{f}^1}{\partial z^n} & \frac{\partial \overline{f}^1}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^1}{\partial \overline{z}^n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \overline{f}^m}{\partial z^1} & \cdots & \frac{\partial \overline{f}^m}{\partial z^n} & \frac{\partial \overline{f}^m}{\partial \overline{z}^1} & \cdots & \frac{\partial \overline{f}^m}{\partial \overline{z}^n}, \end{pmatrix} \tag{1.32}$$

where $(f^1, \dots, f^m) = f$ now denotes the complex components of f. This is equivalent to saying that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^1}{\partial \overline{z}^k} d\overline{z}^k. \tag{1.33}$$

Notice that (1.32) is of the form

$$\mathcal{J}_{\mathbb{C}} f = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \tag{1.34}$$

which is equivalent to the condition that the complex mapping is the complexification of a real mapping.

What we have done here is to embed

$$Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.35}$$

where $\mathbb{C}$-linear means with respect to i (not J_0), via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \frac{1}{2} \begin{pmatrix} A + D + i(C - B) & A - D + i(B + C) \\ A - D - i(B + C) & A + D - i(C - B) \end{pmatrix}. \tag{1.36}$$

Notice that if f is holomorphic, the condition that f_* commutes with J_0 says that the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \tag{1.37}$$

This corresponds to the embeddings

$$Hom_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}), \tag{1.38}$$

where the left $\mathbb{C}$-linear is with respect to J_0, via

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A + iB & 0 \\ 0 & A - iB \end{pmatrix}. \tag{1.39}$$

Note that since the latter embedding is just a change of basis, if $m = n$, then

$$\det(\mathcal{J}_{\mathbb{R}}) = \det(A + iB) \det(A - iB) = |\det(A + iB)|^2 \geq 0, \tag{1.40}$$

which implies that holomorphic maps are orientation-preserving. Note also that f is holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(1,0)}. \tag{1.41}$$

Notice that if f is anti-holomorphic, which is the condition that f_* anti-commutes with J_0, then the real Jacobian must have the form

$$(f_*)_{\mathbb{R}} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \tag{1.42}$$

This corresponds to the embeddings

$$Hom_{\overline{\mathbb{C}}}(\mathbb{C}^n, \mathbb{C}^m) \subset Hom_{\mathbb{R}}(\mathbb{R}^{2n}, \mathbb{R}^{2m}) \subset Hom_{\mathbb{C}}(\mathbb{C}^{2n}, \mathbb{C}^{2m}) \tag{1.43}$$

via

$$A + iB \mapsto \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A + iB \\ A - iB & 0 \end{pmatrix}. \tag{1.44}$$

We see that f is anti-holomorphic if and only if

$$f_*(T^{(1,0)}) \subset T^{(0,1)}. \tag{1.45}$$

Note that if f is antiholomorphic, then is it holomorphic with respect to the complex structure $-J_0$ on the domain (but still J_0 on the range).

Note that we can decompose $f_* = f_*^C + f_*^A$, where

$$f_*^C = \frac{1}{2}\left(f_* - J f_* J\right) \tag{1.46}$$

$$f_*^A = \frac{1}{2}\left(f_* + J f_* J\right), \tag{1.47}$$

and f_*^C is holomorphic, while f_*^A is anti-holomorphic. In block matrix form, this just says that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A + D & B - C \\ C - B & A + D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A - D & B + C \\ B + C & D - A \end{pmatrix}. \tag{1.48}$$

2 Lecture 2

2.1 Cauchy's formula in one complex variable

For now, just consider $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is an open set. Assume that f, as a mapping from $\mathbb{R}^2 \to \mathbb{R}^2$, is differentiable. This means that, for each $z \in U$, there exists a linear mapping $L_z : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - L_z h\|}{\|h\|} = 0. \tag{2.1}$$

This implies that the partial derivatives of f exist. Conversely, if the partial derivatives exists and are continuous at z, then the mapping L_z exists.

We say that f is holomorphic in U if it is C^1 and satisfies the Cauchy-Riemann equations. Writing $f = u + iv$, then Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Longleftrightarrow \frac{\partial}{\partial \bar{z}} f = 0, \tag{2.2}$$

Note that the linear mapping L is given by

$$L = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}. \tag{2.3}$$

If we consider h as a complex number, then f being holomorphic is equivalent to

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - \frac{\partial f}{\partial z} h\|}{\|h\|} = 0. \tag{2.4}$$

Definition 2.1. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{2.5}$$

which converges absolutely and uniformly in a disc $\Delta(z_0, \epsilon)$ around z_0, for some $\epsilon > 0$.

Proposition 2.2 (Cauchy-Pompieu Formula)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded domain in $\mathbb{C}$ with C^1 boundary. For $z \in \Omega$ and $f \in C^1(\overline{\Omega})$, we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)dw}{w - z} + \frac{1}{2\pi i} \int_{\Omega} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}, \tag{2.6}$$

where the boundary has the counterclockwise orientation.

Proof. The 1-form

$$\eta = \frac{1}{2\pi i} \frac{f(w)dw}{w - z}, \tag{2.7}$$

satisfies

$$d\eta = -\frac{1}{2\pi i}\frac{\partial f}{\partial \bar{w}}\frac{d\bar{w}\wedge dw}{w-z}. \tag{2.8}$$

Apply Stokes' Theorem to the annular domain $\Omega\setminus\Delta(z,\epsilon)$, to get

$$\int_{\partial(\Omega\setminus\Delta(z,\epsilon))}\eta = \int_{\Omega\setminus\Delta(z,\epsilon)} d\eta. \tag{2.9}$$

The left hand side of (2.9) is

$$\int_{\partial\Omega}\eta - \int_{\partial\Delta(z,\epsilon)}\eta, \tag{2.10}$$

and the right hand side of (2.9) is

$$\int_{\Omega} d\eta - \int_{\Delta(z,\epsilon)} d\eta, \tag{2.11}$$

since $d\eta$ is obviously in $L^1(\Omega)$. A calculation shows that the inner boundary integral limits to $f(z)$, and the error term in the solid integral goes to 0 as $\epsilon\to 0$. For details, see [GH78, page 3]. □

Proposition 2.3. *Let U be an open set in $\mathbb{C}$. Then f is holomorphic in U if and only if f is complex analytic in U.*

Proof. If f is holomorphic in U the Cauchy-Pompieu formula in a small disc $\Delta=\Delta(z_0,\epsilon)$ yields for $z\in\Delta$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{w-z}. \tag{2.12}$$

Then expand

$$\frac{1}{z-w} = \frac{1}{z-z_0+z_0-w} = \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \tag{2.13}$$

$$= \frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k, \tag{2.14}$$

with the sum converging absolutely and uniformly in any smaller disc. So the above yield the power series expansion

$$f(z) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{(w-z_0)^{k+1}}\right)(z-z_0)^n, \tag{2.15}$$

which also converges absolutely and uniformly in any smaller disc.

For the converse, if f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula without solid integral. So then f does also by uniform convergence. So we have

$$\frac{\partial}{\partial\overline{z}}f(z) = \frac{\partial}{\partial\overline{z}}\left(\frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{w-z}\right) = \frac{1}{2\pi i}\int_{\partial\Delta}f(w)\left(\frac{\partial}{\partial\overline{z}}\frac{1}{w-z}\right)dw = 0. \tag{2.16}$$

For more details, see [GH78, page 4]. □

Definition 2.4. We wil let $\Omega \subset \mathbb{C}$ be a bounded domain with C^1 boundary. If u is holomorphic in an open set Ω, then we write $u \in \mathcal{O}(\Omega)$.

First, let's recall the basic result about differentiating under an integral.

Proposition 2.5. *Let*

$$f(z) = \int_\Omega a(z,w) dw \wedge d\overline{w}. \tag{2.17}$$

(Note this notation does not mean that f is holomorphic in z or that a is holomorphic as a function of 2 variables!). Assume that

1. *$a(z,w) \in L^1(\Omega)$, in the w variable.*
2. *$\frac{\partial a}{\partial z}$ and $\frac{\partial a}{\partial \overline{z}}$ exist for all z, for almost every $w \in \Omega$.*
3. *$\left|\frac{\partial a}{\partial z}\right| + \left|\frac{\partial a}{\partial \overline{z}}\right| \leq h(w)$, where $h \in L^1(\Omega)$.*

Then

$$\frac{\partial f}{\partial z} = \int_\Omega \frac{\partial}{\partial z}\big(a(z,w)\big) dw \wedge d\overline{w} \tag{2.18}$$

$$\frac{\partial f}{\partial \overline{z}} = \int_\Omega \frac{\partial}{\partial \overline{z}}\big(a(z,w)\big) dw \wedge d\overline{w}. \tag{2.19}$$

Proof. Recall that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big). \tag{2.20}$$

The real part of the left hand side of (2.18) is

$$Re\Big(\frac{\partial f}{\partial z}\Big) = \frac{1}{2}\Big(\frac{\partial Ref}{\partial x} + \frac{\partial Imf}{\partial y}\Big) \tag{2.21}$$

The real part of the right hand side of (2.18) is

$$\int_\Omega \frac{1}{2}\Big(\frac{\partial Re(a(x+iy,w))}{\partial x} + \frac{\partial Im(a(x+iy,w))}{\partial y}\Big) dw \wedge d\overline{w}. \tag{2.22}$$

Therefore we can consider real-valued functions, and prove for partials with respect to the real variables x and y. We have that

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\delta \to 0} \frac{f(x+\delta,y) - f(x,y)}{\delta} \tag{2.23}$$

For $\delta \neq 0$, consider

$$\frac{f(x+\delta,y) - f(x,y)}{\delta} = \int_\Omega \frac{a(x+\delta+iy,w) - a(x+iy,w)}{\delta} dw \wedge d\overline{w}. \tag{2.24}$$

By the mean value theorem, given $\delta > 0$, there exists x' on the line segment from (x, y) to $(x + \delta, y)$ such that

$$a(x + \delta + iy, w) - a(x + iy, w) = \frac{\partial a}{\partial x}(x' + iy, w)\delta, \tag{2.25}$$

so

$$\begin{aligned}\Big|\frac{a(x + \delta + iy, w) - a(x + iy, w)}{\delta}\Big| &\leq \Big|\frac{\partial a}{\partial x}(x' + iy, w)\Big| \\ &\leq \Big|\frac{\partial a}{\partial z}(x' + iy, w)\Big| + \Big|\frac{\partial a}{\partial \bar{z}}(x' + iy, w)\Big| \leq |h(w)|.\end{aligned} \tag{2.26}$$

We can do this for any sequence $\delta_n \to 0$, so the result follows from Lebesgue's dominated convergence theorem. The proof for the other derivative (2.19) is similar. $\square$

We next go through several corollaries of the Cauchy-Pompieu formula; see [Hör90, Chapter 1] for more details.

Corollary 2.6. *Let $K \subset \Omega$ be a compact subset. Then there exist constant C_k, depending only upon K and Ω such that*

$$\sup_{z \in K} \Big|\Big(\frac{\partial}{\partial z}\Big)^k u(z)\Big| \leq C_k \|u\|_{L^1(\Omega)}, \tag{2.27}$$

for all $u \in \mathcal{O}(\Omega)$.

Proof. Choose a $\psi \in C_0^\infty(\Omega)$ (compact support) such that $\psi \equiv 1$ in a neighborhood of K. If $u \in \mathcal{O}(\Omega)$, then

$$\frac{\partial}{\partial \bar{z}}(\psi u) = u \frac{\partial}{\partial \bar{z}} \psi. \tag{2.28}$$

Now we apply (2.6) to ψu in Ω to get

$$\psi(z) u(z) = \frac{1}{2\pi i} \int_\Omega u(w) \frac{\partial \psi(w)}{\partial \overline{w}} \frac{dw \wedge d\bar{w}}{w - z}, \tag{2.29}$$

Now consider

$$a(z, w) = u(w) \frac{\partial \psi(w)}{\partial \overline{w}} \frac{1}{w - z}, \tag{2.30}$$

If $z \in K$, then $|w - z| > \delta > 0$, since the support of $\partial \psi / \partial \overline{w}$ is at a positive distance from K. So using Proposition 2.5, we can differentiate under the integral as many times as we like, and obtain

$$\Big(\frac{\partial}{\partial z}\Big)^k (\psi u(z)) = \frac{1}{2\pi i} \int_\Omega u(w) \frac{\partial \psi(w)}{\partial \overline{w}} \Big(\frac{\partial}{\partial z}\Big)^k \Big(\frac{1}{w - z}\Big) dw \wedge d\bar{w} \tag{2.31}$$

If $z \in K$, then ψu is equal to u in a neighborhood of z, so (2.27) follows. $\square$

Corollary 2.7. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \to u$ converges uniformly to u in the C^0 norm as $n \to \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Proof. Let $K \subset \Omega$, be a compact subset. Then given $\epsilon > 0$, there exist N such that

$$\sup_{z \in K} |u_m(z) - u_n(z)| < \epsilon, \tag{2.32}$$

for $m, n \geq N$. The difference $u_m - u_n \in \mathcal{O}(\Omega)$. Corollary 5.11 implies that

$$\sup_{z \in K} \left| \frac{\partial}{\partial z}(u_m - u_n)(z) \right| \leq C\epsilon. \tag{2.33}$$

This says that $\partial u_n / \partial z$ converges uniformly on K. But $\partial u_n / \partial \overline{z} = 0$, so the real partial derivatives $\partial u_n / \partial x$ and $\partial u_n / \partial y$ converge uniformly. It is an elementary result that if a sequence of functions converges uniformly, and the derivatives converge uniformly, then the limit of the derivatives is the derivative of the limit. This implies that $u \in C^1$ and $\partial u / \partial \overline{z} = \lim_{n \to \infty} \partial u_n / \partial \overline{z} = 0$. ☐

Corollary 2.8. *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

Proof. Corollary 5.11 yield a uniform bound on derivatives of u_n on any compact subset. By Arzela-Ascoli Theorem, some subsequence converges to a limit u uniformly on compact subsets. Then the previous corollary yields that $u \in \mathcal{O}(\Omega)$. ☐

3 Lecture 3

3.1 The $\overline{\partial}$-equation in domains in $\mathbb{C}$

Theorem 3.1. *If $\Omega \subset \mathbb{C}$ is any bounded domain, then $H^{0,1}_{\overline{\partial}}(\Omega) = 0$.*

Recall that

$$H^{0,1}_{\overline{\partial}}(\Omega) = \frac{Ker\{\overline{\partial} : \Lambda^{0,1} \to \Lambda^{0,2}\}}{Im\{\overline{\partial} : \Lambda^{0,0} \to \Lambda^{0,1}\}}, \tag{3.1}$$

where these are spaces of C^∞ forms. Since $n = 1$, $\Lambda^{0,2} = \{0\}$. So $\omega \in Ker\{\overline{\partial} : \Lambda^{0,1} \to \Lambda^{0,2}\}$ just means that $\omega = g d\overline{z}$ for $g \in C^\infty(\Omega)$. Also, $f \in \Lambda^{0,0}$ is just a function, and $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} d\overline{z}$. So the theorem is equivalently stated as the following.

Theorem 3.2. *If $\Omega \subset \mathbb{C}$ is any bounded domain, and $g \in C^\infty(\Omega)$, then there exists $g \in C^\infty(\Omega)$ with $\frac{\partial}{\partial \overline{z}} f = g$.*

Our goal of this lecture is to prove this result. If we want to solve $\frac{\partial}{\partial \overline{z}} f = g$, it is natural to guess that

$$f(z) = \frac{1}{2\pi i} \int_\Omega \frac{g(w) dw \wedge d\bar{w}}{w - z}, \tag{3.2}$$

is a solution. However, letting $a(z,w) = g(w)/(w-z)$, we have

$$\frac{\partial a(z,w)}{\partial z} = g(w)\frac{-1}{(w-z)^2}, \tag{3.3}$$

so the assumptions of Proposition 2.5 are NOT satisfied, so we cannot directly differentiate under the integral sign! Another problem is that g is only assumed to be in $C^\infty(\Omega)$, so it is not in $L^1(\Omega)$ and (3.2) is not necessarily defined. We first give a preliminary result, with a stronger assumption on g.

Proposition 3.3. *If $g \in C^1(\overline{\Omega})$ then the function*

$$f(z) = \frac{1}{2\pi i}\int_\Omega \frac{g(w)dw \wedge d\bar{w}}{w-z}, \tag{3.4}$$

satisfies $f \in C^1(\overline{\Omega})$ and $\partial f/\partial \overline{z} = g$ in Ω.

Proof. We fix a point $z_0 \in \Omega$. Choose a C^∞ cutoff function $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ on a neighborhood $V \subset \Omega$ containing z_0. We then write $f = f_1 + f_2$, where

$$f_1(z) = \frac{1}{2\pi i}\int_\Omega \frac{\psi(w)g(w)dw \wedge d\bar{w}}{w-z} \tag{3.5}$$

$$f_2(z) = \frac{1}{2\pi i}\int_\Omega \frac{(1-\psi(w))g(w)dw \wedge d\bar{w}}{w-z}. \tag{3.6}$$

For z in a small neighborhood of z_0, the integrand in f_2 does not have a singularity. We can therefore differentiate under the integral sign to see that $\partial f_2/\partial \overline{z} = 0$. So we just need to prove that $\partial f_1/\partial \overline{z} = g$ in V. Since ψ has compact support, we can extend ψg to all of $\mathbb{C}$, and write

$$f_1(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(w)g(w)dw \wedge d\bar{w}}{w-z} \tag{3.7}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\psi(\xi+z)g(\xi+z)d\xi \wedge d\bar{\xi}}{\xi}, \tag{3.8}$$

where we used the change of variables $w = \xi + z$. Note that

$$\frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial z} = \frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial \xi} \tag{3.9}$$

$$\frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial \overline{z}} = \frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial \bar{\xi}}. \tag{3.10}$$

This shows that the z and $\overline{z}$ partials of the integrand are uniformly in L^1, so we can differentiate under the integral sign, to obtain

$$\frac{\partial f_1(z)}{\partial \overline{z}} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi(\xi+z)g(\xi+z))}{\partial \bar{\xi}}\frac{d\xi \wedge d\bar{\xi}}{\xi} \tag{3.11}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{\partial(\psi(w)g(w))}{\partial \overline{w}}\frac{dw \wedge d\overline{w}}{w-z}. \tag{3.12}$$

Now apply the Cachy-Pompieu formula in a very large ball in $\mathbb{C}$, to conclude the right hand side is equal to $\psi(z)g(z)$ which is $g(z)$ if $z \in V$. $\square$

This result does not directly help us in proving Theorem 4.2. But notice that in the proof, we also proved the following result.

Proposition 3.4. *If $g \in C_c^\infty(\Omega)$, then there exists $f \in C^\infty(\mathbb{C})$ such that $\partial f/\partial \overline{z} = g$.*

Proof. Above, we proved that there is a solution $f \in C^1(\mathbb{C})$, but the same argument allows us to differentiate f_1 infinitely many times, provided g is infinitely differentiable. □

Now we can prove a special case of Theorem 4.2.

Proposition 3.5. *If $\Omega = \Delta(z, r)$ is a disc in $\mathbb{C}$, then $H^{0,1}_{\overline{\partial}}(\Omega) = 0$.*

Proof. Take a sequence $0 < r_1 < r_2 < \cdots < r$ such that $\lim_{j\to\infty} r_i = r$. Let $0 \leq \psi_k \in C_0^\infty(\Delta(z, r_{k+1}))$ and $\psi_k \equiv 1$ on $\Delta(z, r_k)$. Then $g_k = \psi_k g \in C_0^\infty(\Delta(z, r_{k+1}))$, and by Proposition 3.4, we can find $f_k \in C^\infty(\mathbb{C})$ such that $\partial f_k = g_k$, which is equal to g in $\Delta(z, r_k)$.

Now there is no reason that the sequence f_k will converge to a limit, so we need to modify as follows. We claim that we can choose f_k so that

$$\sup_{z\in\Delta(z,r_{k-1})} |f_{k+1}(z) - f_k(z)| \leq 2^{-k}. \tag{3.13}$$

Given f_2, the difference $f_3 - f_2$ is holomorphic in $\overline{\Delta(z, r_1)}$. So there exists a polynomial P_2 such that

$$\sup_{z\in\Delta(z,r_1)} |f_3(z) - f_2(z) - P_2(z)| \leq 2^{-2}. \tag{3.14}$$

So we redefined f_3 to be $f_3 - P_2$. We then proceed by induction. Given f_k, the difference $f_{k+1} - f_k$ is holomorphic in $\overline{\Delta(z, r_{k-1})}$, so we can find a polynomial P_{k+1} such that

$$\sup_{z\in\Delta(z,r_{k-1})} |f_{k+1}(z) - f_k(z) - P_{k+1}(z)| \leq 2^{-k}, \tag{3.15}$$

and we redefine f_{k+1} to be $f_{k+1} - P_{k_1}$.

The sequence of functions f_k will be a Cauchy sequence in any disc $\Delta(z, r')$, when $r' < r$. So there exists a uniform limit f. Fixing any m, then $f - f_m$ is then a uniform limit of holomorphic functions in $\Delta(z, r_{m-1})$, so is holomorphic by Corollary 2.7, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m/\partial \overline{z} \to \partial f/\partial \overline{z} = g, \tag{3.16}$$

and the proof is finished. □

To prove for a general domain, we require the following result.

Theorem 3.6 (Runge's approximation Theorem, first version)**.** *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z\in K} |f(z) - f_\epsilon(z)| < \epsilon, \tag{3.17}$$

and such that poles of f_ϵ are contained in $\mathbb{C} \setminus K$.

Proof. The proof is from [Sar07, Theorem IX.15], we just give an outline. From elementary arguments, there exists a contour $\gamma : S^1 \to U \setminus K$ such that $K \subset \text{Int}\Gamma \subset U$, which has winding number 1 around any point $z_0 \in K$. Note that K might have several components, so γ will also. Since the winding number is 1, by Cauchy's Integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw \tag{3.18}$$

for any $z \in K$. By dividing the plane into a sufficiently fine grid, we can assume that γ is piecewise smooth and $\gamma = \gamma_1 + \cdots \gamma_n$, with each γ_j a line segment parallel to one of the coordinate axes. Consider each term

$$f_k(z) = \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw \tag{3.19}$$

We can approximate this arbitrarily closely with a Riemann sum R_k, which will be of the form

$$\frac{c_1}{w_1 - z} + \cdots + \frac{c_l}{w_l - z}, \tag{3.20}$$

where the w_j are points on γ_k. Doing this for every γ_j, the proof is complete. $\square$

4 Lecture 4

4.1 Runge's Theorem

Theorem 4.1 (Runge's approximation Theorem, second version). *Let $K \subset \mathbb{C}$ be a compact subset, and $f \in \mathcal{O}(U)$ for some open set U with $K \subset U$. Let $S \subset \mathbb{C} \setminus K$ which contains at least one point from each connected component of $\mathbb{C} \setminus K$. Given any $\epsilon > 0$, there exists a rational function f_ϵ with*

$$\sup_{z \in K} |f(z) - f_\epsilon(z)| < \epsilon, \tag{4.1}$$

and such that poles of f_ϵ are contained in S. If $\mathbb{C} \setminus K$ is connected, the rational functions can be taken to be polynomials.

Proof. The proof is from [Sar07, Theorem IX.17]. In the proof of Theorem 3.6, each term in the approximation was of the form $c/(w - z)$, where $w \in \gamma$. Now choose a picewise linear path α from w to any point w_0 in the same connected component of $\mathbb{C} \setminus K$. Choose points w_i on α so that

$$|w_{i-1} - w_i| < \text{dist}(\gamma, K). \tag{4.2}$$

We show that any rational R_{j-1} function with a pole only at w_{j-1} may be uniformly approximated on K by a rational function R_j with a poles only at w_j. But this follows from consdering the Laurent series expansion of R_{j-1} centered at w_j: R_{j-1} is holomorphic in the region $U = \mathbb{C} \setminus \Delta(w_j, |w_j - w_{j-1}|)$, so the Laurent series of R_{j-1} centered at w_j converges uniformly on compact subsets of U. Then we can approximate R_{j-1} by a rational function with a pole only at w_j, uniformly on K, since K is a compact subset of U, which follows from (4.2).

If $\mathbb{C} \setminus K$ is connected, by the above argument, we can move the pole of the rational function R_j to a single point z_0 so that $K \subset \Delta(0, |z_0|)$. The Talyor series of R_j converges uniformly on K, so we can approximate by the partial sums of the Taylor series.

□

Theorem 4.2. *If $\Omega \subset \mathbb{C}$ is any domain, and $g \in C^\infty(\Omega)$, then there exists $f \in C^\infty(\Omega)$ with $\frac{\partial}{\partial \overline{z}} f = g$.*

Proof. We choose a sequence of compact sets $K_1 \subset K_2 \subset K_3 \subset \cdots$, so that $\overline{K_j} \subset \mathrm{Int}K_{j+1}$ and $\cup K_j = \Omega$. Note that $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, and we can assume that for large j, each component of $\mathbb{C} \setminus K_j$ contains a component of $\mathbb{C} \setminus \Omega$. Let $0 \leq \psi_j \in C_0^\infty(K_{j+1})$ and $\psi_k \equiv 1$ on K_j. Then $g_j = \psi_j g \in C_0^\infty(K_{j+1})$, and by Proposition 3.4, we can find $f_j \in C^\infty(\mathbb{C})$ such that $\partial f_j = g_j$.

We claim that we can choose $f_j \in C^\infty(\Omega)$ so that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z)| \leq 2^{-j}. \tag{4.3}$$

We proceed by induction. Given f_j, the difference $f_{j+1} - f_j$ is holomorphic in

$$U \equiv \mathrm{Int}K_j \supset K = \overline{K_{j-1}}. \tag{4.4}$$

We have $\mathbb{C} \setminus \Omega \subset \mathbb{C} \setminus K_j$, so by Theorem 4.1, there exists a rational function R_{j+1} such that it poles are in $\mathbb{C} \setminus \Omega$ and such that

$$\sup_{z \in K_{j-1}} |f_{j+1}(z) - f_j(z) - R_{j+1}(z)| \leq 2^{-j}, \tag{4.5}$$

and we redefine f_{j+1} to be $f_{j+1} - P_{j-1}$.

The sequence of functions f_j will be a Cauchy sequence in any subset K_m for fixed m. So there exists a limit f, with uniform convergence on compact subsets. Fixing any m, then $f - f_m$ is then a uniform limit of holomorphic functions in K_{m-1}, so is holomorphic by Corollary 2.7, and the convergence is in C^1 of any compact subset. So we can differentiate to show that

$$\partial f_m / \partial \overline{z} \to \partial f / \partial \overline{z} = g, \tag{4.6}$$

and the proof is finished.

□

4.2 Meromorphic Functions

The above solution of the inhomogeneous Cauchy-Riemann equations has many corollaries. We give some applications to the theory of meromorphic functions. References for this section are [Hör90, Chapter 1] and [Eps91, Section 1.6].

Definition 4.3. For a domain $\Omega \subset \mathbb{C}$, we say that $f \in \mathcal{M}(\Omega)$, or f is meromorphic in Ω, if there is an open covering U_j of Ω such that $f|_{U_j} = \frac{g_j}{h_j}$, where g_j and h_j are in $\mathcal{O}(U_j)$.

Note this is equivalent to saying that f has a Laurent series expansion near any $z_0 \in \Omega$ with only finitely many negative terms. That is we have

$$f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k. \tag{4.7}$$

The finite sum of the negative terms is called the principal part of f at z_0. Note the set of poles will be some discrete subset $\{w_j\}$ of Ω.

Theorem 4.4 (Mittag-Leffler)**.** *Let Ω be a domain in $\mathbb{C}$ and $\{w_j\}$ a discrete subset of Ω. Let P_j be any principal sum at w_j. Then there exists a meromorphic function $h \in \mathcal{M}(\Omega)$ such that the principal part of h at w_j is P_j.*

Proof. Let ψ_j be a cutoff function supported in a small neighborhood of w_j which doesn't contain any other points in the discrete subset. Consider $g = \sum_j \psi_j P_j$. Then $\partial g / \partial \bar{z} \in C^{\infty}(\Omega)$. By Theorem 4.2, there exists a solution $f \in C^{\infty}(\Omega)$ of $\partial f / \partial \overline{z} = \partial g / \partial \bar{z}$. Then $h = g - f$ satisfies $\partial h / \partial \overline{z} = 0$, and the principal part of h at w_j is P_j. □

Definition 4.5. The order of $f \in \mathcal{M}(\Omega)$ at $z_0 \in \Omega$ is the least integer n such that the coefficient $a_n \neq 0$ in (4.7)

Theorem 4.6 (Weierstrass)**.** *Let Ω be a domain in $\mathbb{C}$, $\{w_j\}$ a discrete subset of Ω, and $n_j \in \mathbb{Z}$. Then there exists a meromorphic function $f \in \mathcal{M}(\Omega)$ with the order of f at w_j equal to n_j.*

Proof. We cover Ω by discs $U_i = \Delta(z_i, r_i)$ such that each w_i is contained in exactly one of these discs. Define the function $f_i = (z - w_j)^{n_j}$ if $w_j \in U_i$, and let $f_i = 1$ if U_i doesn't contain any of the discrete points. On $U_i \cap U_j$, let $f_{ij} = f_i / f_j$. Then $f_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ is a non-vanishing holomorphic function. Since $U_i \cap U_j$ is simply-connected, we can define $g_{ij} = \log f_{ij}$. Note that $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ is only defined up to adding an integer multiple of $2\pi i$. Since

$$f_{ik} = \frac{f_i}{f_k} = \frac{f_i}{f_j} \frac{f_j}{f_k} = f_{ij} f_{jk}, \tag{4.8}$$

the g_{ij} satisfy on triple intersections $U_i \cap U_j \cap U_k$

$$g_{ij} - g_{ik} + g_{jk} = 2\pi i n_{ijk}, \tag{4.9}$$

where $n_{ijk} \in \mathbb{Z}$. The n_{ijk} satisfy the condition on intersections $U_i \cap U_j \cap U_k \cap U_l$,

$$n_{jkl} - n_{ikl} + n_{ijl} - n_{ijk} = 0. \tag{4.10}$$

So $n_{ijk} \in H^2(\mathfrak{U}, \mathbb{Z})$ defines a Čech 2-cocycle. Since $\mathfrak{U}$ is a good cover of Ω, $H^2(\mathfrak{U}, \mathbb{Z}) = H^2(\Omega, \mathbb{Z})$. However, since Ω is a domain in $\mathbb{C}$, it is in particular a non-compact 2-manifold, so this latter group vanishes. Therefore, there exists integers n_{ij} such that

$$n_{ijk} = n_{jk} - n_{ik} + n_{ij}. \tag{4.11}$$

Now we define $g'_{ij} = g_{ij} - 2\pi i n_{ij}$, which now satisfy

$$g'_{ij} - g'_{ik} + g'_{jk} = 0 \tag{4.12}$$

Now choose a partition of unity ψ_i subordinate to U_i, and define

$$h_i = \sum_j g'_{ij}\psi_j, \tag{4.13}$$

which satisfies $h_i \in C^\infty(U_i)$. On $U_i \cap U_j$, we have

$$h_i - h_j = \sum_k (g'_{ik} - g'_{jk})\psi_k = \sum_k g'_{ij}\psi_k = g'_{ij}. \tag{4.14}$$

We then have that

$$\frac{\partial}{\partial \overline{z}}(h_i - h_j) = \frac{\partial}{\partial \overline{z}} g'_{ij} = 0, \tag{4.15}$$

So we can define $h \in C^\infty(\Omega)$ by letting

$$h|_{U_i} = \frac{\partial h_i}{\partial \overline{z}} \tag{4.16}$$

By Theorem 4.2, we can solve the equation $\partial f/\partial \overline{z} = h$ for $f \in C^\infty(\Omega)$. Then we redefine $h'_i = h_i - f$. These now satisfy $h'_i \in \mathcal{O}(U_i)$, and $h'_i - h'_j = g'_{ij}$.

So going back to the above, we define $f'_i = e^{-h'_i} f_i$. On overlaps, we now have

$$\frac{f'_i}{f'_j} = \frac{e^{-h'_i} f_i}{e^{-h'_j} f_j} = e^{-h'_i + h'_j}\frac{f_i}{f_j} = e^{-g_{ij'}}\frac{f_i}{f_j} = e^{-g_{ij}}\frac{f_i}{f_j} = 1, \tag{4.17}$$

so the f'_i patch together to define $f \in \mathcal{M}(\Omega)$. Since we only multiplied the f_i by a non-zero holomorphic function, the order of f at w_j is equal to n_j. □

Corollary 4.7. *If $f \in \mathcal{M}(\Omega)$, then there exists $g, h \in \mathcal{O}(\Omega)$ such that $f = g/h$ in all of Ω.*

Proof. If f has poles of order n_j at w_j, then by the Weierstrass Theorem, there exists a holomorphic function $h \in \mathcal{O}(\Omega)$ which has a zero of order n_j at w_j. Then $g = hf$ has no poles so $g \in \mathcal{O}(\Omega)$. □

5 Lecture 5

5.1 Power series in several variables

We review some basic facts about power series in several variables. Some good references for this material are [FG02, Chapter 1], [JP08, Chapter 1], or [KP02, Chapter 2.1], We write a point $z = (z_1, \dots, z_n)$. The open polydisc with polyradius $r = (r_1, \dots, r_n)$ about a point $z_0 = (z_1^0, \dots, z_n^0)$ is the set

$$\Delta(z_0, r) = \{z \mid |z_j - z_j^0| < r_j,\ j = 1 \dots n\}. \tag{5.1}$$

We will let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ denote a multi-index, where $\mathbb{Z}_+$ denotes the non-negative integers. Define

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \tag{5.2}$$

$$|z|^\alpha = |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \tag{5.3}$$

$$\alpha! = \alpha_1! \cdots \alpha_n! \tag{5.4}$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{5.5}$$

Definition 5.1. The series $\sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha (z - z_0)^\alpha$ converges at z if some rearrangement converges, that is, give some bijection $\phi : \mathbb{Z}_+ \to (\mathbb{Z}_+)^n$, the series

$$\sum_{j=0}^{\infty} a_{\phi(j)} (z - z_0)^{\phi(j)} \tag{5.6}$$

converges. The *domain of convergence* of the power series is the interior of the set of points of convergence.

In 1 variable we know that domains of convergence are discs. Regions of convergence in several variable can be more complicated.

Example 5.2. The domain of convergence of the series $\sum_{k=0}^{\infty} z^k w^k$ is $\{(z, w) \mid |zw| < 1\}$.

Example 5.3 (Boas)**.** The series $\sum_{n=1}^{\infty} z^n w^{n!}$ converges in the 3 sets

$$U_1 = \{(z, w) \mid |w| < 1\},\ U_2 = \{(0, w)\},\ U_3 = \{(z, w) \mid |z| < 1 \text{ and } |w| = 1\}. \tag{5.7}$$

Only U_1 is an open set; the sets U_2 and U_3 are 1 dimensional, and are not domains. The domain of convergence is U_1.

Lemma 5.4 (Abel)**.** *If $\sum_\alpha a_\alpha z^\alpha$ (centered at $z = 0$) converges at the point z' then it converges uniformly and absolutely for any point z of the form $z_j = \rho_j z_j'$ where $|\rho_j| < 1$. Furthermore, a point p belongs to the domain of convergence of the power series $\sum_\alpha a_\alpha z^\alpha$ if and only if there exists a neighborhood U of p, a constant C, and $r < 1$ such that $|a_\alpha z^\alpha| \le C r^{|\alpha|}$ for all $z \in U$.*

Proof. Since the series converges at the point z', the terms must be bounded, so there exists a constant C so that $|a_\alpha||z'|^\alpha \le C$. Let $\rho = \max\{|\rho_1|, \dots, |\rho_n|\} < 1$, and consider any point $z = (z_1, \dots, z_n)$ so that $|z_j| < \rho |z_j'|$. We then have

$$|a_\alpha||z|^\alpha \le |a_\alpha| \rho^{|\alpha|} |z'|^\alpha \le C \rho^{|\alpha|}. \tag{5.8}$$

So given an integer $N > 0$, we have

$$\begin{aligned} \sum_{|\alpha| \le N} |a_\alpha||z|^\alpha &= \sum_{j=0}^{N} \sum_{|\alpha| = j} |a_\alpha||z|^\alpha \\ &\le \sum_{j=0}^{N} \sum_{|\alpha| = j} C \rho^j \end{aligned} \tag{5.9}$$

How many multi-indices of length j are there? This is counting the number of non-negative integer solutions of

$$\alpha_1 + \cdots + \alpha_n = j. \tag{5.10}$$

To see this, let $\alpha' = \alpha_1 + 1$, then we are interested in the number of positive integer solutions to

$$\alpha'_1 + \cdots + \alpha'_n = j + n. \tag{5.11}$$

So we have a total of $j + n$ integers, dividing this up into n integers is the same as putting $n - 1$ partitions somewhere in the spaces between them, so the number is

$$\binom{j+n-1}{n-1}. \tag{5.12}$$

Continuing with the above calculation,

$$\begin{aligned}
\sum_{|\alpha| \leq N} |a_\alpha||z|^\alpha &\leq C \sum_{j=0}^{N} \binom{j+n-1}{n-1} \rho^j \\
&= C \sum_{j=0}^{N} \frac{(j+n-1)!}{j!(n-1)!} \rho^j \\
&= \frac{C}{(n-1)!} \sum_{j=0}^{N} (j+n-1)(j+n-2)\cdots(j+1)\rho^j \\
&\leq C_n \sum_{j=0}^{N} j^n \rho^j.
\end{aligned} \tag{5.13}$$

Applying the ratio test, we have

$$\lim_{j\to\infty} \frac{(j+1)^n \rho^{j+1}}{j^n \rho^j} = \lim_{j\to\infty} \left(\frac{j+1}{j}\right)^n \rho = \rho, \tag{5.14}$$

so the series converges provided $\rho < 1$.

If p belongs to the domain of convergence, then by definition the series converges in a neighborhood of p. Then by the first part it converges in some polydisc around the origin containing z, and we follow the first part of the proof.

□

Definition 5.5. We say that f is complex analytic in U if for each $z_0 \in U$, there exists a power series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n_+} a_\alpha (z - z_0)^\alpha \tag{5.15}$$

which converges absolutely and uniformly in a polydisc $\Delta(z_0, \hat{\epsilon})$ around z_0, for some positive polyradius $\hat{\epsilon}$.

5.2 Cauchy's formula in several complex variables

Basic reference are [GH78, Hör90, Nog16].

Definition 5.6. We say that f is holomorphic in U if it is $C^1(U)$ and satisfies the Cauchy-Riemann equations,

$$\frac{\partial f}{\partial \bar{z}^j} = 0,\ j = 1 \cdots n. \tag{5.16}$$

Proposition 5.7. *Let U be an open set in $\mathbb{C}$. Then f is holomorphic in U if and only if f is complex analytic in U.*

Proof. Consider $n = 2$, the higher-dimensional case is similar. We assume that $U = \Delta(0, r_1) \times \Delta(0, r_2)$ is a polydisc, and $f \in C^1(\overline{U})$. If f is holomorphic in U, then for fixed z_1, the slice $f(z_1, z_2)$ is a 1-variable holomorphic function for $z_2 \in \Delta(0, r_2)$. This holds similarly for the other variable, so the Cauchy-Pompieu formula applied twice yields

$$\begin{aligned} f(z_1, z_2) &= \frac{1}{2\pi i} \int_{|w_2|=r_2} \frac{f(z_1, w_2)dw}{w_2 - z_2} \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{|w_2|=r_2} \int_{|w_1|=r_1} \frac{f(w_1, w_2)dw}{(w_1 - z_1)(w_2 - z_2)}. \end{aligned} \tag{5.17}$$

For any $(z_1^0, z_2^0) \in U$, we expand

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0 + z_1^0 - z_1} = \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \frac{1}{1 - \frac{z_1 - z_1^0}{w_1 - z_1^0}} \tag{5.18}$$

$$= \frac{1}{w_2 - z_2} \frac{1}{w_1 - z_1^0} \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0}\right)^k \tag{5.19}$$

$$= \frac{1}{(w_1 - z_1^0)(w_2 - z_2^0)} \sum_{l=0}^{\infty} \left(\frac{z_2 - z_2^0}{w_2 - z_2^0}\right)^l \sum_{k=0}^{\infty} \left(\frac{z_1 - z_1^0}{w_1 - z_1^0}\right)^k \tag{5.20}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}. \tag{5.21}$$

We next show that we are justified in the last step. Let $(z_1^0, z_2^0) \in \Delta(0, r_1') \times \Delta(0, r_2')$ with $r_1' < r_1$ and $r_2' < r_2$. Then we have $|w_1 - z_1^0| > r_1 - r_1'$, and $|w_2 - z_2^0| > r_2 - r_2'$. For $|z_1 - z_1^0| < (r_1 - r_1')/2$ and $|z_2 - z_2^0| < (r_2 - r_2')/2$, we then have

$$|a_{kl}| = \left|\frac{(z_1 - z_1^0)^k (z_2 - z_2^0)^l}{(w_1 - z_1^0)^{k+1} (w_2 - z_2^0)^{l+1}}\right| \leq \frac{1}{(r_1 - r_1')(r_2 - r_2')} 2^{-k} 2^{-l}, \tag{5.22}$$

so the sum converges absolutely and uniformly in any smaller polydisc by Lemma 5.4. Interchanging the integration and summation in (5.17) then yields a power series expansion for f.

The converse is similar to the 1-variable case. If f has a power series expansion, then each term in the power series satisfies the Cauchy integral formula (5.17). So then f does also by uniform convergence. Then we can differentiate under the integral to see that f is holomorphic. For more details, see [GH78, page 6]. $\square$

Remark 5.8. Note that the integral in (5.17) is just over a 2-dimensional torus contained in the boundary of the polydisc. The topological boundary of the polydisc is 3-dimensional, but it is not a manifold, it is $\partial(\Delta \times \Delta) = S^1 \times \Delta \cup \Delta \times S^1$, and these 2 sets intersect along the torus.

Similar to the 1 variable case, we have the following corollaries. In the proofs, we just consider the case of 2 dimensions, the higher dimensional cases are similar.

Corollary 5.9. *If f is analytic at z_0 then f is infinitely differentiable at z_0 and*

$$a_\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f(z_0)}{\partial z^\alpha} = \frac{1}{\alpha_1!\alpha_2!}\frac{\partial^{\alpha_1+\alpha_2} f(z_0)}{\partial z_1^{\alpha_1}\partial z_2^{\alpha_2}} \tag{5.23}$$

Corollary 5.10 (The maximum principle)**.** *Let $\Omega \subset \mathbb{C}^n$, and $f \in \mathcal{O}(\Omega) \cap C^0(\overline{\Omega})$. Then $|f|$ does not assume its maximum at an interior point unless f is constant.*

Proof. Assume that $|f|$ attains local maximum at some interior point $z_0 \in \Omega$. Since f is holomorphic, it admits a power series expansion $f(z_0) = \sum_\alpha a_\alpha (z - z_0)^\alpha$ which converges uniformly in the closure of a polydisc $\Delta(z_0, r) = \Delta(z_0^1, r_1) \times \Delta(z_0^2, r_2)$. Write $S^1 \times S^1 = \partial\Delta(z_0^1, r_1) \times \partial\Delta(z_0^2, r_2)$. Then $|f(z)|^2 \le |f(z_0)|^2$ for $z \in \Delta(z_0, r)$, so

$$|f(z_0)|^2 \ge \left(\frac{1}{2\pi}\right)^2 \int_{S^1\times S^1} |f(z_1, z_2)|^2 dV_{S^1\times S^1} = \sum_{k,l=0}^{\infty} |a_{kl}|^2 r_1^{2k} r_2^{2l} \ge |a_{00}|^2 = |f(z_0)|^2. \tag{5.24}$$

This implies that a_{kl} is zero except for a_{00} and therefore f is constant. $\square$

Corollary 5.11. *Let $K \subset \Omega$ be a compact subset. Then there exist constants $C_{|\alpha|}$, depending only upon K and Ω such that*

$$\sup_{z\in K}\left|\frac{\partial^\alpha f(z)}{\partial z^\alpha}\right| \le C_{|\alpha|} \sup_{z\in\Omega} |f(z)|. \tag{5.25}$$

for all $u \in \mathcal{O}(\Omega)$.

Proof. Again, we just consider the case of 2 dimensions, the higher dimensional case is similar. Fix $(z_1^0, z_2^0) \in \Omega$, and let $\Delta(z^0, (r_1, r_2)) \subset \Omega$ be a polydisc. Then for $(z_1, z_2) \in \Delta(z^0, (r_1', r_2'))$ with $r_1' < r_1$ and $r_2' < r_2$, we have

$$f(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{kl}(z_1 - z_1^0)^k (z_2 - z_2^0)^l, \tag{5.26}$$

where

$$a_{kl} = \left(\frac{1}{2\pi i}\right)^2 \int_{|w_2|=r_2}\int_{|w_1|=r_1} \frac{f(w_1, w_2)dw}{(w_1 - z_1^0)^{k+1}(w_2 - z_2^0)^{l+1}}. \tag{5.27}$$

We then get *Cauchy's inequalities*

$$\left|\frac{\partial^{k+l} f}{\partial z_1^k \partial z_2^l}(z_1^0, z_2^0)\right| = k!l!|a_{kl}| \le \frac{k!l!r_1 r_2}{(r_1 - r_1')^k (r_2 - r_2')^l} \sup_{w=(w_1,w_2),|w_1|=r_1,|w_2|=r_2} |f(w)|. \tag{5.28}$$

The claim follows by covering K with finitely many polydiscs contained in Ω. $\square$

The following corollaries are proved exactly as before.

Corollary 5.12. *If $u_n \in \mathcal{O}(\Omega)$ and $u_n \to u$ converges uniformly to u in the C^0 norm as $n \to \infty$ on compact subsets, then $u \in \mathcal{O}(\Omega)$.*

Corollary 5.13. *If $u_n \in \mathcal{O}(\Omega)$ and $|u_n|$ is uniformly bounded on every compact subset $K \subset \Omega$, then some subsequence u_{n_j} converges uniformly on compact subsets to a limit $u \in \mathcal{O}(\Omega)$.*

6 Lecture 6

6.1 The operators ∂ and $\overline{\partial}$ in $\mathbb{C}^n$

Using the coordinates

$$(z^1, \dots, z^n) = (x^1 + iy^1, \dots, x^n + iy^n), \tag{6.1}$$

recall that that $T^{1,0}$ is spanned by

$$\frac{\partial}{\partial z^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\Big), \tag{6.2}$$

$T^{0,1}$ is spanned by

$$\frac{\partial}{\partial \bar{z}^j} \equiv \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\Big), \tag{6.3}$$

$\Lambda^{1,0}$ is spanned by

$$dz^j \equiv dx^j + idy^j, \tag{6.4}$$

and $\Lambda^{0,1}$ is spanned by

$$d\bar{z}^j \equiv dx^j - idy^j, \tag{6.5}$$

for $j = 1 \dots n$.

We define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{6.6}$$

We define $\Omega^k, \Omega^k_{\mathbb{C}}, \Omega^{p,q}$ to be the space of sections of $\Lambda^k, \Lambda^k \otimes \mathbb{C}, \Lambda^{p,q}$, respectively. So we have that

$$\Omega^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}. \tag{6.7}$$

If $\alpha \in \Omega^{p,q}(U)$, then we can write

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\bar{z}^J, \tag{6.8}$$

where I and J are multi-indices of length p and q, respectively, and $\alpha_{I,J} : U \to \mathbb{C}$ are complex-valued functions.

The real operator $d : \Omega^k_{\mathbb{R}} \to \Omega^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{6.9}$$

by complexification.

Proposition 6.1. *We have*

$$d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}. \tag{6.10}$$

Proof. Recall that

$$df^j = \sum_k \frac{\partial f^j}{\partial z^k} dz^k + \sum_k \frac{\partial f^1}{\partial \overline{z}^k} d\overline{z}^k. \tag{6.11}$$

Applying d to (6.8), we obtain

$$d\alpha = \sum_{I,J} \Big(\sum_k \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k + \sum_k \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \Big) \wedge dz^I \wedge d\overline{z}^J, \tag{6.12}$$

and we are done. $\square$

We can therefore define operators

$$\partial : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{6.13}$$

$$\overline{\partial} : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{6.14}$$

by

$$\partial\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial z^k} dz^k \wedge dz^I \wedge d\overline{z}^J \tag{6.15}$$

$$\overline{\partial}\alpha = \sum_{I,J,k} \frac{\partial \alpha_{I,J}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J. \tag{6.16}$$

using (9.17) and we have

$$\partial|_{\Omega^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \tag{6.17}$$

$$\overline{\partial}|_{\Omega^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \tag{6.18}$$

Corollary 6.2. *We have $d = \partial + \overline{\partial}$ which satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{6.19}$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2. \tag{6.20}$$

If we plug in a form of type (p, q) the first term is of type $(p+2, q)$, the middle terms are of type $(p+1, q+1)$, and the last term is of type $(p, q+2)$. Since (9.17) is a direct sum, the claim follows. $\square$

6.2 Dolbeault cohomology

Definition 6.3. Let $U \subset \mathbb{C}^n$ be a domain. For $0 \leq p, q \leq n$, the (p,q) Dolbeault cohomology group is

$$H^{p,q}_{\overline{\partial}}(U) = \frac{\{\alpha \in \Omega^{p,q}(U) | \overline{\partial}\alpha = 0\}}{\overline{\partial}(\Omega^{p,q-1}(U))}, \tag{6.21}$$

where the forms have C^∞ regularity.

The Dolbeault cohomology groups enjoy the following functorality properties.

Proposition 6.4. *Let $U \subset \mathbb{C}^m, V \subset \mathbb{C}^n, W \subset \mathbb{C}^l$ be domains. Let $f : U \to V$ be a C^1 mapping which is holomorphic, that is*

$$f_* \circ J_0 = J_0 \circ f_*. \tag{6.22}$$

Then there are induced mappings

$$f^* : H^{p,q}(V) \to H^{p,q}(U). \tag{6.23}$$

If $g : V \to W$ is C^1 holomorphic, then so is $g \circ f : U \to W$ and

$$(g \circ f)^* = f^* \circ g^* : H^{p,q}(W) \to H^{p,q}(U). \tag{6.24}$$

In particular, if f is a biholomorphism (one-to-one, onto, with holomorphic inverse), then the Dolbeault cohomologies of U and V are isomorphic.

Proof. The equation (6.22) implies that

$$f^* : \Omega^{p,q}(V) \to \Omega^{p,q}(U). \tag{6.25}$$

To see this, let $\alpha^{p,q} \in \Omega^{p,q}(V)$, then for vectors $X_1, \ldots, X_{p+q}$ we have

$$f^*\alpha^{p,q}(X_1, \ldots, X_{p+q}) = \alpha^{p,q}(f_*X_1, \ldots, f_*X_{p+q}) \tag{6.26}$$

Note that if $X \in T^{1,0}(U)$, then $J_U X = iX$, so then

$$J_V f_* X = f_* J_U X = f_* i X = i f_* X, \tag{6.27}$$

therefore $f_*X \in T^{1,0}(V)$. Similarly, if $X \in T^{0,1}(X)$ then $f_*X \in T^{0,1}(V)$. If more than p of the X_j are of type $(1,0)$ or more than q of the X_j are of type $(1,0)$, then the same is true for the f_*X_j, and the claim follows.

We also know that the exterior derivative commutes with pullback,

$$d_U \circ f^* = f^* \circ d_V. \tag{6.28}$$

This is equivalent to

$$(\partial_U + \overline{\partial}_U) \circ f^* = f^* \circ (\partial_V + \overline{\partial}_V) \tag{6.29}$$

If we plug in $\alpha^{p,q} \in \Omega^{p,q}(V)$, we have 2 equations

$$\partial_U \circ f^* \alpha^{p,q} = f^* \circ \partial_V \alpha^{p,q} \tag{6.30}$$
$$\overline{\partial}_U \circ f^* \alpha^{p,q} = f^* \circ \overline{\partial}_V \alpha^{p,q} \tag{6.31}$$

The second equation implies that f^* induces a well-defined mapping on cohomology $f^* : H^{p,q}(V) \to H^{p,q}(U)$ by the following. If $[\alpha^{p,q}] \in H^{p,q}(V)$ is represented by a form $\alpha^{p,q}$, such that $\overline{\partial}_V \alpha^{p,q} = 0$, then we have

$$\overline{\partial}_U f^* \alpha^{p,q} = f^* \overline{\partial}_V \alpha^{p,q} = f^* 0 = 0, \tag{6.32}$$

so we can define $f^*[\alpha^{p,q}] = [f^* \alpha^{p,q}]$, that is, map to the cohomology class of the pullback of any representative form. To see that this is well-defined,

$$f^*(\alpha^{p,q} + \overline{\partial}_V \beta^{p,q-1}) = f^* \alpha^{p,q} + f^* \overline{\partial}_V \beta^{p,q-1} = f^* \alpha^{p,q} + \overline{\partial}_U f^* \beta^{p,q-1}, \tag{6.33}$$

so we have

$$[f^*(\alpha^{p,q} + \overline{\partial}_V \beta^{p,q-1})] = [f^* \alpha^{p,q} + \overline{\partial}_U f^* \beta^{p,q-1}] = [f^* \alpha^{p,q}]. \tag{6.34}$$

The next part follows since

$$(g \circ f)^* = f^* \circ g^* \tag{6.35}$$

holds on the level of forms. Finally, if f is a pseudo-biholomorphism, then f^{-1} exists and is pseudo-holomorphic, so we have

$$f \circ f^{-1} = id_V, \quad f^{-1} \circ f = id_U, \tag{6.36}$$

and the induced mappings on cohomology satisfy

$$f^* \circ (f^{-1})^* = id_{H^{p,q}(U)}, \quad (f^{-1})^* \circ f^* = id_{H^{p,q}(V)}, \tag{6.37}$$

□

Definition 6.5. A form $\alpha \in \Omega^{p,0}(U)$ is *holomorphic* if $\overline{\partial} \alpha = 0$.

Remark 6.6. We only talk about forms of type $(p, 0)$ being holomorphic, we never call a (p, q)-form holomorphic if $q > 0$. Also, we have (trivially)

$$H^{p,0}(U) = \{\alpha \in \Omega^{p,0}(U) \mid \alpha \text{ is holomorphic}\}. \tag{6.38}$$

Proposition 6.7. *A p-form $\alpha \in \Omega^{p,0}(U)$ is holomorphic if and only if it can be written as*

$$\alpha = \sum_{|I|=p} \alpha_I dz^I, \tag{6.39}$$

where the $\alpha_I : U \to \mathbb{C}$ are holomorphic functions.

Proof. We have

$$\overline{\partial}\alpha = \sum_{|I|=p,k} \frac{\partial \alpha_I}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I. \tag{6.40}$$

So $\overline{\partial}\alpha = 0$ if and only if the α_I are holomorphic. $\square$

Remark 6.8. So for $U \subset \mathbb{C}^n$ a domain, $\dim_{\mathbb{C}} H^{p,0}(U) = \infty$ is always infinite-dimensional for $0 \leq p \leq n$, in particular because any polynomial function in the z-variables is holomorphic.

Example 6.9. Let's review the case of a domain $U \subset \mathbb{C}$. First, $H^{0,0}_{\overline{\partial}}(U) = \mathcal{O}(U)$. Theorem 4.2 shows that $H^{0,1}_{\overline{\partial}}(U) = \{0\}$. The space $H^{1,0}_{\overline{\partial}}(U)$ consists of holomorphic 1-forms, but since $n = 1$, any holomorphic 1-form is of the form $f(z)dz$, where $f \in \mathcal{O}(U)$. So $H^{1,0}_{\overline{\partial}}(U) \cong \mathcal{O}(U)$. Finally,

$$H^{1,1}_{\overline{\partial}}(U) = \frac{\operatorname{Ker} \overline{\partial} : \Omega^{1,1} \to \Omega^{1,2}}{\operatorname{Image} \overline{\partial} : \Omega^{1,0} \to \Omega^{1,1}} = \frac{g dz \wedge d\overline{z}}{(\partial f / \partial \overline{z}) d\overline{z} \wedge dz} = \{0\}, \tag{6.41}$$

which also follows from Theorem 4.2.

7 Lecture 7

7.1 The $\overline{\partial}$-equation for $(0,1)$-forms and Hartogs' Theorem

A reference for this section is [HL84, Section 1.2]. For $n \geq 2$, and $g \in \Omega^{0,1}(U)$, the equation $\overline{\partial} f = g$ is not always solvable. This follows from (9.30): applying $\overline{\partial}$ yields a compatibility condition $\overline{\partial} g = 0$. The following is in sharp contrast to the case $n = 1$.

Proposition 7.1. *Let $g \in \Omega^{0,1}_0(\mathbb{C}^n)$ (compact support) have C^∞ regularity and satisfy $\overline{\partial} g = 0$. Then there exists a smooth $f \in \Omega^0_0(\mathbb{C}^n)$ (also having compact support) with $\overline{\partial} f = g$. Furthermore, $f \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus supp(g)$.*

Proof. We write $g = \sum_{j=1}^n g_j d\overline{z}^j$. Define

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(w, z_2, \ldots, z_n)}{w - z_1} dw \wedge d\overline{w}. \tag{7.1}$$

The integral is defined since g_1 has compact support. Make the change of variable $\xi = w - z_1$, and we can write f as

$$f(z_1, \ldots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g_1(\xi + z_1, z_2, \ldots, z_n)}{\xi} d\xi \wedge d\bar{\xi}. \tag{7.2}$$

This shows that we can differentiate under the integral sign to conclude that f has C^∞ regularity. Furthermore,

$$\begin{aligned}
\frac{\partial f(z_1, \ldots, z_n)}{\partial \overline{z}^1} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(\xi + z_1, z_2, \ldots, z_n)}{\partial \overline{z}^1} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(\xi + z_1, z_2, \ldots, z_n)}{\partial \bar{\xi}} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(w, z_2, \ldots, z_n)}{\partial \overline{w}} \frac{1}{w - z_1} dw \wedge d\overline{w} = g_1(z_1, \ldots, z_n),
\end{aligned} \tag{7.3}$$

by the Cauchy-Pompieu formula applied to a large ball containing the support of g. The condition that $\overline{\partial} g = 0$ means that

$$0 = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial g_j}{\partial \overline{z}_k} d\overline{z}^k \wedge d\overline{z}^j, \tag{7.4}$$

so

$$\frac{\partial g_j}{\partial \overline{z}_k} = \frac{\partial g_k}{\partial \overline{z}_j} \tag{7.5}$$

for all $1 \leq j, k \leq n$. Then differentiating (7.6) for $j \geq 2$, we obtain

$$\begin{aligned} \frac{\partial f(z_1, \ldots, z_n)}{\partial \overline{z}_j} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_1(\xi + z_1, z_2, \ldots, z_n)}{\partial \overline{z}^j} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_j(\xi + z_1, z_2, \ldots, z_n)}{\partial \overline{z}^1} \frac{1}{\xi} d\xi \wedge d\bar{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial g_j(w, z_2, \ldots, z_n)}{\partial \overline{w}} \frac{1}{w - z_1} dw \wedge d\overline{w} = g_j(z_1, \ldots, z_n). \end{aligned} \tag{7.6}$$

So the equation $\overline{\partial} f = g$ is satisfied everywhere. Finally, since g has compact support, it follows that f is holomorphic on the complement of a large ball $B_r(0)$ containing the support of g. But (7.1) shows that f vanishes when $\max\{|z_2|, \ldots, |z_n|\} > r$. Therefore f is a holomorphic function on $\mathbb{C}^n \setminus \overline{B_r(0)}$ which vanishes on the open subset $V = \{\max\{|z_2|, \ldots, |z_n|\} > r\}$. By unique continuation, $f \equiv 0$ on the unbounded component of $\mathbb{C}^n \setminus \mathrm{supp}(g)$. □

Theorem 7.2 (Hartogs). *Let $n \geq 2$, U a domain, and $K \subset U$ a compact subset of U such that $U \setminus K$ is connected. Then if $u \in \mathcal{O}(U \setminus K)$, there exists $\tilde{u} \in \mathcal{O}(U)$ with $\tilde{u}|_{U \setminus K} = u$.*

Proof. Let $0 \leq \chi \in C_0^\infty(U)$ and $\chi \equiv 1$ on K. Define $g = \overline{\partial}(\chi \cdot u)$. Since

$$\overline{\partial}(\chi u) = u \overline{\partial}(\chi) + \chi \overline{\partial}(u) = u \overline{\partial}(\chi) + 0, \tag{7.7}$$

we see that g extends smoothly to U, that is, $g \in \Omega_0^{0,1}(U)$, and $\overline{\partial} g = 0$. By Proposition 7.1, there exists $f \in C_0^\infty(\mathbb{C}^n)$ with $\overline{\partial} f = g$. So then we let $\tilde{u} = (1 - \chi) u + f$. This satisfies

$$\overline{\partial} \tilde{f} = -g + \overline{\partial}(f) = 0, \tag{7.8}$$

so $\tilde{u} \in \mathcal{O}(U)$. Let V denote the unbounded component of the complement of the support of χ. Since $\mathrm{supp}(g) \subset \mathrm{supp}(\chi)$, from Proposition 7.1, we have that $f \equiv 0$ in V, so $\tilde{u} = u$ in $U \cap V$. But since $U \setminus K$ is connected and $V \cap (U \setminus K) \neq \emptyset$, we have $\tilde{u} = u$ in $U \setminus K$ from unique continuation. □

Example 7.3. For example, point singularities are removable for $n \geq 2$. Even polydics are removable: if u is holomorphic on $\Delta \setminus \Delta'$, where $\Delta' \subset \Delta$ are polydiscs with $\overline{\Delta'} \subset \Delta$, then u extends to a holomorphic function on Δ. Same for $B_{r_1}(0) \subset B_{r_2}(0)$ with $r_1 < r_2$.

7.2 Dolbeault cohomology of a polydisc

Some references for this section are [GH78, Section 0.2] or [Nog16, Section 3.6].

Proposition 7.4. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), and $\omega \in \Omega^{p,q}(U)$ satisfies $\overline{\partial}\omega = 0$ for $q \geq 1$, then given any polyradius $s < r$, there exists $\eta \in \Omega^{p,q-1}(\Delta(r))$ with $\overline{\partial}\eta = \omega$ satisfied in $\Delta(s)$.*

Proof. Step 1: reduce to case of $\Omega^{0,q}$. If $\omega \in \Omega^{p,q}(U)$,

$$\omega = \sum_{|I|=p,|J|=q} \omega_{IJ} dz^I \wedge d\overline{z}^J. \tag{7.9}$$

Define

$$\omega_I = \sum_{|I|=p} \omega_{IJ} d\overline{z}^j. \tag{7.10}$$

Then $\omega_I \in \Omega^{0,q}$, and $\overline{\partial}\omega_I = 0$. If $\omega_I = \overline{\partial}\eta_I$, then

$$\overline{\partial}(dz^I \wedge \eta_I) = (-1)^p dz^I \wedge \overline{\partial}\eta_I = (-1)^p dz^I \wedge \omega_I, \tag{7.11}$$

and we are done with Step 1.

Step 2. Given $s < r$, if $\omega \in \Omega^{0,q}(\Delta(r))$ and $\overline{\partial}\omega = 0$ in $\Delta(r)$, then there exists $\eta \in \Omega^{0,q-1}(\Delta(r))$ with $\overline{\partial}\eta = \omega$ satisfied in $\Delta(s)$. Choose cutoff functions $0 \leq \chi_j(t) \leq 1$ so that

$$\chi_i(t) = \begin{cases} 1 & t \leq s_j \\ 0 & t \geq r_j \end{cases}. \tag{7.12}$$

We begin with $q = 1$. Note that $\omega \in \Omega^{0,1}(\Delta(r))$, but it does not have compact support, so we proceed differently than in the proof of Proposition 7.1. Write

$$\omega = \sum_k \omega_k d\overline{z}^k, \tag{7.13}$$

and define

$$\eta_1(z^1, \ldots, z^n) = \frac{1}{2\pi i} \int_{|w^j| \leq r_j} \frac{\chi_1(w_1)\omega_j(w^1, z^2, \ldots, z^n)}{w^1 - z^1} dw^1 \wedge d\overline{w}^1. \tag{7.14}$$

Then $\partial\eta_1 / \partial\overline{z}^1 = \chi_1\omega_1$, and we have

$$\overline{\partial}\eta_1 = \sum_l \frac{\partial\eta_1}{\partial\overline{z}^l} d\overline{z}^l = \chi_1\omega_1 d\overline{z}^1 + \sum_{j>1} \frac{\partial\eta_1}{\partial\overline{z}^j} d\overline{z}^j. \tag{7.15}$$

That is, we have solved the $d\overline{z}^1$-term, modulo terms involving $d\overline{z}^j$ for $j > 1$ (we have not even used the fact that $\overline{\partial}\omega = 0$ yet!) Next, we consider the case

$$\omega = \sum_{k>1} \omega_k d\overline{z}^k, \tag{7.16}$$

Since $\overline{\partial}\omega = 0$, this tells us that $\partial\omega_2/\partial\overline{z}^1 = 0$. Next, we define

$$\eta_2(z^1,\ldots,z^n) = \frac{1}{2\pi i}\int_{|w^2|\le r_2} \frac{\chi_2(w_2)\omega_2(z^1,w^2,z^3,\ldots,z^n)}{w^2-z^2} dw^2 \wedge d\overline{w}^2. \tag{7.17}$$

Then $\partial\eta_2/\partial\overline{z}^2 = \chi_2\omega_2$ and $\partial\eta_2/\partial\overline{z}^1 = 0$, so we have

$$\overline{\partial}\eta_2 = \sum_l \frac{\partial\eta_2}{\partial\overline{z}^j} d\overline{z}^j = \chi_2\omega_2 d\overline{z}^2 + \sum_{j>2}\frac{\partial\eta_2}{\partial\overline{z}^j} d\overline{z}^j. \tag{7.18}$$

Assume that we can solve all the terms involving $d\overline{z}^k$ for $k \le l$, and

$$\omega = \sum_{k>l}\omega_k d\overline{z}^k. \tag{7.19}$$

Since $\overline{\partial}\omega = 0$, this tells us that $\partial\omega_{l+1}/\partial\overline{z}^j = 0$ for $j \le l$. Then we define

$$\eta_{l+1}(z^1,\ldots,z^n) = \frac{1}{2\pi i}\int_{|w^{l+1}|\le r_{l+1}} \frac{\chi_{l+1}(w_{l+1})\omega_{l+1}(z^1,\ldots,w^{l+1},\ldots,z^n)}{w^{l+1}-z^{l+1}} dw^{l+1} \wedge d\overline{w}^{l+1}. \tag{7.20}$$

Then $\partial\eta_{l+1}/\partial\overline{z}^{l+1} = \chi_{l+1}\omega_{l+1}$ and $\partial\eta_{l+1}/\partial\overline{z}^j = 0$ for $j \le l$, so we have

$$\overline{\partial}\eta_{l+1} = \sum_{j>l}\frac{\partial\eta_{l+1}}{\partial\overline{z}^j} d\overline{z}^j = \chi_{l+1}\omega_{l+1} d\overline{z}^{l+1} + \sum_{j>l+1}\frac{\partial\eta_{l+1}}{\partial\overline{z}^j} d\overline{z}^j. \tag{7.21}$$

By induction, we are done with the case of $q = 1$.

Next, consider the case of $q = 2$. Then

$$\omega = \sum_{1\le k<l}\omega_{kl} d\overline{z}^k \wedge d\overline{z}^l = \sum_{1<l}\omega_{1l} d\overline{z}^1 \wedge d\overline{z}^l + \sum_{1<k<l}\omega_{kl} d\overline{z}^k \wedge d\overline{z}^l. \tag{7.22}$$

Define $\eta = \sum_{1<k}\eta_{1k} d\overline{z}^k$, where

$$\eta_{1k}(z^1,\ldots,z^n) = \frac{1}{2\pi i}\int_{|w^1|\le r_1} \frac{\chi_1(w^1)\omega_{1k}(w^1,z^2,\ldots,z^n)}{w^1-z^1} dw^1 \wedge d\overline{w}^1. \tag{7.23}$$

Then η_{1k} solves $\partial\eta_{1k}/\partial\overline{z}^1 = \chi_1\omega_{1k}$. So then

$$\overline{\partial}\eta = \sum_{1<k}\frac{\partial\eta_{1k}}{\partial\overline{z}^l} d\overline{z}^l \wedge d\overline{z}^k = \sum_{1<k}\chi_1\omega_{1k} d\overline{z}^1 \wedge d\overline{z}^k + R \tag{7.24}$$

where R doesn't include any $d\overline{z}^1$-s. So we have solved the terms in ω involving $d\overline{z}^1$-s. We next assume that ω is of the form

$$\omega = \sum_{1<k<l}\omega_{kl} d\overline{z}^k \wedge d\overline{z}^l = \sum_{2<l}\omega_{2l} d\overline{z}^2 \wedge d\overline{z}^l + \sum_{2<k<l}\omega_{kl} d\overline{z}^k \wedge d\overline{z}^l. \tag{7.25}$$

Let $\eta = \sum_{2<k} \eta_{2k} d\bar{z}^k$ where

$$\eta_{2k} = \frac{1}{2\pi i} \int_{|w^2| \leq r_2} \frac{\chi_2(w^2)\omega_{2k}(z^1, w^2, z^3, \dots z^n)}{w^2 - z^2} dw^2 \wedge d\bar{w}^2. \tag{7.26}$$

Then $\partial \eta_{2k}/\partial \bar{z}^2 = \chi_2 \omega_{2k}$. Furthermore, since $\bar{\partial}\omega = 0$, $\partial \eta_{2k}/\partial \bar{z}^1 = 0$. So then

$$\bar{\partial}\eta = \sum_{2<k} \bar{\partial}(\eta_{2k} d\bar{z}^k) = \sum_{2<k} \sum_{2 \leq l} \frac{\partial \eta_{2k}}{\partial \bar{z}^l} d\bar{z}^l \wedge d\bar{z}^k = \sum_{2<k} \chi_2 \omega_{2k} d\bar{z}^2 \wedge d\bar{z}^k + R, \tag{7.27}$$

where R only has terms $d\bar{z}^k \wedge d\bar{z}^l$ for $k, l \geq 3$. So we have solved as the term in ω having $d\bar{z}^1$-s or $d\bar{z}^2$-s. By a similar induction argument as in the $q = 1$ case, we can solve all terms in this manner. The case of $q > 2$ is similar, and details left as an exercise. □

Theorem 7.5. *If $U = \Delta(r)$ is polydisc (with some radii allowed to be infinite), then $H^{p,q}_{\bar{\partial}}(U) = \{0\}$ for $q \geq 1$.*

Proof. Choose a monotone increasing sequence of polyradii $r_1 < r_2 < \dots$ with $\lim_{j\to\infty} r_j = r$. Given $\omega \in \Omega^{0,q}(\Delta(r)$, by Step 2, we can find $\eta_j \in \Omega^{0,q-1}(\Delta(r))$ with $\bar{\partial}\eta_j = \omega$ on $\Delta(r_j)$. We do not know that the sequence η_j will converge. However, $\bar{\partial}(\eta_{j+1} - \eta_j) = 0$ in $\Delta(r_j)$. If $q \geq 2$, then by Step 2, we can find $\beta_{j+1} \in \Omega^{0,q-2}(\Delta(r_j))$ solving $\bar{\partial}(\beta_{j+1}) = \eta_{j+1} - \eta_j$ in $\Delta(r_{j-1})$. We then consider the sequence $\eta'_{j+1} = \eta_{j+1} - \bar{\partial}(\beta_{j+1})$. Then $\eta'_{j+1} \in \Omega^{0,q-2}(\Delta(r_j)$ and

$$\bar{\partial}(\eta'_{j+1}) = \bar{\partial}\eta_{j+1} - \bar{\partial}^2(\beta_{j+1}) = \omega \tag{7.28}$$

in $\Delta(r_{j-1})$, and this new sequence now obviously converges to a solution $\eta \in \Omega^{0,q}(\Delta(r)$ with $\bar{\partial}\eta = \omega$ in $\Delta(r)$.

If $q = 1$, then we prove exactly like we did in the case of $n = 1$, by approximating the difference $\eta_{j+1} - \eta_j$ by a polynomial P_{j+1} to obtain a sequence so that

$$\sup_{z\in K} |\eta_{j+1}(z) - \eta_j(z)| < 2^{-j}, \tag{7.29}$$

and we obtain a sequence converging on compact subsets to a solution. □

Remark 7.6. Using Laurent series instead of polynomials, a similar proof works to prove that Theorem 7.5 also holds for products $\Delta^*(r_1) \times \cdots \times \Delta^*(r_k) \times \Delta(r_{k+1}) \times \cdots \Delta(r_{k+l})$, that is, we can allow punctured 1-dimensional disks. With a lot more work, one can also show that Theorem 7.5 holds for $\Omega_1 \times \cdots \times \Omega_n$ with $\Omega_j \subset \mathbb{C}$ are domains. Note the result is NOT true for a punctured polydisc $\Delta(0, r) \setminus \{0\}$ for $n \geq 2$, but we cannot prove that yet.

Remark 7.7. Theorem 7.5 also holds for a ball $B(0, r) \subset \mathbb{C}^n$. However, this is difficult to prove directly. One could use the Bochner-Martinelli kernel instead of the Cauchy kernel to prove Proposition 7.4. Then one would also need to prove that the $B(0, r)$ is a Runge domain, that is, $\mathcal{O}(B(0, r))$ can be approximated by holomorphic polynomials uniformly on compact subsets. However, it seems actually easier to prove this more generally for any pseudoconvex domain (using Hörmander's L^2 methods), and then show that $B(0, r)$ is pseudoconvex.

8 Lecture 8

8.1 Almost complex manifolds

Definition 8.1. An *almost complex manifold* is a real manifold with an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$.

The following lemma shows that we can always take J to be standard at any *point*.

Lemma 8.2. *Let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear mapping satisfying $J^2 = -Id$. Then there exists an invertible matrix A such that $A^{-1}JA = J_{Euc}$.*

Proof. For $X \in \mathbb{R}^{2n}$, define

$$(a + ib)X = aX + bJX. \tag{8.1}$$

Then $\mathbb{R}^{2n}$ becomes an n-dimensional complex vector space. Let $X_1, \dots, X_n$ be a complex basis. Then $X_1, JX_1, \dots, X_n, JX_n$ is a basis of $\mathbb{R}^{2n}$ as a real vector space, and J is obviously standard in this basis. □

Remark 8.3. The Newlander-Nirenberg Theorem deals with the following question: when can we make J standard in a *neighborhood* of a point? As we will see shortly, this cannot possibly be true for an arbitrary almost complex structure; there is an *integrability condition* which must be satisfied.

All of the linear algebra we discussed above in $\mathbb{C}^n$ can be done on an almost complex manifold (M, J). We can decompose

$$TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}, \tag{8.2}$$

where

$$T^{1,0} = \{X - iJX, X \in T_pM\} \tag{8.3}$$

is the i-eigenspace of J and

$$T^{0,1} = \{X + iJX, X \in T_pM\} \tag{8.4}$$

is the $-i$-eigenspace of J.

The map J also induces an endomorphism of 1-forms by

$$J(\omega)(v_1) = \omega(Jv_1).$$

We then have

$$T^* \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{8.5}$$

where

$$\Lambda^{1,0} = \{\alpha - iJ\alpha, \alpha \in T_p^*M\} \tag{8.6}$$

is the i-eigenspace of J, and

$$\Lambda^{0,1} = \{\alpha + iJ\alpha, \alpha \in T_p^*M\} \tag{8.7}$$

is the $-i$-eigenspace of J.

Next, we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q} \tag{8.8}$$

decomposes as a direct sum.

Remark 8.4. This gives a necessary topological obstruction for existence of an almost complex structure: the bundle of complex k-forms must decompose into to a direct sum of subbundles as in (8.8).

We can extend $J : \Lambda^k \otimes \mathbb{C} \to \Lambda^k \otimes \mathbb{C}$ by letting

$$J\alpha = i^{p-q}\alpha, \tag{8.9}$$

for $\alpha \in \Lambda^{p,q}$, $p + q = k$. Note we can also extend J to k-forms by

$$J\alpha(X_1, \ldots, X_k) = \alpha(JX_1, \ldots, JX_k). \tag{8.10}$$

Exercise 8.5. Check that these two definitions of J on k-forms agree.

Definition 8.6. A triple (M, J, g) where J is an almost complex structure, and g is a Riemannian metric is *almost Hermitian* if

$$g(X, Y) = g(JX, JY) \tag{8.11}$$

for all $X, Y \in TM$. We also say that g is *compatible* with J.

Proposition 8.7. *Given a linear J with $J^2 = -Id$ on $\mathbb{R}^{2n}$, and a positive definite inner product g on $\mathbb{R}^{2n}$ which is compatible with J, there exist elements $\{X_1, \ldots X_n\}$ in $\mathbb{R}^{2n}$ so that*

$$\{X_1, JX_1, \ldots, X_n, JX_n\} \tag{8.12}$$

is an ONB for $\mathbb{R}^{2n}$ with respect to g.

Proof. We use induction on the dimension. First we note that if X is any unit vector, then JX is also unit, and

$$g(X, JX) = g(JX, J^2X) = -g(X, JX), \tag{8.13}$$

so X and JX are orthonormal. This handles $n = 1$. In general, start with any X_1, and let W be the orthogonal complement of $\mathrm{span}\{X_1, JX_1\}$. We claim that $J : W \to W$. To see this, let $X \in W$ so that $g(X, X_1) = 0$, and $g(X, JX_1) = 0$. Using J-invariance of g, we see that $g(JX, JX_1) = 0$ and $g(JX, X_1) = 0$, which says that $JX \in W$. Then use induction since W is of dimension $2n - 2$. $\square$

Definition 8.8. To an almost Hermitian structure (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y) \tag{8.14}$$

called the *Kähler form* or *fundamental* 2-*form*.

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{8.15}$$

Furthermore, since

$$\omega(JX, JY) = \omega(X, Y), \tag{8.16}$$

this form is a real form of type $(1, 1)$. That is, $\omega \in \Gamma(\Lambda^{1,1}_{\mathbb{R}})$, where $\Lambda^{1,1}_{\mathbb{R}} \subset \Lambda^{1,1}$ is the real subspace of elements satisfying $\overline{\omega} = \omega$.

In Euclidean space $(\mathbb{R}^{2n}, J_0, g_{Euc})$, the fundamental 2-form is

$$\omega_{Euc} = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{8.17}$$

We note the following formula for the volume form:

$$\left(\frac{i}{2}\right)^n dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n \tag{8.18}$$

Note that this defines an orientation on $\mathbb{C}^n$, which we will refer to as the natural orientation. Note also that

$$\omega^n_{Euc} = n! \cdot dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n. \tag{8.19}$$

Proposition 8.9. *If (M, J) is almost complex, then* $\dim(M)$ *is even and M is orientable.*

Proof. If M is of real dimension m, and admits an almost complex structure, then

$$(\det(J))^2 = \det(J^2) = \det(-I) = (-1)^m, \tag{8.20}$$

which implies that m is even. We will henceforth write $m = 2n$. Next, let g be any Riemannian metric on M. Then define

$$h(X, Y) = g(X, Y) + g(JX, JY). \tag{8.21}$$

Then $h(JX, JY) = h(X, Y)$ is J-invariant, so (M, J, h) is almost Hermitian. We then consider the fundamental 2-form

$$\omega(X, Y) = h(JX, Y). \tag{8.22}$$

This is a form of type $(1, 1)$, so $\omega^n \in \Lambda^{n,n}_{\mathbb{R}} \cong \Lambda^{2n}_{\mathbb{R}}$ is a top degree $2n$-form. It is nowhere-vanishing since at any point $x \in M$ by Proposition 8.7 we can assume that both $J_x = J_{Euc}$ and $g_x = g_{Euc}$, so $\omega^n(x) \neq 0$ by (8.19). Therefore, ω gives a globally defined orientation on M. $\square$

Example 8.10. For example, $\mathbb{RP}^n$ does not admit any almost complex structure, since it is non-orientable for n even.

Definition 8.11. A smooth mapping between $f : M \to N$ between almost complex manifolds (M, J_M) and (N, J_N) is *pseudo-holomorphic* if

$$f_* \circ J_M = J_N \circ f_* \tag{8.23}$$

We have a useful characterization of pseudo-holomorphic mappings.

Proposition 8.12. *A mapping $f : M \to N$ between almost complex manifolds (M, J_M) and (N, J_N) is pseudo-holomorphic if and only if*

$$f_*(T^{1,0}(M)) \subset T^{1,0}(N), \tag{8.24}$$

if and only if

$$f^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M). \tag{8.25}$$

8.2 Complex manifolds

We next define a complex manifold.

Definition 8.13. A *complex manifold* of dimension n is a smooth manifold of real dimension $2n$ with a collection of coordinate charts (U_α, ϕ_α) covering M, such that $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ and with overlap maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ satisfying the Cauchy-Riemann equations.

Example 8.14. Since holomorphic mappings are orientation-preserving by (1.40), any complex manifold is necessarily orientable. For example, $\mathbb{RP}^n$ does not admit any complex structure. Note that we knew from Example 8.10 above that there is no almost complex structure.

Complex manifolds have a uniquely determined compatible almost complex structure on the tangent bundle:

Proposition 8.15. *In any coordinate chart, define $J_\alpha : TM_{U_\alpha} \to TM_{U_\alpha}$ by*

$$J(X) = (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* X. \tag{8.26}$$

Then $J_\alpha = J_\beta$ on $U_\alpha \cap U_\beta$ and therefore gives a globally defined almost complex structure $J : TM \to TM$ satisfying $J^2 = -Id$.

Proof. On overlaps, the equation

$$(\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* = (\phi_\beta)_*^{-1} \circ J_0 \circ (\phi_\beta)_* \tag{8.27}$$

can be rewritten as

$$J_0 \circ (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} = (\phi_\alpha)_* \circ (\phi_\beta)_*^{-1} \circ J_0. \tag{8.28}$$

Using the chain rule this is

$$J_0 \circ (\phi_\alpha \circ \phi_\beta^{-1})_* = (\phi_\alpha \circ \phi_\beta^{-1})_* \circ J_0, \tag{8.29}$$

which is exactly the condition that the overlap maps satisfy the Cauchy-Riemann equations.

Obviously,

$$\begin{aligned} J^2 &= (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \circ (\phi_\alpha)_*^{-1} \circ J_0 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ J_0^2 \circ (\phi_\alpha)_* \\ &= (\phi_\alpha)_*^{-1} \circ (-Id) \circ (\phi_\alpha)_* = -Id. \end{aligned}$$

□

The next proposition follows from the above discussion on Cauchy-Riemann equations.

Proposition 8.16. *If (M, J_M) and (N, J_N) are complex manifolds, then $f : M \to N$ is pseudo-holomorphic if and only if is a holomorphic mapping in local holomorphic coordinate systems.*

Definition 8.17. An almost complex structure J is said to be a *complex structure* if J is induced from a collection of holomorphic coordinates on M.

Proposition 8.18. *An almost complex structure J is a complex structure if and only if for any $x \in M$, there is a neighborhood U of x and a pseudo-holomorphic mapping $\phi : (U, J) \to (\mathbb{C}^n, J_0)$ which has non-vanishing Jacobian at x. Equivalently, there exist n pseudo-holomorphic functions $f^j : U \to \mathbb{C}, j = 1 \dots n$, with linearly independent differentials at x.*

Proof. By the inverse function theorem, ϕ gives a coordinate system in a possible smaller neighborhood of of x. The overlap mappings are pseudo-holomorphic mappings with respect to J_0, so they satisfy the Cauchy-Riemann equations, and are therefore holomorphic. The components of ϕ are functions $f^j, j = 1 \dots n$ with linearly independent differentials, and conversely, $\phi = (f^1, \dots, f^n)$ is a local coordinate system. □

Proposition 8.19. *A real 2-dimensional manifold admits an almost complex structure if and only if it is oriented.*

Proof. We have already proved the forward direction. Let M^2 be any oriented surface, and choose any Riemannian metric g on M. Then $* : \Lambda^1 \to \Lambda^1$ satisfies $*^2 = -Id$, and using the metric to identify $\Lambda^1 \cong TM$, we obtain an endomorphism $J : TM \to TM$ satisfying $J^2 = -Id$, which is an almost complex structure. □

Remark 8.20. In this case, any such J is necessarily a complex structure. This is equivalent to the problem of existence of isothermal coordinates, we will prove this soon.

9 Lecture 9

9.1 The Nijenhuis tensor

When does an almost complex structure arise from a true complex structure? To answer this question, we define the following tensor associated to an almost complex structure.

Proposition 9.1. *The* Nijenhuis *tensor of an almost complex structure defined by*

$$N(X,Y) = 2\{[JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]\} \tag{9.1}$$

is in $\Gamma(T^*M \otimes T^*M \otimes TM)$ *and satisfies*

$$N(Y,X) = -N(X,Y), \tag{9.2}$$
$$N(JX,JY) = -N(X,Y), \tag{9.3}$$
$$N(X,JY) = N(JX,Y) = -J(N(X,Y)). \tag{9.4}$$

Proof. Given a function $f : M \to \mathbb{R}$, we compute

$$\begin{aligned} N(fX,Y) &= 2\{[J(fX),JY] - [fX,Y] - J[fX,JY] - J[J(fX),Y]\} \\ &= 2\{[fJX,JY] - [fX,Y] - J[fX,JY] - J[fJX,Y]\} \\ &= 2\{f[JX,JY] - (JY(f))JX - f[X,Y] + (Yf)X \\ &\quad - J(f[X,JY] - (JY(f))X) - J(f[JX,Y] - (Yf)JX)\} \\ &= fN(X,Y) + 2\{-(JY(f))JX + (Yf)X + (JY(f))JX + (Yf)J^2X\}. \end{aligned}$$

Since $J^2 = -I$, the last 4 terms vanish. A similar computation proves that $N(X,fY) = fN(X,Y)$. Consequently, N is a tensor. The skew-symmetry in X and Y (9.2) is obvious, and (9.3) follows easily using $J^2 = -Id$. For (9.4)

$$N(X,JY) = -N(JX,J^2Y) = N(JX,Y), \tag{9.5}$$

and

$$\begin{aligned} N(X,JY) &= 2\{[JX,J^2Y] - [X,JY] - J[X,J^2Y] - J[JX,JY]\} \\ &= 2\{-[JX,Y] - [X,JY] + J[X,Y] - J[JX,JY]\} \\ &= 2J\{J[JX,Y] + J[X,JY] + [X,Y] - [JX,JY]\} \\ &= -2J\{N(X,Y)\}. \end{aligned} \tag{9.6}$$

□

Proposition 9.2. *For a* C^1 *almost complex structure* J,

$$N_J \in \Gamma\Big(\big\{\big(\Lambda^{2,0} \otimes T^{0,1}\big) \oplus \big(\Lambda^{0,2} \otimes T^{1,0}\big)\big\}_{\mathbb{R}}\Big). \tag{9.7}$$

Consequently, if $\dim(M) = 2n$, *then the Nijenhuis tensor has* $n^2(n-1)$ *independent real components. In particular, if* $n = 1$, *then* $N_J \equiv 0$.

Proof. If we complexify, just using (9.2), we have

$$
\begin{aligned}
N_J &\in \Gamma((\Lambda^2 \otimes TM) \otimes \mathbb{C})) \\
&= \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}\big) \otimes \big(T^{1,0} \oplus T^{0,1}\big)\Big).
\end{aligned} \tag{9.8}
$$

But (9.3) says that the $\Lambda^{1,1}$ component vanishes. So we have

$$
N_J \in \Gamma\Big(\big(\Lambda^{2,0} \oplus \Lambda^{0,2}\big) \otimes \big(T^{1,0} \oplus T^{0,1}\big)\Big). \tag{9.9}
$$

Using (9.4), for $X', Y' \in \Gamma(TM)$, we have

$$
\begin{aligned}
&N_J(X' - iJX', Y' - iJY') \\
&\quad = N_J(X', Y') - N_J(JX', JY') - iN_J(JX', Y') - iN_J(X', JY') \\
&\quad = N_J(X', Y') + N_J(X', Y') + iJN_J(X', Y') + iJN_J(X', Y') \\
&\quad = 2N_J(X', Y') + 2iJN_J(X', Y'),
\end{aligned} \tag{9.10}
$$

which lies in $T^{0,1}$. This shows that the $\Lambda^{2,0} \otimes T^{1,0}$ component vanishes, so the $\Lambda^{0,2} \otimes T^{0,1}$ component also vanishes, and (9.7) follows since N_J is a real tensor. □

We have the following local formula for the Nijenhuis tensor.

Proposition 9.3. *In local coordinates, the Nijenhuis tensor is given by*

$$
N^i_{jk} = 2\sum_{h=1}^{2n} (J^h_j \partial_h J^i_k - J^h_k \partial_h J^i_j - J^i_h \partial_j J^h_k + J^i_h \partial_k J^h_j) \tag{9.11}
$$

Proof. We compute

$$
\begin{aligned}
&\frac{1}{2} N(\partial_j, \partial_k) = [J\partial_j, J\partial_k] - [\partial_j, \partial_k] - J[\partial_j, J\partial_k] - J[J\partial_j, \partial_k] \\
&= [J^l_j \partial_l, J^m_k \partial_m] - [\partial_j, \partial_k] - J[\partial_j, J^l_k \partial_l] - J[J^l_j \partial_l, \partial_k] \\
&= I + II + III + IV.
\end{aligned}
$$

The first term is

$$
\begin{aligned}
I &= J^l_j \partial_l (J^m_k \partial_m) - J^m_k \partial_m (J^l_j \partial_l) \\
&= J^l_j (\partial_l J^m_k)\partial_m + J^l_j J^m_k \partial_l \partial_m - J^m_k (\partial_m J^l_j)\partial_l - J^m_k J^l_j \partial_m \partial_l \\
&= J^l_j (\partial_l J^m_k)\partial_m - J^m_k (\partial_m J^l_j)\partial_l.
\end{aligned}
$$

The second term is obviously zero. The third term is

$$
III = -J(\partial_j (J^l_k)\partial_l) = -\partial_j (J^l_k) J^m_l \partial_m. \tag{9.12}
$$

Finally, the fourth term is

$$
III = \partial_k (J^l_j) J^m_l \partial_m. \tag{9.13}
$$

Combining these, we are done. □

Definition 9.4. If J is an almost complex structure of class C^1 satisfying $N_J \equiv 0$, then we say that J is *integrable.*

Corollary 9.5. *If (M, J) arises from a complex structure, then J is integrable.*

Proof. In local holomorphic coordinates $J = J_0$ is a constant tensor, and $N(J) = 0$ follows from Proposition 9.3. □

Next, we have an alternative characterization of the vanishing of the Nijenhuis tensor.

Proposition 9.6. *For an almost complex structure J the Nijenhius tensor $N(J) = 0$ if and only if for any 2 vector fields $X, Y \in \Gamma(T^{1,0})$, their Lie bracket $[X, Y] \in \Gamma(T^{1,0})$.*

Proof. To see this, if X and Y are both sections of $T^{1,0}$ then we can write $X = X' - iJX'$ and $Y = Y' - iJY'$ for real vector fields X' and Y'. The commutator is

$$[X' - iJX', Y' - iJY'] = [X', Y'] - [JX', JY'] - i([X', JY'] + [JX', Y']). \tag{9.14}$$

But this is also a $(1, 0)$ vector field if and only if

$$[X', JY'] + [JX', Y'] = J[X', Y'] - J[JX', JY'], \tag{9.15}$$

applying J, and moving everything to the left hand side, this says that

$$[JX', JY'] - [X', Y'] - J[X', JY'] - J[JX', Y'] = 0, \tag{9.16}$$

which is exactly the vanishing of the Nijenhuis tensor. □

9.2 The operators ∂ and $\overline{\partial}$

Recall that on any almost complex manifold (M, J), we can define $\Lambda^{p,q} \subset \Lambda^{p+q} \otimes \mathbb{C}$ to be the span of forms which can be written as the wedge product of exactly p elements in $\Lambda^{1,0}$ and exactly q elements in $\Lambda^{0,1}$. We have that

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}. \tag{9.17}$$

We define $\Omega^k, \Omega^k_{\mathbb{C}}, \Omega^{p,q}$ to be the space of sections of $\Lambda^k, \Lambda^k \otimes \mathbb{C}, \Lambda^{p,q}$, respectively. The real operator $d : \Omega^k_{\mathbb{R}} \to \Omega^{k+1}_{\mathbb{R}}$, extends to an operator

$$d : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{9.18}$$

by complexification.

Proposition 9.7. *For a C^1 almost complex structure J*

$$d(\Omega^{p,q}) \subset \Omega^{p+2,q-1} \oplus \Omega^{p+1,q} \oplus \Omega^{p,q+1} \oplus \Omega^{p-1,q+2}, \tag{9.19}$$

and $N_J = 0$ if and only if

$$d(\Omega^{p,q}) \subset \Omega^{p+1,q} \oplus \Omega^{p,q+1}. \tag{9.20}$$

if and only if

$$d(\Omega^{1,0}) \subset \Omega^{2,0} \oplus \Omega^{1,1} \tag{9.21}$$

if and only if

$$d(\Omega^{0,1}) \subset \Omega^{1,1} \oplus \Omega^{0,2} \tag{9.22}$$

Proof. Let $\alpha \in \Omega^{p,q}$, and write $p + q = r$. Then we have the basic formula

$$\begin{aligned} d\alpha(X_0, \dots, X_r) = &\sum (-1)^j X_j \alpha(X_0, \dots, \hat{X}_j, \dots, X_r) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r). \end{aligned} \tag{9.23}$$

This is easily seen to vanish if more than $p+2$ of the X_j are of type $(1,0)$ or if more than $q+2$ are of type $(0,1)$, and (9.19) follows.

Next, assume that (9.22) is satisfied. Let $\alpha \in \Omega^{0,1}$, then

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \tag{9.24}$$

then implies that if both X and Y are in $T^{1,0}$ then so is their bracket $[X,Y]$. Proposition 9.6 implies that $N(J) \equiv 0$. Conversely, if $N(J) \equiv 0$, then we can reverse the steps in this argument to obtain (9.22). Equation (9.21) is just the conjugate of (9.22).

Recall that if $\alpha \in \Omega^k$ and $\beta \in \Omega^l$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{9.25}$$

The formula (9.20) then follows from this. $\square$

If $N_J = 0$, we can therefore define operators

$$\partial : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{9.26}$$

$$\overline{\partial} : \Omega^k_{\mathbb{C}} \to \Omega^{k+1}_{\mathbb{C}} \tag{9.27}$$

using (9.17) and

$$\partial|_{\Omega^{p,q}} = \Pi_{\Lambda^{p+1,q}} d \tag{9.28}$$

$$\overline{\partial}|_{\Omega^{p,q}} = \Pi_{\Lambda^{p,q+1}} d. \tag{9.29}$$

Corollary 9.8. *For a C^1 almost complex structure J with $N_J = 0$, $d = \partial + \overline{\partial}$ which satisfy*

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{9.30}$$

Proof. The equation $d^2 = 0$ implies that

$$0 = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2. \tag{9.31}$$

If we plug in a form of type (p,q) the first term is of type $(p+2,q)$, the middle terms are of type $(p+1,q+1)$, and the last term is of type $(p,q+2)$. Since (9.17) is a direct sum, the claim follows. $\square$

10 Lecture 10

Recall that for $n = 1$, any almost complex structure J satisfies $N_J = 0$, so there is no integrability condition. Let's look at various forms of the equations.

10.1 Real form of the equations

We just look in an open set in real coordinates (x, y), and then we have

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}. \tag{10.1}$$

The only condition is

$$-I = J^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} \tag{10.2}$$

If we assume that J is not too far from J_0, then $b \sim -1$ and $c \sim 1$, so we must have

$$a + d = 0, \quad a^2 + bc = -1. \tag{10.3}$$

Note that since $b \sim -1$, we can solve $c = -(1 + a^2)/b$, but we won't need to do this now. So we just consider

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix}. \tag{10.4}$$

We want to find a pseudo-holomorphic mapping

$$\phi : (U, J) \to (\mathbb{C}, J_0) \tag{10.5}$$

which has non-vanishing Jacobian at 0. So we want to solve

$$\phi_* \circ J = J_0 \circ \phi_* \tag{10.6}$$

If we write

$$\phi(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}, \tag{10.7}$$

then the pseudoholomorphic condition is

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{10.8}$$

which yields the 4 equations

$$\begin{matrix} au_x + cu_y = -v_x & bu_x - au_y = -v_y \\ av_x + cv_y = u_x & bv_x - av_y = u_y \end{matrix}. \tag{10.9}$$

This looks like 4 first-order equations for 2 unknown functions, so one wouldn't expect a solution. However, the first two equations imply the second two:

$$av_x + cv_y = a(-au_x - cu_y) + c(-bu_x + au_y) = (-a^2 - bc)u_x = u_x, \tag{10.10}$$

and

$$bv_x - av_y = b(-au_x - cu_y) + a(bu_x - au_y) = (-bc - a^2)u_y = u_y, \tag{10.11}$$

using the condition that $a^2 + bc = -1$.

Example 10.1. Let's now do an example. Consider

$$J = \begin{pmatrix} 2x & -1 \\ 1+4x^2 & -2x \end{pmatrix}. \tag{10.12}$$

We have

$$J^2 = \begin{pmatrix} 2x & -1 \\ 1+4x^2 & -2x \end{pmatrix} \begin{pmatrix} 2x & -1 \\ 1+4x^2 & -2x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10.13}$$

so this is indeed an almost complex structure.

From (10.9), the pseudoholomorphic equations are

$$2xu_x + (1+4x^2)u_y = -v_x \tag{10.14}$$
$$-u_x - 2xu_y = -v_y. \tag{10.15}$$

If a sufficiently smooth solution exists, then we have $v_{xy} = v_{yx}$, which yields

$$(2xu_x + (1+4x^2)u_y)_y = -(u_x + 2xu_y)_x \tag{10.16}$$

This can be rewritten as

$$u_{xx} + 4xu_{xy} + (1+4x^2)u_{yy} + 2u_y = 0. \tag{10.17}$$

By inspection, we find that $u = x$ is obviously a solution. We then return to the pseudo-holomorphic equations, and find that

$$v_x = -2x, \quad v_y = 1, \tag{10.18}$$

so we can choose $v = -x^2 + y$. So our solution is $\phi = (u,v) = (x, y - x^2)$. The Jacobian at the origin is clearly non-degenerate, so we have found a holomorphic coordinate system. Note that the mapping $\phi : \mathbb{R}^2 \to \mathbb{C}$ is defined everywhere. It is injective: if we have $(x_1, y_1 - x_1^2) = (x_2, y_2 - x_2^2)$ then the first component says that $x_1 = x_2$ and the second component then implies that $y_1 = y_2$. It is also surjective: given any $(u,v) \in \mathbb{C}$, we let $x_2 = u$, and then we need to solve $y - u^2 = v$, which obviously has a solution $y = -u^2 + v$. Thus we have found that

$$\phi : (\mathbb{R}^2, J) \to (\mathbb{C}, J_0) \tag{10.19}$$

is a global biholomorphism! Note that any function of the form $f(x,y) = h(x + i(y - x^2))$, where h is a holomorphic function with respect to J_0, is then holomorphic for J_0, for example

$$f(x,y) = e^x(\cos(y - x^2) + i\sin(y - x^2)). \tag{10.20}$$

10.2 Complex form of the equations

In the basis $\{\partial/\partial x, \partial/\partial y\}$ we have J of the form

$$J = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix} \tag{10.21}$$

satisfying $a^2 + bc = -1$. Using (1.36) to change to the complex basis $\{\partial/\partial z, \partial/\partial \overline{z}\}$, then we have

$$J = \frac{1}{2}\begin{pmatrix} i(c-b) & 2a + i(b+c) \\ 2a - i(b+c) & -i(c-b) \end{pmatrix} \tag{10.22}$$

For a complex valued function w, the equation $\overline{\partial}_J w = 0$ is $\Pi_{\Lambda^{0,1}} dw = 0$, which is

$$\begin{aligned} 0 &= dw + iJdw = w_z dz + w_{\overline{z}} d\overline{z} + iJ(w_z dz + w_{\overline{z}} d\overline{z}) \\ &= w_z dz + w_{\overline{z}} d\overline{z} + i w_z J dz + i w_{\overline{z}} J d\overline{z}. \end{aligned} \tag{10.23}$$

Note that we need to use $J : \Lambda^1 \to \Lambda^1$ here, which is the transpose matrix of the above J. So we have

$$\begin{aligned} 0 &= w_z dz + w_{\overline{z}} d\overline{z} + \frac{i}{2} w_z (i(c-b)dz + (2a + i(b+c))d\overline{z}) + \frac{i}{2} w_{\overline{z}}((2a - i(b+c))dz - i(c-b)d\overline{z}) \\ &= \Big(w_z + \frac{1}{2}(b-c)w_z + \frac{1}{2}(2ai + b + c)w_{\overline{z}}\Big)dz + \Big(w_{\overline{z}} + \frac{1}{2}(2ai - b - c)w_z + \frac{1}{2}(c-b)w_{\overline{z}}\Big)d\overline{z}. \end{aligned} \tag{10.24}$$

Let's look only at the second equation which is

$$\Big(1 + \frac{1}{2}(c-b)\Big)w_{\overline{z}} = -\frac{1}{2}(2ai - b - c)w_z. \tag{10.25}$$

If $b - c \neq 2$, which is certainly the case if J is close to J_0, then the leading coefficient is non-zero, and we can divide to get

$$w_{\overline{z}} = -\frac{2ai - b - c}{2 + c - b} w_z \tag{10.26}$$

Note that the first equation is

$$\Big(1 + \frac{1}{2}(b-c)\Big)w_z = -\frac{1}{2}(2ai + b + c)w_{\overline{z}}. \tag{10.27}$$

If $2ai - b - c \neq 0$, then we can divide to get

$$w_{\overline{z}} = -\frac{2 + b - c}{2ai + b + c} w_z. \tag{10.28}$$

I claim these are the same equation. For this, we would need

$$\frac{2ai - b - c}{2 + c - b} = \frac{2 + b - c}{2ai + b + c}, \tag{10.29}$$

which yields

$$(2ai - b - c)(2ai + b + c) = (2 + c - b)(2 + b - c), \tag{10.30}$$

which is

$$-4a^2 - (b + c)^2 = 4 - (c - b)^2. \tag{10.31}$$

Expanding this out

$$-4a^2 - b^2 - 2bc - c^2 = 4 - c^2 + 2bc - b^2, \tag{10.32}$$

which is true since $a^2 + bc = -1$!

Definition 10.2. The equation

$$w_{\overline{z}} - \mu(z, \overline{z}) w_z = 0 \tag{10.33}$$

is called the *Beltrami equation*.

10.3 Method of characteristics

This is a general method for solving linear PDE by solving nonlinear ODEs, we just explain for the Beltrami equation. Let's solve the nonlinear ODE

$$\frac{\partial z}{\partial s} = -\mu(z, s), \;\; z(0) = w. \tag{10.34}$$

The solution will depend on the independent variable s and the initial conditions w, call the solution $\Phi(s, w)$, and we write

$$z = \Phi(s, w). \tag{10.35}$$

By the implicit function theorem, we can write $w = w(z, s)$ in a neighborhood of $(s, w) = (0, 0)$, provided that $\frac{\partial \Phi}{\partial w}|_{0,0} \neq 0$. But this is

$$\frac{\partial \Phi}{\partial w}\Big|_{0,0} = \lim_{h \to 0} \frac{\Phi(0, h) - \Phi(0, 0)}{h} = 1. \tag{10.36}$$

So we have

$$z = \Phi(s, w(z, s)). \tag{10.37}$$

Taking the partial derivative of (10.37) with respect to z yields

$$1 = \frac{\partial \Phi}{\partial w} \frac{\partial w}{\partial z}. \tag{10.38}$$

Taking the partial derivative of (10.37) with respect to s yields

$$0 = \frac{\partial \Phi}{\partial s} + \frac{\partial \Phi}{\partial w} \frac{\partial w}{\partial s}, \tag{10.39}$$

which is

$$0 = -\mu(z,s) + \left(\frac{\partial w}{\partial z}\right)^{-1} \frac{\partial w}{\partial s}, \tag{10.40}$$

which is the Beltrami equation upon letting $s = \overline{z}$.

Let's return to Example (10.1), and solve using this method. Recall

$$J = \begin{pmatrix} 2x & -1 \\ 1+4x^2 & -2x \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{10.41}$$

So we need to solve the Beltrami equation with

$$\mu = -\frac{2ai - b - c}{2 + c - b} = -\frac{4xi + 1 - 1 - 4x^2}{2 + 1 + 4x^2 + 1} = -\frac{xi - x^2}{1 + x^2} = \frac{x}{i + x}. \tag{10.42}$$

Since $x = (z + \overline{z})/2$, we have

$$\mu(z, \overline{z}) = \frac{z + \overline{z}}{2i + z + \overline{z}}. \tag{10.43}$$

Let's solve the ODE

$$\frac{dz}{ds} = -\mu(z,s), \ \ z(0) = w. \tag{10.44}$$

For our example, this is

$$\frac{dz}{ds} = -\frac{z + s}{2i + z + s}. \tag{10.45}$$

To solve this, let's make a change of variables $p = z + s$. Then

$$\frac{dp}{ds} - 1 = -\frac{p}{2i + p}, \tag{10.46}$$

which gives

$$\frac{dp}{ds} = 1 - \frac{p}{2i + p} = \frac{2i}{2i + p}, \tag{10.47}$$

or

$$(2i + p)dp = 2ids, \tag{10.48}$$

which integrates to

$$2ip + \frac{1}{2}p^2 = 2is + C, \tag{10.49}$$

which is

$$2i(z + s) + \frac{1}{2}(z + s)^2 = 2is + C. \tag{10.50}$$

Our initial conditions are $z(0) = w$, so we get

$$2i(z + s) + \frac{1}{2}(z + s)^2 = 2is + 2iw + \frac{1}{2}w^2. \tag{10.51}$$

This is

$$w^2 + 4iw - 4iz - (z + s)^2 = 0 \tag{10.52}$$

Using the quadratic formula and letting $s = \overline{z}$ yields

$$w = -2i + \sqrt{-4 + 4iz + (z + \overline{z})^2}, \tag{10.53}$$

and we take the branch of the square root satisfying $\sqrt{-4} = 2i$. Note that this does not agree with the above method, but this is because the initial conditions are different. The above solution satisfies $w(z, 0) = z$, but the solution found in the previous section was

$$w = z - ix^2 = z - \frac{i}{4}(z + \overline{z})^2, \tag{10.54}$$

which satisfies $w(z, 0) = z - \frac{i}{4}z^2$.

11 Lecture 11

11.1 Another example

This example will be crucial in proving convergence in the analytic case, and is called a *Cauchy majorant.*

Proposition 11.1. *For $\rho > 0$, and $C > 0$, let*

$$\mu^* = C\Big(\frac{1}{1 - (z + \overline{z})\rho^{-1}} - 1\Big) = C\frac{z + \overline{z}}{\rho - z - \overline{z}}. \tag{11.1}$$

which is analytic in the polydisc $P(\rho) = \{(z, \overline{z}) \mid |z| < \rho/2, |\overline{z}| < \rho/2\}$. Then there is a solution w^* of the Beltrami equation $w^*_{\overline{z}} - \mu^*(z, \overline{z})w^*_z = 0$ satisfying $w^*(z, 0) = z$ which is analytic in some polydisc $P(\rho')$ for some $\rho' > 0$.

Proof. We use the method of characteristics from the previous example: solve the ODE

$$\frac{dz}{ds} = -\mu^*(z, s) \tag{11.2}$$

with initial condition $z(0) = w$. Note, by scaling the coordinates, without loss of generaliity, we can assume that $\rho = 1$. So we need to solve the ODE

$$\frac{dz}{ds} = -C\frac{z + s}{1 - z - s}, \; z(0) = w. \tag{11.3}$$

Letting $p = z + s$, and the equation becomes

$$\frac{dp}{ds} = 1 - \frac{Cp}{1-p} = \frac{1-(C+1)p}{1-p},\ p(0) = w. \tag{11.4}$$

This is separable, so we rewrite as

$$\frac{1-p}{1-(C+1)p} \cdot dp = ds \tag{11.5}$$

We can write

$$\frac{1-p}{1-(C+1)p} = \frac{1}{C+1}\Big(1 + \frac{C}{1-(C+1)p}\Big). \tag{11.6}$$

So the equation is

$$\Big(1 + \frac{C}{1-(C+1)p}\Big) = (C+1)ds \tag{11.7}$$

Integrating yields

$$p - \frac{C}{C+1}\log(1-(C+1)p) = (C+1)s + C_1. \tag{11.8}$$

Plugging in the initial conditions gives

$$p - \frac{C}{C+1}\log(1-(C+1)p) = (C+1)s + w - \frac{C}{C+1}\log(1-(C+1)w). \tag{11.9}$$

In terms of z, this is

$$z - \frac{C}{C+1}\log(1-(C+1)(z+s)) = Cs + w - \frac{C}{C+1}\log(1-(C+1)w). \tag{11.10}$$

Rewrite this as

$$w - \frac{C}{C+1}\log(1-(C+1)w) = z - Cs - \frac{C}{C+1}\log(1-(C+1)(z+s)). \tag{11.11}$$

Near $(z, s) = (0, 0)$, the right hand side is an analytic function. If we let

$$f(w) = w - \frac{C}{C+1}\log(1-(C+1)w) \tag{11.12}$$

Then f is analytic near $w = 0$. Also,

$$f'(0) = 1 + C \neq 0. \tag{11.13}$$

By the holomorphic inverse function theorem, f^{-1} exists and is analytic near 0; see [Pal91, Theorem VIII.1.8]. So then we have

$$w = f^{-1}\Big(z - Cs - \frac{C}{C+1}\log(1-(C+1)(z+s))\Big) \tag{11.14}$$

is analytic. Setting $s = \bar{z}$, we have

$$w = f^{-1}\Big(z - C\overline{z} - \frac{C}{C+1}\log(1-(C+1)(z+\overline{z}))\Big) \tag{11.15}$$

which is analytic. Note also that

$$w(z,0) = f^{-1}\Big(z - \frac{C}{C+1}\log(1-(C+1)z)\Big) = f^{-1}(f(z)) = z, \tag{11.16}$$

so the correct initial conditions are satisfied. $\square$

11.2 Equivalence of J and μ

The following proposition gives another way to think about almost complex structures for $n = 1$.

Proposition 11.2. *If J is defined in an open set U which induces the standard orientation on U, then there exists a unique complex valued function $\mu : U \to B(0,1) \subset \mathbb{C}$ so that*

$$T_J^{0,1} = \{v + \mu\overline{v} \mid v \in T_{J_0}^{0,1}\} \subset T_{\mathbb{C}}U. \tag{11.17}$$

Explicitly, if

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{11.18}$$

with $a^2 + bc = -1$, then

$$\mu = \frac{2ai - b - c}{2 + c - b}. \tag{11.19}$$

Conversely, given a function $\mu : U \to B(0,1) \subset \mathbb{C}$, writing $\mu = f + ig$, there is a uniquely determined almost complex structure J given by

$$J = \frac{1}{1 - f^2 - g^2} \begin{pmatrix} 2g & -(1+f)^2 - g^2 \\ g^2 + (1-f)^2 & -2g \end{pmatrix} \tag{11.20}$$

which has $T_J^{0,1}$ given by the above.

Proof. Given any such J, then we have previously defined

$$T_J^{0,1} = \{X \in T_{\mathbb{C}}U \mid JX = -iX\} = \{X' + iJX' \mid X \in T_{\mathbb{R}}U\}. \tag{11.21}$$

We next claim that the projection $\pi : T_J^{0,1} \to T_{J_0}^{0,1}$ is a complex linear isomorphism. These are two 1-dimensional complex subspaces of the 2-dimensional space $TU \otimes \mathbb{C}$, so there is a complex linear projection mapping, which is given by

$$X' + iJX' \mapsto X' + iJX' + iJ_0(X' + iJX') = (X' - J_0JX') + i(J + J_0)X'. \tag{11.22}$$

Since both spaces are 1-dimensional, and π is complex linear, it is an isomorphism provided it is not the zero map. Obviously, from (11.22), if $J \neq -J_0$ then it is not the zero mapping. We may therefore write $T_J^{0,1}$ as a graph over $T_{J_0}^{0,1}$. To do this, we compute like last time: using (1.36) to change to the complex basis $\{\partial/\partial z, \partial/\partial\overline{z}\}$, then we have

$$J = \frac{1}{2} \begin{pmatrix} i(c-b) & 2a + i(b+c) \\ 2a - i(b+c) & -i(c-b) \end{pmatrix}. \tag{11.23}$$

Then a basis for the 1-dimensional space $T_J^{0,1}$ is given by

$$\frac{\partial}{\partial\overline{z}} + iJ\Big(\frac{\partial}{\partial\overline{z}}\Big) = \frac{\partial}{\partial\overline{z}} + \frac{i}{2}\Big(i(b-c)\frac{\partial}{\partial\overline{z}} + (2a + i(b+c))\frac{\partial}{\partial z}\Big) \tag{11.24}$$

$$= \Big(1 + \frac{c-b}{2}\Big)\frac{\partial}{\partial\overline{z}} + \frac{1}{2}(2ai - b - c)\frac{\partial}{\partial z}. \tag{11.25}$$

From this, we find that

$$\mu = \frac{2ai - b - c}{2 + c - b}, \tag{11.26}$$

as claimed. Using $a^2 + bc = -1$, we compute

$$|\mu|^2 = \frac{4(-1 - bc) + (b + c)^2}{(2 + c - b)^2} = \frac{2 + b - c}{-2 + b - c}. \tag{11.27}$$

To show that $|\mu| < 1$, we use the orientation condition. Notice that the condition $bc = -1 - a^2$ says that $bc < 0$, so there are 2 components to the set of almost complex structures, determined by the sign of b: if $b < 0$, then this is the component inducing the standard orientation. In this case, we have

$$\frac{2 + b - c}{-2 + b - c} < 1 \tag{11.28}$$

is equivalent to

$$2 + b - c > -2 + b - c, \tag{11.29}$$

which is obviously true.

Next, given any such function μ, we define

$$T^{0,1}_{\mu} = \mathrm{span}\Big\{ \frac{\partial}{\partial \overline{z}} + \mu \frac{\partial}{\partial z} \Big\}. \tag{11.30}$$

Define

$$T^{1,0}_{\mu} = \mathrm{span}\Big\{ \frac{\partial}{\partial z} + \overline{\mu} \frac{\partial}{\partial \overline{z}} \Big\}. \tag{11.31}$$

We claim that $T^{1,0}_{\mu} \cap T^{0,1}_{\mu} = \{0\}$. To see this, if the intersection was non-zero, then there would exist $\alpha \in \mathbb{C}$ so that

$$\frac{\partial}{\partial \overline{z}} + \mu \frac{\partial}{\partial z} = \alpha \Big(\frac{\partial}{\partial z} + \overline{\mu} \frac{\partial}{\partial \overline{z}} \Big). \tag{11.32}$$

This clearly implies that $\alpha = \mu$ and then $|\mu|^2 = 1$. But we have assumed that $|\mu| < 1$, so the claim follows. To find the corresponding almost complex structure J, we must have

$$\frac{\partial}{\partial \overline{z}} + \mu \frac{\partial}{\partial z} = X' + iJX', \tag{11.33}$$

for some real tangent vector X'. We then write the real and imaginary parts of the left hand side:

$$\begin{aligned} \frac{\partial}{\partial \overline{z}} + \mu \frac{\partial}{\partial z} &= \frac{1}{2}\Big(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \Big) + (f + ig) \frac{1}{2} \Big(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \Big) \\ &= \frac{1}{2} \Big((1 + f) \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \Big) + \frac{i}{2} \Big(g \frac{\partial}{\partial x} + (1 - f) \frac{\partial}{\partial y} \Big). \end{aligned} \tag{11.34}$$

So we must have

$$J\Big((1+f)\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}\Big)=g\frac{\partial}{\partial x}+(1-f)\frac{\partial}{\partial y}, \tag{11.35}$$

and since $J^2=-Id$,

$$J\Big(g\frac{\partial}{\partial x}+(1-f)\frac{\partial}{\partial y}\Big)=-\Big((1+f)\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}\Big). \tag{11.36}$$

A simple change of basis computation shows that

$$J=\frac{1}{1-f^2-g^2}\begin{pmatrix} 2g & -(1+f)^2-g^2 \\ g^2+(1-f)^2 & -2g \end{pmatrix}. \tag{11.37}$$

□

This gives another way to understand the Beltrami equation. Given $\mu : U \to \mathbb{C}$ with $|\mu|<1$, then since

$$T^{0,1}_{\mu}=\mathrm{span}\Big\{\frac{\partial}{\partial \overline{z}}+\mu\frac{\partial}{\partial z}\Big\}, \tag{11.38}$$

a function $w : U \to \mathbb{C}$ is holomorphic if and only if

$$\Big(\frac{\partial}{\partial \overline{z}}+\mu\frac{\partial}{\partial z}\Big)w=0, \tag{11.39}$$

or

$$w_{\overline{z}}+\mu w_z=0, \tag{11.40}$$

which is exactly the Beltrami equation. We can just completely forget about the matrix version of J, and parametrize almost complex structures by a single function $\mu : U \to B(0,1)$.

Remark 11.3. This proposition also shows us that the regularity of $J : U \to \mathrm{GL}(2,\mathbb{R})$ is the same as the regularity of $\mu : U \to B(0,1)$. That is, J is $C^{k,\alpha}, C^{\infty}, C^{\omega}$ if and only if μ is also.

Remark 11.4. The complex structures inducing the reversed orientation correspond $|\mu|>1$ together with the point at infinity, which corresponds to the complex structure $-J_0$.

12 Lecture 12

12.1 The Beltrami equation: analytic case

Now we consider the Beltrami equation

$$w_{\overline{z}}=\mu(z,\overline{z})w_z \tag{12.1}$$

Assuming μ is analytic, we have a convergent power series expansion

$$\mu(z,\overline{z}) = \sum_{j,k} \mu_{j\bar{k}} z^j \overline{z}^k = \sum_{l=0}^{\infty} \sum_{j+k=l} \mu_{j\bar{k}k} z^j \overline{z}^k = \sum_{l=0}^{\infty} \mu_l. \tag{12.2}$$

Using Lemma 8.2, we can make the ACS standard at the origin, which implies that $\mu_0 = 0$, that is, μ has no constant term. We also write

$$w = \sum_{j,k} w_{j\bar{k}} z^j \overline{z}^k = \sum_{l=0}^{\infty} \sum_{j+k=l} w_{j\bar{k}} z^j \overline{z}^k = \sum_{l=0}^{\infty} w_l. \tag{12.3}$$

We want to find a holomorphic coordinate system, so we make the assumption that $w_0 = 0$ and $w_1 = z$.

We then have

$$w_{\overline{z}} = \sum_{l=2}^{\infty} \partial_{\overline{z}} w_l \tag{12.4}$$

$$w_z = 1 + \sum_{l=2}^{\infty} \partial_z w_l. \tag{12.5}$$

We then want to solve

$$w_{\overline{z}} = \sum_{l=2}^{\infty} \partial_{\overline{z}} w_l = \mu w_z = \Big(\sum_{l=1}^{\infty} \mu_l\Big)\Big(1 + \sum_{k=2}^{\infty} \partial_z w_k\Big) = \Big(\sum_{l=1}^{\infty} \mu_l\Big) + \sum_{l=2}^{\infty} \sum_{j+k=l, j\geq 1, k\geq 2} \mu_j \partial_z w_k. \tag{12.6}$$

We then find the recursion relation

$$\partial_{\overline{z}} w_{l+1} = \mu_l + \sum_{j+k=l+1, j\geq 1, k\geq 2} \mu_j \partial_z w_k. \tag{12.7}$$

Note that in the sum on the right hand side, we must have $k \leq l$, so this in indeed a recursion relation, provided that we can solve for w_{l+1}.

Fixing l, the right hand side is just a homogeneous polynomial of degree l in the variables z and $\overline{z}$. In general, if $f_l = \sum_{j+k=l, j\geq 0, k\geq 0} h_{j\bar{k}} z^j \overline{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j\geq 0, k\geq 0} \frac{1}{k+1} h_{j\bar{k}} z^j \overline{z}^{k+1} \tag{12.8}$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\overline{z}} F = f$.

Remark 12.1. Notice that our "inverse" of the $\overline{\partial}$-operator on homogeneous polynomials of degree l does not contain any terms proportional to z^{l+1}. Our inverse operator is unique with this condition. If we had not imposed this condition, one could have chosen $w_l = l! z^l + O(\overline{z})$, in which case our series would definitely not converge! Also, if we view our series as a power series in 2 complex variables, then formally $w(z, 0) = z$ exactly because of this choice of inverse to $\overline{\partial}$.

Proposition 12.2. *The coefficients $w_{j\bar{k}}$ for $j+k=l$ are a polynomial of degree $l-1$ in the $\mu_{p\bar{q}}$ for $p+q<l$ with all coefficients non-negative rational numbers.*

Proof. Let us examine the first few steps of the iteration. We have $w_{00}=1$, $w_{10}=1$, and $w_{0\bar{1}}=0$. The term w_2 is determined by

$$\partial_{\overline{z}} w_2 = \mu_1 = \mu_{1\bar{0}} z + \mu_{0\bar{1}} \overline{z}, \tag{12.9}$$

so

$$w_2 = \mu_{1\bar{0}} z\overline{z} + \frac{1}{2}\mu_{0\bar{1}} \overline{z}^2, \tag{12.10}$$

so

$$w_{2\bar{0}} = 0, \quad w_{1\bar{1}} = \mu_{1\bar{0}}, \quad w_{0\bar{2}} = \frac{1}{2}\mu_{0\bar{1}}. \tag{12.11}$$

To illustrate, let's do one more step. The term w_3 is determined by

$$\begin{aligned} \partial_{\overline{z}} w_3 = \mu_2 + \mu_1 \partial_z w_2 &= \mu_{2\bar{0}} z^2 + \mu_{1\bar{1}} z\overline{z} + \mu_{0\bar{2}}\overline{z}^2 + (\mu_{1\bar{0}} z + \mu_{0\bar{1}}\overline{z})(\mu_{1\bar{0}}\overline{z}) \\ &= \mu_{2\bar{0}} z^2 + (\mu_{1\bar{1}} + \mu_{1\bar{0}}^2) z\overline{z} + (\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}})\overline{z}^2. \end{aligned} \tag{12.12}$$

so

$$w_3 = \mu_{2\bar{0}} z^2\overline{z} + \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2) z\overline{z}^2 + \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}})\overline{z}^3. \tag{12.13}$$

so

$$w_{3\bar{0}} = 0, \quad w_{2\bar{0}} = \mu_{2\bar{0}}, \quad w_{1\bar{2}} = \frac{1}{2}(\mu_{1\bar{1}} + \mu_{1\bar{0}}^2), \quad w_{0\bar{3}} = \frac{1}{3}(\mu_{0\bar{2}} + \mu_{0\bar{1}}\mu_{1\bar{0}}), \tag{12.14}$$

and the claim is evidently true.

To do the general case, we prove by induction: assume the claim is true up to for $0,\dots,l$, and we prove for $l+1$. Recall that

$$\partial_{\overline{z}} w_{l+1} = \mu_l + \sum_{j+k=l+1, j\geq 1, k\geq 2} \mu_j \partial_z w_k. \tag{12.15}$$

By induction, the coefficients of w_k for $k\leq l$ are polynomials with non-negative coefficients in the $\mu_{p\bar{q}}$ with $p+\bar{q}<k<l$, so that $\partial_z w_k$ is also of this form. Then since

$$\mu_j = \sum_{k+l=j} \mu_{k\bar{l}} z^k \overline{z}^l, \tag{12.16}$$

any term $\mu_j \partial_z w_k$ is also a polynomial in the $\mu_{k\bar{l}}$ with non-negative coefficients.

To get w_{l+1}, recall that if $f_l = \sum_{j+k=l, j\geq 0, k\geq 0} h_{jk} z^j \overline{z}^k$, then

$$F_{l+1} = \sum_{j+k=l, j\geq 0, k\geq 0} \frac{1}{k+1} h_{jk} z^j \overline{z}^{k+1} \tag{12.17}$$

is a homogeneous polynomial of degree $l+1$, which satisfies $\partial_{\overline{z}} F = f$. Clearly, this preserves non-negativity of the coefficients, and we are done. □

Theorem 12.3. *If $\mu(z,\overline{z})$ is analytic in the closed polydisc $|z| \leq \rho, |\overline{z}| \leq \rho$, there there exists a unique solution of the Beltrami equation*

$$w_{\overline{z}} = \mu(z,\overline{z}) w_z \tag{12.18}$$

which is analytic in the polydisc $|z| < \rho', |\overline{z}| < \rho'$ for some $\rho' > 0$, and satisfies the Cauchy data

$$w(z,0) = z. \tag{12.19}$$

Proof. By assumption, the series

$$\mu = \sum_{j,k} \mu_{j\bar{k}} z^j \overline{z}^k \tag{12.20}$$

converges for any point in the polydisc

$$P(\rho) = \{(z,\overline{z}) \mid |z| < \rho, |\overline{z}| < \rho\}, \tag{12.21}$$

with uniform convergence in the polydisc $\overline{P(\rho')}$, for any $\rho' < \rho$. So for any $(z,\overline{z}) \in \overline{P(\rho')}$, there exists a constant $C > 0$ so that

$$|\mu_{j\bar{k}} z^j \overline{z}^k| < C \text{ (no summation)}. \tag{12.22}$$

Choosing $(z,\overline{z}) = (\rho', \rho')$, this implies that

$$|\mu_{j\bar{k}}| < C(\rho')^{-j-k}. \tag{12.23}$$

To simplify notation, let's call ρ' by ρ. Then we define

$$\mu^* = C\Big(\frac{1}{1-(z+\overline{z})\rho^{-1}} - 1\Big) = C\frac{z+\overline{z}}{\rho - z - \overline{z}}, \tag{12.24}$$

which is analytic in the polydisc $P(\rho) = \{(z,\overline{z}) \mid |z| < \rho, |\overline{z}| < \rho\}$. We have

$$\mu^* = C\sum_{j\geq 1}(z+\overline{z})^j \rho^{-j} = C \sum_{(k,l)\neq(0,0)} \rho^{-k-l}\frac{(k+l)!}{k!l!} z^k \overline{z}^l. \tag{12.25}$$

Since the multinomial coefficients are at least 1, we therefore have

$$|\mu_{j\bar{k}}| \leq C\rho^{-j-k} \leq \frac{(j+k)!}{j!k!}\rho^{-j-k} = \mu^*_{j\bar{k}}. \tag{12.26}$$

Recall from Proposition 11.1 that there is a solution w^* of the Beltrami equation for μ^* satisfying $w(z,0) = z$ which is analytic in $P(\rho')$ for some $\rho' > 0$. Write the power series expansion for w^* as

$$w^*(z,\overline{z}) = \sum_{(j,k)\neq(0,0)} w^*_{j\bar{k}} z^j \overline{z}^k. \tag{12.27}$$

Recall that our formal power series solves

$$w_{j\bar{k}} = P_{j\bar{k}}(\mu_{*\bar{*}}), \tag{12.28}$$

where $P_{j\bar{k}}$ is a polynomial with positive coefficients depending only upon $\mu_{p\bar{q}}$ for $p+q < j+k$. Since w^* is an analytic solution of the Beltrami equation with μ^*, we must also have

$$w^*_{j\bar{k}} = P_{j\bar{k}}(\mu^*_{*\bar{*}}), \tag{12.29}$$

where $P_{j\bar{k}}$ is the *same* polynomial since $\mu^*(0,0) = 0$ and $w^*(z,0) = z$. We then estimate

$$|w_{j\bar{k}}| = |P_{j\bar{k}}(\mu_{*\bar{*}})| \leq P_{j\bar{k}}(|\mu_{*\bar{*}}|) \leq P_{j\bar{k}}(\mu^*_{*\bar{*}}) = w^*_{j\bar{k}}. \tag{12.30}$$

The inequalities hold since $P_{j\bar{k}}$ is a polynomial with real non-negative coefficients, and using (12.26). This shows that our power series is majorized by the power series of w^*, which implies that the power series for w also converges in the open polydisc $P(\rho')$, by the comparison test. $\square$

13 Lecture 13

13.1 Reduction to the analytic case

In the subsection, we will discuss a method of Malgrange, which transforms the smooth case into the analytic case [Mal69], [Nir73, Section I.4]. We want to change coordinates $\xi = \xi(z, \overline{z})$ so that such that our solution of the Beltrami equation in the z-coordinates

$$w_{\overline{z}} + \mu(z, \overline{z}) w_z \tag{13.1}$$

transforms into another Beltrami equation,

$$W_{\bar{\xi}} + \tilde{U}(\xi, \bar{\xi}) W_{\xi} = 0, \tag{13.2}$$

with $\tilde{U}$ analytic. Note that we want a real change of coordinates, so if we write $\xi = \xi_1 + i\xi_2$, we need

$$\det \begin{pmatrix} \frac{\partial \xi_1}{\partial x} & \frac{\partial \xi_1}{\partial y} \\ \frac{\partial \xi_2}{\partial x} & \frac{\partial \xi_2}{\partial y} \end{pmatrix} (0,0) \neq 0. \tag{13.3}$$

As we know, after a change of basis, this is

$$\det \begin{pmatrix} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial \overline{z}} \\ \frac{\partial \bar{\xi}}{\partial z} & \frac{\partial \bar{\xi}}{\partial \overline{z}} \end{pmatrix} (0,0) = \left| \frac{\partial \xi}{\partial z}(0,0) \right|^2 - \left| \frac{\partial \xi}{\partial \overline{z}}(0,0) \right|^2 \neq 0. \tag{13.4}$$

Write

$$w(z, \overline{z}) = W(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})) \tag{13.5}$$
$$\mu(z, \overline{z}) = U(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})). \tag{13.6}$$

Then

$$\frac{\partial w}{\partial \overline{z}} = \frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial \overline{z}} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial \overline{z}} \tag{13.7}$$

$$\frac{\partial w}{\partial z} = \frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial z}. \tag{13.8}$$

So the Beltrami equation becomes

$$\frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial \overline{z}} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial \overline{z}} = -U(\xi,\bar{\xi})\Big(\frac{\partial W}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial W}{\partial \bar{\xi}}\frac{\partial \bar{\xi}}{\partial z}\Big), \tag{13.9}$$

which we can write as

$$\frac{\partial W}{\partial \bar{\xi}} = -\left(\frac{\frac{\partial \xi}{\partial \overline{z}} + U(\xi,\bar{\xi})\frac{\partial \xi}{\partial z}}{\frac{\partial \bar{\xi}}{\partial \overline{z}} + U(\xi,\bar{\xi})\frac{\partial \bar{\xi}}{\partial z}}\right)\frac{\partial W}{\partial \xi}, \tag{13.10}$$

which is another Beltrami equation with a new right hand side

$$\tilde{U}(\xi,\bar{\xi}) = \frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}. \tag{13.11}$$

Let us try to find the coordinates so that

$$\frac{\partial}{\partial \xi}\tilde{U}(\xi,\bar{\xi}) = 0. \tag{13.12}$$

Then then new $\tilde{U}$ will be anti-holomorphic and therefore analytic by the Cauchy integral formula. From the chain rule, we have

$$\frac{\partial}{\partial \xi} = \frac{\partial z}{\partial \xi}\frac{\partial}{\partial z} + \frac{\partial \overline{z}}{\partial \xi}\frac{\partial}{\partial \overline{z}}, \tag{13.13}$$

and we have

$$\begin{aligned}\frac{\partial}{\partial \xi}\tilde{U}(\xi,\bar{\xi}) &= \frac{\partial}{\partial \xi}\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big) \\ &= \Big(z_\xi \partial_z + \overline{z}_\xi \partial_{\overline{z}}\Big)\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big).\end{aligned} \tag{13.14}$$

By the inverse function theorem, we have

$$\begin{pmatrix} z_\xi & z_{\bar{\xi}} \\ \overline{z}_\xi & \overline{z}_{\bar{\xi}} \end{pmatrix} = \begin{pmatrix} \xi_z & \xi_{\bar{z}} \\ \bar{\xi}_z & \bar{\xi}_{\bar{z}} \end{pmatrix}^{-1} = \frac{1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}\begin{pmatrix} \bar{\xi}_{\bar{z}} & -\xi_{\bar{z}} \\ -\bar{\xi}_z & \xi_z \end{pmatrix}, \tag{13.15}$$

so

$$z_\xi = \frac{1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}\bar{\xi}_{\bar{z}} \tag{13.16}$$

$$\overline{z}_\xi = \frac{-1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}\bar{\xi}_z. \tag{13.17}$$

We therefore have

$$\frac{\partial}{\partial \xi}\tilde{U}(\xi,\bar{\xi}) = \frac{1}{|\xi_z|^2 - |\xi_{\overline{z}}|^2}(\bar{\xi}_{\bar{z}}\partial_z - \bar{\xi}_z\partial_{\overline{z}})\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big). \tag{13.18}$$

If we multiply through by the leading factor, we want to solve

$$0 = (\bar{\xi}_{\bar{z}}\partial_z - \bar{\xi}_z\partial_{\overline{z}})\Big(\frac{\xi_{\overline{z}} + U(\xi,\bar{\xi})\xi_z}{\bar{\xi}_{\overline{z}} + U(\xi,\bar{\xi})\bar{\xi}_z}\Big) \tag{13.19}$$

but keep in mind that we need to find a solution with $|\xi_z|^2(0,0) - |\xi_{\overline{z}}|^2(0,0) \neq 0$. Converting the $U(\xi,\bar{\xi})$ term back to the $(z,\overline{z})$ coordinates, we have

$$0 = (\bar{\xi}_{\bar{z}}\partial_z - \bar{\xi}_z\partial_{\overline{z}})\Big(\frac{\xi_{\overline{z}} + \mu(z,\overline{z})\xi_z}{\bar{\xi}_{\overline{z}} + \mu(z,\overline{z})\bar{\xi}_z}\Big). \tag{13.20}$$

The equation (13.20) is *quasilinear* of the form

$$F(D^2\xi, D\xi, \xi, z, \overline{z}) = 0. \tag{13.21}$$

Definition 13.1. The linearization of F at a function ξ is given by

$$F'_{\xi}(h) = \frac{d}{dt}F(D^2(\xi+th), D(\xi+th), \xi+th, z, \overline{z})\Big|_{t=0}. \tag{13.22}$$

The linearization is too complicated to write down in general, but the following is all that we really need.

Proposition 13.2. *Assuming $\mu \in C^1$, then the linearization of F at $\xi = z$ is*

$$F'_z(h) = \partial_z\Big(h_{\overline{z}} + \mu(h_z - \bar{h}_{\overline{z}} - \mu\bar{h}_z)\Big) + \bar{h}_{\bar{z}}\mu_z - \bar{h}_z\mu_{\overline{z}}. \tag{13.23}$$

If $\mu(0,0) = 0$, then we have

$$F'_z(h)(0,0) = \frac{1}{4}\Delta h + c_1 h_z + c_2\bar{h}_z + c_3 h_{\overline{z}} + c_4\bar{h}_{\overline{z}}. \tag{13.24}$$

for some constants c_1, c_2, c_3, c_4. If μ has sufficiently small $C^{1,\alpha}$, norm then F'_z is an elliptic operator with Hölder coefficients bounded in C^α.

Proof. We write out

$$F(D^2(\xi+th), D(\xi+th), \xi+th, z, \overline{z}) \tag{13.25}$$

$$= \Big((\overline{\xi+th})_{\bar{z}}\partial_z - (\overline{\xi+th})_z\partial_{\overline{z}}\Big)\Big(\frac{(\xi+th)_{\overline{z}} + \mu(z,\overline{z})(\xi+th)_z}{(\overline{\xi+th})_{\overline{z}} + \mu(z,\overline{z})(\overline{\xi+th})_z}\Big). \tag{13.26}$$

Letting $\xi = z$, this becomes

$$F(D^2(z+th), D(z+th), z+th, z, \overline{z}) \tag{13.27}$$

$$= \Big((1+t\bar{h}_{\bar{z}})\partial_z - t\bar{h}_z\partial_{\overline{z}}\Big)\Big(\frac{th_{\overline{z}} + \mu(z,\bar{z})(1+th_z)}{(1+t\bar{h}_{\overline{z}}) + \mu(z,\overline{z})t\bar{h}_z}\Big). \tag{13.28}$$

We also see that

$$F'_z(h) = \partial_z\Big(h_{\overline{z}} + \mu(h_z - \bar{h}_{\overline{z}} - \mu\bar{h}_z)\Big) + \bar{h}_{\bar{z}}\mu_z - \bar{h}_z\mu_{\overline{z}}. \tag{13.29}$$

Noting that

$$\frac{\partial^2 h}{\partial z \partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)h = \frac{1}{4}\Delta h, \tag{13.30}$$

the proposition follows from this. □

We next need the inverse function theorem in Banach spaces.

Lemma 13.3. *Let $\mathscr{F} : \mathcal{B}_1 \to \mathcal{B}_2$ be a C^1-map between two Banach spaces such that $\mathscr{F}(x) = \mathscr{F}(0) + \mathscr{L}(x) + \mathscr{Q}(x)$, where the operator $\mathscr{L} : \mathcal{B}_1 \to \mathcal{B}_2$ is linear and $\mathscr{Q}(0) = 0$. Assume that*

1. *$\mathscr{L}$ is an isomorphism with inverse T satisfying $\|T\| \leq C_1$,*
2. *there are constants $r > 0$ and $C_2 > 0$ with $r < \frac{1}{3C_1C_2}$ such that*
   (a) *$\|\mathscr{Q}(x) - \mathscr{Q}(y)\|_{\mathcal{B}_2} \leq C_2 \cdot (\|x\|_{\mathcal{B}_1} + \|y\|_{\mathcal{B}_1}) \cdot \|x - y\|_{\mathcal{B}_1}$ for all $x, y \in B_r(0) \subset \mathcal{B}_1$,*
   (b) *$\|\mathscr{F}(0)\|_{\mathcal{B}_2} \leq \frac{r}{3C_1}$.*

Then there exists a unique solution to $\mathscr{F}(x) = 0$ in $\mathcal{B}_1$ such that

$$\|x\|_{\mathcal{B}_1} \leq 3C_1 \cdot \|\mathscr{F}(0)\|_{\mathcal{B}_2}. \tag{13.31}$$

Proof. Writing $x = Tf$, we can write the equation $\mathscr{F}(x) = 0$ as

$$\mathscr{F}(0) + f + \mathscr{Q}(Tf) = 0, \tag{13.32}$$

that is

$$f = -\mathscr{Q}(Tf) - \mathscr{F}(0). \tag{13.33}$$

So we would like to find a fixed point of the operator $S : \mathcal{B}_2 \to \mathcal{B}_2$ defined by

$$Sf = -\mathscr{Q}(Tf) - \mathscr{F}(0). \tag{13.34}$$

We next claim that under the assumptions, S is a contraction mapping from $B_{r/C_1}(0) \subset \mathcal{B}_2$. To see this, we compute

$$\begin{aligned} \|Sf_1 - Sf_2\|_{\mathcal{B}_2} &= \|\mathscr{Q}(Tf_1) - \mathscr{Q}(Tf_2)\|_{\mathcal{B}_2} \\ &\leq C_2(\|Tf_1\|_{\mathcal{B}_1} + \|Tf_2\|_{\mathcal{B}_1})(\|Tf_1 - Tf_2\|_{\mathcal{B}_1} \\ &\leq C_2(2C_1r/C_1)C_1\|f_1 - f_2\|_{\mathcal{B}_2} \leq \frac{2}{3}(\|f_1 - f_2\|_{\mathcal{B}_2}). \end{aligned} \tag{13.35}$$

We then let $f_0 = 0$, and define $f_{j+1} = Sf_j$. If $n \geq m$, we have

$$
\begin{aligned}
\|f_n - f_m\|_{\mathcal{B}_2} &\leq \sum_{j=m+1}^{n} \|f_j - f_{j-1}\|_{\mathcal{B}_2} \\
&= \sum_{j=m+1}^{n} \|S^{j-1} f_1 - S^{j-1} f_0\|_{\mathcal{B}_2} \\
&\leq \sum_{j=m+1}^{n} \left(\frac{2}{3}\right)^{j-1} \|f_1 - f_0\|_{\mathcal{B}_2} \\
&\leq \frac{(2/3)^m}{1 - 2/3} \|f_1 - f_0\|_{\mathcal{B}_2}.
\end{aligned}
\tag{13.36}
$$

The right hand side limits to 0 as $m \to \infty$. This proves that the sequence f_j is a Cauchy sequence in the Banach space $\mathcal{B}_2$, which therefore converges to a limit f_∞. Since S is continuous, we therefore have

$$
Sf_\infty = S \lim_{j\to\infty} f_j = \lim_{j\to\infty} Sf_j = \lim_{j\to\infty} f_{j+1} = f_\infty. \tag{13.37}
$$

Take $m = 1$ in (13.36) to get

$$
\|f_n - f_1\|_{\mathcal{B}_2} \leq 2\|f_1 - f_0\|_{\mathcal{B}_2} = 2\|f_1\|_{\mathcal{B}_2} \tag{13.38}
$$

Letting $n \to \infty$ yields

$$
\|f_\infty\|_{\mathcal{B}_2} - \|f_1\|_{\mathcal{B}_2} \leq \|f_\infty - f_1\|_{\mathcal{B}_2} \leq 2\|f_1\|_{\mathcal{B}_2}. \tag{13.39}
$$

Then $x_\infty = Tf_\infty$ is a solution to $\mathscr{F}(x_\infty) = 0$ and

$$
\|x_\infty\|_{\mathcal{B}_1} \leq C_1 \|f_\infty\|_{\mathcal{B}_2} \leq 3C_1 \|\mathscr{F}(0)\|_{\mathcal{B}_2} \tag{13.40}
$$

which implies (13.31). If x is any solution satisfying (13.31), then letting $f = \mathscr{L}x$, we estimate

$$
\|f\|_{\mathcal{B}_2} \leq \frac{1}{C_1}\|x\|_{\mathcal{B}_1} \leq \frac{1}{C_1} 3C_1 \frac{r}{3C_1} = \frac{r}{C_1} \tag{13.41}
$$

and uniqueness then follows from (13.35). $\square$

Remark 13.4. In the above statment, $\mathscr{Q}$ is a nonlinear operator. But it is easy to see that the result also holds if $\mathscr{Q}$ is linear, but with sufficiently small operator norm satisfying $\|\mathscr{Q}\| < \|T\|$.

Back to our problem, we define

$$
\begin{aligned}
\mathcal{B}_1 &= C^{2,\alpha}_0(B_1(0)) = \{u \in C^{2,\alpha}(B_1(0)) \mid u = 0 \text{ on } \partial B_1(0)\} &(13.42)\\
\mathcal{B}_2 &= C^{0,\alpha}(B_1(0)) &(13.43)\\
\mathscr{F}(h) &= F(z + h) &(13.44)\\
\mathscr{L}(h) &= F'_z(h) &(13.45)\\
\mathscr{Q}(h) &= F(z + h) - F(z) - F'_z(h) = \mathscr{F}(h) - \mathscr{F}(0) - \mathscr{L}(h). &(13.46)
\end{aligned}
$$

By scaling the coordinates $z = \epsilon z'$, and letting $w'(z', \overline{z}') = w(\epsilon z', \epsilon \overline{z}')$, then we have

$$w'_{\overline{z}'} = \mu(\epsilon z', \epsilon \overline{z}') w'_{z'}. \tag{13.47}$$

So by choosing ϵ small, we can assume that

$$\|\mu\|_{C^{1,\alpha}(B(0,1))} < C\epsilon^{1+\alpha}. \tag{13.48}$$

From Proposition 20.2 we see that $\mathscr{L}$ is an elliptic operator. The mapping $T : C^{0,\alpha} \to C^{2,\alpha}_0$ is defined to be the unique solution to the Dirichlet problem

$$\mathscr{L}(Tf) = f \text{ in } B(0,1), \quad Tf = 0 \text{ on } \partial B(0,1). \tag{13.49}$$

From basic elliptic theory, there exists a constant C so that

$$\|Tf\|_{C^{2,\alpha}(B(0,1))} \leq C\|f\|_{C^{0,\alpha}(B(0,1))}. \tag{13.50}$$

This constant will be uniformly bounded for sufficiently small ϵ since the elliptic estimates only depend upon the $C^{0,\alpha}$ norm of the coefficients. (Note: since $\mathscr{L}$ differs from the Laplacian by lower order terms, it actually suffices to just invert the Laplacian, but we invert $\mathscr{L}$ here for simplicity; more on this later.) We need to estimate

$$\mathscr{Q}(h_2) - \mathscr{Q}(h_2) = F(z + h_2) - F'_z(h_2) - (F(z + h_1) - F'_z(h_1)). \tag{13.51}$$

Consider $f(t) = F(z + th_2 + (1-t)h_1)$. Since $F(z + h) = \tilde{F}(D^2h, Dh, h, z)$, where $\tilde{F}$ is a C^1 function of these variables, then using the fundamental theorem of calculus

$$F(z + h_2) - F(z + h_1) = f(1) - f(0) = \int_0^1 f'(t)dt \tag{13.52}$$

We note that

$$\begin{aligned} f'(t) = \frac{d}{ds}F(z + sh_2 + (1-s)h_1)|_{s=t} &= \frac{d}{ds}F(z + th_1 + (1-t)h_2 + s(h_2 - h_1))|_{s=0} \\ &= F'_{z+th_1+(1-t)h_2}(h_2 - h_1). \end{aligned} \tag{13.53}$$

This gives the expression

$$\begin{aligned} \mathscr{Q}(h_2) - \mathscr{Q}(h_1) &= \Big(\int_0^1 F'_{z+th_1+(1-t)h_2}(h_2 - h_1)dt\Big) - F'_z(h_2 - h_1) \\ &= \Big(\int_0^1 (F'_{z+th_1+(1-t)h_2} - F'_z)dt\Big)(h_2 - h_1). \end{aligned} \tag{13.54}$$

We next claim that for any y and h, we have the estimate

$$\|(F'_{z+h} - F'_z)y\|_{C^0} \leq C\|h\|_{C^2}\|y\|_{C^2}. \tag{13.55}$$

To see this, note the linearized operator is of the form

$$\begin{aligned} F'_u(h) &= \frac{d}{dt}F(u + th)|_{t=0} \\ &= a^{ij}(D^2u, Du, u, z)D_{ij}h + b^i(D^2u, Du, u, z)D_ih + c(D^2u, Du, u, z)D_ih. \end{aligned} \tag{13.56}$$

If $\tilde{F}(D^2h, Dh, h, z)$ is C^2 as in the D^2h, Dh, h variables, and continuous in the z variable, then the coefficients a^{ij}, b^i, c are C^1 as functions of D^2u, Du, u, and we have for example

$$c(D^2(z+h), D(z+h), z+h, z) = c(D^2z, Dz, z, z) + O(|D^2h| + |Dh| + |h|), \tag{13.57}$$

so the estimate (13.55) follows. This implies the quadratic estimate for the C^0-norm.

For the Hölder norm, similar to the above arguments, we see that any y and h, we have the estimate

$$\|(F'_{z+h} - F'_z)y\|_{C^\alpha} \leq C\|h\|_{C^{2,\alpha}}\|y\|_{C^{2,\alpha}}, \tag{13.58}$$

provided that $\tilde{F}$ is $C^{2,\alpha}$ in the D^2h, Dh, h variables and Hölder continuous in the z variable. This finishes the quadratic estimate.

Also, by taking ϵ sufficiently small, we can always arrange so that condition (b) is satisfied. The implicit function theorem yields a solution h with

$$\|h\|_{C^{2,\alpha}(B(0,1))} = o(\epsilon), \tag{13.59}$$

as $\epsilon \to 0$. Obviously, we have

$$|h(0)| = o(\epsilon), \quad |\nabla h|(0) = o(\epsilon), \tag{13.60}$$

as $\epsilon \to 0$. Then if ϵ is sufficiently small, then condition (13.4) will also be satisfied.

Remark 13.5. The minimal regularity required in the above argument is $\mu \in C^{1,\alpha}$. One can actually get away with only assuming $\mu \in C^{0,\alpha}$, but one needs a different method to see this.

14 Lecture 14

A reference for this section in [Tay11, page 376].

14.1 Relation with isothermal coordinates

Given a Riemannian metric g, recall the isomorphism

$$\flat : TM \to T^*M \tag{14.1}$$

defined by $\flat(X)(Y) = g(X, Y)$, with inverse $\# : T^*M \to TM$.

Proposition 14.1. *For $n = 1$, an oriented conformal structure $(M, [g])$ is equivalent to an almost complex structure $J : TM \to TM$. More precisely, given a conformal class $[g]$ and an orientation, choose a Riemannian metric $g \in [g]$. Then the Hodge star operator $*_g : T^*M \to T^*M$ satisfies $*_g^2 = -1$, and then $J = \#_g \circ *_g \circ \flat_g$ is an almost complex structure which is compatible with g (it is an isometry with respect to g). Conversely, given an almost complex structure $J : TM \to TM$, choose any Riemannian metric g compatible with J, and we map $J \to [g]$, with the complex orientation.*

Proof. Choose any representative g of the conformal class $[g]$. The Hodge star operator is an isometry on 1-forms uniquely defined by

$$\alpha \wedge *_g \beta = g(\alpha, \beta) dV_g, \tag{14.2}$$

where dV_g is the oriented volume element. Then $*_g : T^*M \to T^*M$ satisfies $*^2 = -Id$, so the mapping $\#_g \circ *_g \circ \flat_g$ is an almost complex structure. This is clearly conformally invariant, since if $\tilde{g} = fg$, then $dV_{\tilde{g}} = f dV_g$, and $\tilde{g}(\alpha, \beta) = f^{-1} g(\alpha, \beta)$.

Conversely, given an almost complex structure J, we know this determines an orientation. Choose any non-zero 2-form compatible with this orientation, and call it dV_g. Then we define an inner product on 1-forms by

$$\alpha \wedge J^* \beta = g(\alpha, \beta) dV_g, \tag{14.3}$$

It is positive definite because by Lemma 8.2, at any point, we can assume that J is standard. So there is a basis dx, dy of $T^*_p M$ such that $Jdx = -dy$. The complex orientation is $dy \wedge dx$. Writing any form $\alpha = \alpha_1 dx + \alpha_2 dy$, we then have

$$\alpha \wedge * \alpha = (\alpha_1 dx + \alpha_2 dy) \wedge (-\alpha_1 dy + \alpha_2 dx) = (|\alpha_1|^2 + |\alpha_2|^2) dy \wedge dx. \tag{14.4}$$

Clearly, different choices of volume elements lead to conformally equivalent metrics. Note also the mapping J^* will be an isometry, so we have $J = \#_g \circ J^* \circ \flat_g$.

□

Remark 14.2. Instead of using the Hodge star and the sharp and flat operators, one could argue directly as follows. Given an oriented conformal class $[g]$ and a non-zero orientation 2-form ω, choose a Riemannian metric $g \in [g]$ and scale ω so that it is the volume element of g. Then define $J : TM \to TM$ by $\omega(X, JY) = g(X, Y)$. Then J is an almost complex structure. Conversely, given J, then J determines the complex orientation, and let ω be any non-zero 2-form compatible with this orientation. Then define $g(X, Y) = \omega(X, JY)$. The proofs are entirely equivalent the above; see [Har90, Theorem 5.34].

Proposition 14.3. *Given (M_1, J_1) and (M_2, J_2), choose any compatible Riemannian metrics g_1 on M_1 and g_2 on M_2. Let $\phi : M_1 \to M_2$ be a local diffeomorphism. Then ϕ is holomorphic if and only if ϕ is orientation-preserving and conformal.*

Proof. The Cauchy-Riemann equations are

$$\phi_* \circ J_1 = J_2 \circ \phi_*. \tag{14.5}$$

Taking the dual of both sides yields

$$(J_1)^* \circ \phi^* = \phi^* \circ (J_2)^*. \tag{14.6}$$

Since $*_1$ is an isometry, we have $J_1^* = *_1$, and similarly $J_2^* = *_2$, so we have

$$*_1 \circ \phi^* = \phi^* \circ *_2. \tag{14.7}$$

On M_2, we have the equation

$$\alpha \wedge *_2\beta = g_2(\alpha, \beta) dV_2, \tag{14.8}$$

pulling this back under ϕ yields

$$(\phi^*\alpha) \wedge (\phi^* \circ *_2)\beta = (g_2(\alpha, \beta) \circ \phi)\phi^* dV_2. \tag{14.9}$$

Using (14.7), this is

$$(\phi^*\alpha) \wedge (*_1 \circ \phi^*\beta) = (g_2(\alpha, \beta) \circ \phi)\phi^* dV_2. \tag{14.10}$$

Since ϕ is a local diffomorphism, let's replace α with $\phi_*\alpha$ and β with $\phi_*\beta$, and we obtain

$$\alpha \wedge *_1\beta = (g_2(\phi_*\alpha, \phi_*\beta) \circ \phi)\phi^* dV_2, \tag{14.11}$$

but the left hand side is $g_1(\alpha, \beta)dV_1$, so we conclude that $\phi^* g_2 = e^\lambda g_1$, for some function $\lambda : M_1 \to \mathbb{R}$. For the converse, reverse the above argument. $\square$

Corollary 14.4. *The problem of isothermal coordinates for a Riemannian metric g is equivalent to solving the Beltrami equation for the almost complex structure determined by $*_g$. That is, solving the Beltrami equation $w_{\overline{z}} = \mu w_z$ in a neighborhood of a point p with $dw(p) \neq 0$ is equivalent to finding a coordinate system $\phi : U \to \mathbb{R}^2$ so that*

$$\phi_* g = e^{\lambda(x,y)}(dx^2 + dy^2), \tag{14.12}$$

for some function $\lambda : \phi(U) \to \mathbb{R}$.

Proof. We know that a solution of the Beltrami equation $w_{\overline{z}} = \mu(z, \overline{z})w_z$ is a holomorphic function. As long as $\partial_z w(p) \neq 0$, then we know that

$$w : (U, J) \to (\mathbb{C}, J_0) \tag{14.13}$$

is a holomorphic coordinate system. From Proposition 14.3, w must be conformal and orientation preserving. But the conformal class determined by J_0 is the conformal class of the Euclidean metric, so we have found isothermal coordinates.

Conversely, given an isothermal coordinate system $\phi : (U, [g]) \to (\mathbb{R}^2, [g_{Euc}])$. Then $[g]$ induces a unique J, and by the above, ϕ must be pseudoholomorphic with respect to J, so yields a solution of the Beltrami equation. $\square$

We can write down explicity the above in coordinates.

Proposition 14.5. *Given $g = g_{ij}dx^i \otimes dx^j$, then*

$$*_g = \frac{\pm 1}{\sqrt{\det(g)}} \begin{pmatrix} g_{12} & -g_{11} \\ g_{22} & -g_{12} \end{pmatrix}, \tag{14.14}$$

depending upon choice of orientation. Consequently, if $\tilde{g} = fg$, then $*_{\tilde{g}} = *_g$. Conversely, given any

$$J = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{14.15}$$

with $a^2 + bc = -1$, and a choice of volume form $f dx^1 \wedge dx^2$, we define a Riemannian metric up to scaling by

$$g = \pm f \begin{pmatrix} -b & a \\ a & c \end{pmatrix}, \tag{14.16}$$

for the sign choice which makes this positive definite.

Proof. We choose a coordinate system $\{x^1, x^2\}$, and write

$$g(\partial_i, \partial_j) = g_{ij}. \tag{14.17}$$

We then have

$$g(dx^i, dx^j) = g^{ij}, \tag{14.18}$$

where g^{ij} are the components of the inverse matrix of g_{ij}. Also, we have

$$dV_g = \sqrt{\det(g)} dx^1 \wedge dx^2. \tag{14.19}$$

In matrix form, we write

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad g^{-1} = \frac{1}{\det(g)} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix}. \tag{14.20}$$

We write

$$*dx^1 = a_{11} dx^1 + a_{21} dx^2, \quad *dx^2 = a_{12} dx^1 + a_{22} dx^2, \tag{14.21}$$

which in matrix form is just

$$* = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{14.22}$$

We then have

$$dx^1 \wedge *dx^1 = dx^1 \wedge (a_{11} dx^1 + a_{21} dx^2) = a_{21} dx^1 \wedge dx^2 \tag{14.23}$$
$$dx^1 \wedge *dx^2 = dx^1 \wedge (a_{12} dx^1 + a_{22} dx^2) = a_{22} dx^1 \wedge dx^2 \tag{14.24}$$
$$dx^2 \wedge *dx^1 = dx^2 \wedge (a_{11} dx^1 + a_{21} dx^2) = -a_{11} dx^1 \wedge dx^2 \tag{14.25}$$
$$dx^2 \wedge *dx^2 = dx^2 \wedge (a_{12} dx^1 + a_{22} dx^2) = -a_{12} dx^1 \wedge dx^2. \tag{14.26}$$

On the other hand, by definition of the Hodge star operator, these must be equal to

$$g(dx^1, dx^1) dV_g = g^{11} \sqrt{\det(g)} dx^1 \wedge dx^2 = \frac{g_{22}}{\sqrt{\det(g)}} \tag{14.27}$$
$$g(dx^1, dx^2) dV_g = g^{12} \sqrt{\det(g)} dx^1 \wedge dx^2 = -\frac{g_{12}}{\sqrt{\det(g)}} \tag{14.28}$$
$$g(dx^2, dx^1) dV_g = g^{21} \sqrt{\det(g)} dx^1 \wedge dx^2 = -\frac{g_{21}}{\sqrt{\det(g)}} \tag{14.29}$$
$$g(dx^2, dx^2) dV_g = g^{22} \sqrt{\det(g)} dx^1 \wedge dx^2 = \frac{g_{11}}{\sqrt{\det(g)}}. \tag{14.30}$$

Comparing these equations, we obtain

$$*_g = \frac{\pm 1}{\sqrt{\det(g)}} \begin{pmatrix} g_{21} & -g_{11} \\ g_{22} & -g_{12} \end{pmatrix}. \tag{14.31}$$

This expression is obviously conformally invariant.

Conversely, given J and a volume element $dV = f dx^1 \wedge dx^2$, we define an inner product by

$$g(\alpha, \beta) dV_g = \alpha \wedge J\beta. \tag{14.32}$$

We then have

$$g(dx^i, dx^j) dV_g = g^{ij} f dx^1 \wedge dx^2, \tag{14.33}$$

and by the above, we see that

$$g^{-1} = \frac{1}{f} \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix}, \tag{14.34}$$

so then

$$g = f \begin{pmatrix} -b & a \\ a & c \end{pmatrix}. \tag{14.35}$$

□

14.2 Reduction to harmonic functions

We have already solved the Beltrami equation using the Malgrange method, we will next present an alternative proof using harmonic functions.

Proposition 14.6. *If (M^2, J) is a real 2-dimensional almost complex manifold with J of class C^2, then J is a complex 1-manifold.*

Proof. As before, choose a compatible Riemannian metric g, and let $*$ be the Hodge star operator with respect to the almost complex orientation. Then on 1-forms, $J = *$. Given any point x in M, by Proposition 17.2, we need to find a function $f : U \to \mathbb{C}$ where U is a neighborhood of x satisfying $\overline{\partial}_J f = 0$ in U, and $\partial_J f(x) \neq 0$. This equation is

$$0 = \overline{\partial}_J f = df + iJdf = df + i * df. \tag{14.36}$$

Let us write $f = u + iv$, where u and v are real-valued. Then we need

$$0 = du + idv + iJ(du + idv) = (du - *dv) + i(dv + *du). \tag{14.37}$$

Note that applying the Hodge star to $du = *dv$, results in $dv = - * du$, so if we solve the single equation

$$du = *dv, \tag{14.38}$$

then $f = u + iv$ will be pseudo-holomorphic. Note that

$$\partial f = (du + *dv) + i(dv - *du), \tag{14.39}$$

so if $du(x) \neq 0$, then $\partial f(x) \neq 0$. To solve (14.38), we apply $*d$ to get

$$*d * dv = \delta dv = \Delta_g v = 0. \tag{14.40}$$

So if v is harmonic, then $*dv$ is closed, and by the Poincaré Lemma, we can solve $*dv = du$ in any simply-connected neighborhood U of x. To summarize, we have reduced the problem to finding a simply-connected neighborhood U of x, and a harmonic function $v : U \to \mathbb{R}$ with $dv(x) \neq 0$. □

15 Lecture 15

15.1 Inverse function theorem

Let's state another version of the inverse function theorem for linear operators.

Lemma 15.1. *Let $\mathscr{F} : \mathcal{B}_1 \to \mathcal{B}_2$ be a bounded linear mapping between two Banach spaces such that $\mathscr{F}(x) = \mathscr{L}(x) + \mathscr{Q}(x)$, where $\mathscr{L}$ and $\mathscr{Q}$ are both bounded linear mappings. Assume that*

1. *$\mathscr{L}$ is an isomorphism with bounded inverse T.*
2. *$\mathscr{Q}$ satisfies $\|\mathscr{Q}\| \cdot \|T\| = \delta < 1$.*

Then $\mathscr{F}$ is also an isomorphism and

$$\|\mathscr{F}^{-1}\| \leq \frac{1}{1-\delta}\|T\|. \tag{15.1}$$

Proof. Given $f \in \mathcal{B}_2$, we want to solve the equation $\mathscr{F}x = f$ for a unique $x \in \mathcal{B}_1$ with a bound $\|x\|_{\mathcal{B}_1} \leq C\|f\|_{\mathcal{B}_2}$. Writing $x = Ty$, then the equation we want to solve becomes

$$\mathscr{F}(Ty) = (\mathscr{L} + \mathscr{Q})(Ty) = f, \tag{15.2}$$

or

$$y = f - \mathscr{Q}(Ty) \tag{15.3}$$

So we would like to find a fixed point of the operator $S : \mathcal{B}_2 \to \mathcal{B}_2$ defined by

$$Sy = f - \mathscr{Q}(Ty) \tag{15.4}$$

We next claim that under the assumptions, S is a contraction mapping from $\mathcal{B}_2$ to $\mathcal{B}_2$. To see this, we compute

$$\begin{aligned} \|Sy_1 - Sy_2\|_{\mathcal{B}_2} &= \|\mathscr{Q}(Ty_1) - \mathscr{Q}(Ty_2)\|_{\mathcal{B}_2} \\ &\leq \|\mathscr{Q}\| \cdot \|Ty_1 - Ty_2\|_{\mathcal{B}_1} \\ &\leq \|\mathscr{Q}\| \cdot \|T\| \cdot \|y_1 - y_2\|_{\mathcal{B}_2} = \delta\|y_1 - y_2\|_{\mathcal{B}_2}. \end{aligned} \tag{15.5}$$

where $\delta = \|\mathscr{Q}\|\|T\| < 1$ by assumption. We then let $y_0 = 0$, and define $y_{j+1} = Sy_j$. If $n \geq m$, we have

$$\begin{aligned}
\|y_n - y_m\|_{\mathcal{B}_2} &\leq \sum_{j=m+1}^{n} \|y_j - y_{j-1}\|_{\mathcal{B}_2} \\
&= \sum_{j=m+1}^{n} \|S^{j-1}y_1 - S^{j-1}y_0\|_{\mathcal{B}_2} \\
&\leq \sum_{j=m+1}^{n} \delta^{j-1}\|y_1 - y_0\|_{\mathcal{B}_2} \\
&\leq \frac{\delta^m}{1-\delta}\|y_1 - y_0\|_{\mathcal{B}_2}.
\end{aligned} \tag{15.6}$$

The right hand side limits to 0 as $m \to \infty$, so the sequence y_j is a Cauchy sequence in the Banach space $\mathcal{B}_2$, which therefore converges to a limit y_∞. Since S is continuous, we therefore have

$$Sy_\infty = S \lim_{j\to\infty} y_j = \lim_{j\to\infty} Sy_j = \lim_{j\to\infty} y_{j+1} = y_\infty. \tag{15.7}$$

Take $m = 1$ in (15.6) to get

$$\|y_n - y_1\|_{\mathcal{B}_2} \leq \frac{\delta}{1-\delta}\|y_1 - y_0\|_{\mathcal{B}_2} = \frac{\delta}{1-\delta}\|y_1\|_{\mathcal{B}_2} \tag{15.8}$$

Letting $n \to \infty$ yields

$$\|y_\infty\|_{\mathcal{B}_2} - \|y_1\|_{\mathcal{B}_2} \leq \|y_\infty - y_1\|_{\mathcal{B}_2} \leq \frac{\delta}{1-\delta}\|y_1\|_{\mathcal{B}_2}. \tag{15.9}$$

Then $x_\infty = Ty_\infty$ is a solution to $\mathscr{F}(x_\infty) = f$ and

$$\|x_\infty\|_{\mathcal{B}_1} \leq \|T\|\|y_\infty\|_{\mathcal{B}_2} \leq \frac{1}{1-\delta}\|T\| \cdot \|f\|_{\mathcal{B}_2}, \tag{15.10}$$

since $y_1 = S(y_0) = S(0) = f$, which implies (15.1). Finally, uniqueness then follows from (15.5). $\square$

15.2 Harmonic Coordinates

The remaining ingredient we need is the following.

Theorem 15.2 (Harmonic coordinates of Sabitov-Shefel (1976) DeTurck-Kazdan (1981))**.** *If (M^n, g) is any Riemannian manifold with g of class $C^{k,\alpha}$ for $k \geq 1$, and $p \in M$, then there exists a coordinate system $(x_1, \ldots, x_n)$ defined in some neighborhood of p such that $\Delta_g(x_j) = 0$ for $j = 1, \ldots, n$, with x_i of class $C^{k+1,\alpha}$. If g is C^∞ then so are x_i. If g is real analytic, then so are x_i.*

Proof. We will prove the 2-dimensional case. The higher-dimensional case is identical. From above, we have that in local coordinates $\{x, y\}$,

$$* = \sqrt{\det(g)} \begin{pmatrix} -g^{21} & -g^{22} \\ g^{11} & g^{12} \end{pmatrix}. \tag{15.11}$$

Let us write $dv = v_1 dx + v_2 dy$, then

$$*dv = (-g^{21}v_1 - g^{22}v_2)\sqrt{\det(g)}dx + (g^{11}v_1 + g^{12}v_2)\sqrt{\det(g)}dy. \tag{15.12}$$

Then we have

$$*d*dv = \frac{1}{\sqrt{\det(g)}}\Big(\partial_2\big((g^{21}v_1 + g^{22}v_2)\sqrt{\det(g)}\big) + \partial_1\big((g^{11}v_1 + g^{12}v_2)\sqrt{\det(g)}\big)\Big) \tag{15.13}$$

In local coordinates, the Laplacian therefore has the form

$$\Delta v = \frac{1}{\sqrt{\det(g)}}\partial_i\big(g^{ij}u_j\sqrt{\det(g)}\big). \tag{15.14}$$

(This formula holds in any dimension). Expanding this out yields

$$\Delta v = g^{ij}\partial_i\partial_j u + (\partial_i g^{ij})u_j + \partial_i(\log(\sqrt{\det(g)}))g^{ij}u_j. \tag{15.15}$$

Jacobi's formula for the determinant is

$$\frac{1}{2}g^{pq}\partial_i g_{pq} = \partial_i(\log(\sqrt{\det(g)})), \tag{15.16}$$

so we have

$$\Delta v = g^{ij}\partial_i\partial_j u + (\partial_i g^{ij})u_j + \frac{1}{2}g^{pq}\partial_i g_{pq}g^{ij}u_j. \tag{15.17}$$

So we can expand

$$\Delta v = \Delta_0 u + Q(u), \tag{15.18}$$

where

$$Q(u) = a^{ij}\partial_i\partial_j u + b^j u_j \tag{15.19}$$
$$a^{ij} = g^{ij} - \delta^{ij} \tag{15.20}$$
$$b^j = \partial_i g^{ij} + \frac{1}{2}g^{pq}\partial_i g_{pq}g^{ij}. \tag{15.21}$$

Let us assume that $g \in C^{1,\alpha}(B(0,1))$. Using normal coordinates (which are OK under this regularity assumption: the geodesic equation has C^α coefficients), we have that $g_{ij}(p) = \delta_{ij}$ and $\partial_k g_{ij}(p) = 0$. It follows that there exists a constant C so that

$$|g^{ij}(x) - \delta^{ij}| \leq C|x|^{1+\alpha} \tag{15.22}$$
$$|\partial_k g^{ij}(x)| \leq C|x|^\alpha. \tag{15.23}$$

Consider the mapping $\phi_\epsilon : B(0,1) \to B(0,\epsilon)$ defined by $\phi_\epsilon(x') = \epsilon x'$. Then

$$\phi_\epsilon^* g(x') = g_{ij}(\epsilon x')\epsilon^2 dx'_i \otimes dx'_j. \tag{15.24}$$

So the metric $g_\epsilon = \epsilon^{-2}\phi_\epsilon^* g$ has components $(g_\epsilon)_{ij} = g_{ij}(\epsilon x')$ in the x' coordinates. We then have

$$\begin{aligned} |g_\epsilon^{ij}(x') - \delta^{ij}| &\leq C\epsilon^{1+\alpha}|x'|^{1+\alpha} && (15.25)\\ |\partial_k g_\epsilon^{ij}(x')| &\leq C\epsilon^{1+\alpha}|x'|^{\alpha}. && (15.26)\end{aligned}$$

By assumption, there exists a constant C so that

$$|g_{ij}(x) - g_{ij}(y)| \leq C|x-y|^\alpha, \tag{15.27}$$

which implies that

$$|g'_{ij}(x') - g'_{ij}(y')| \leq C\epsilon^\alpha|x'-y'|^\alpha. \tag{15.28}$$

Also by assumption, there exists a constant C so that

$$|\partial_k g_{ij}(x) - \partial_k g_{ij}(y) \leq C|x-y|^\alpha, \tag{15.29}$$

which implies that

$$|\partial_k g'_{ij}(x') - \partial_k g'_{ij}(y')| \leq \epsilon^{1+\alpha}|x'-y'|^\alpha, \tag{15.30}$$

Consequently, we have that

$$\begin{aligned} \|a_\epsilon^{ij}\|_{C^{1,\alpha}(B(0,1))} &\leq C\epsilon^{1+\alpha} && (15.31)\\ \|b_\epsilon^{ij}\|_{C^{0,\alpha}(B(0,1))} &\leq C\epsilon^{\alpha}. && (15.32)\end{aligned}$$

We then have that there exists a constant C so that

$$\begin{aligned} \|Q(f)\|_{C^{0,\alpha}(B(0,1))} &= \|a^{ij}\partial_i\partial_j f + b^j f_j\|_{C^{0,\alpha}(B(0,1))}\\ &\leq \|a^{ij}\|_{C^{0,\alpha}(B(0,1))}\cdot\|\partial_i\partial_j f\|_{C^{0,\alpha}(B(0,1))} + \|b^j\|_{C^{0,\alpha}(B(0,1))}\cdot\|f_j\|_{C^{0,\alpha}(B(0,1))}\\ &\leq C\epsilon^\alpha\|f\|_{C^{2,\alpha}(B(0,1))}. \end{aligned} \tag{15.33}$$

By standard elliptic theory, we know that

$$\Delta_0 : C_0^{2,\alpha}(B_1(0)) \to C^{0,\alpha}(B_1(0)) \tag{15.34}$$

is an isomorphism with bounded inverse, that is, there exists a constant C so that if $\Delta_0 u = f$ and $u = 0$ on the boundary, then

$$\|u\|_{C^{2,\alpha}(B_1(0))} \leq C\|f\|_{C^{0,\alpha}(B_1(0))}. \tag{15.35}$$

So by Lemma 15.1 and (15.33), if ϵ is sufficiently small, then

$$\Delta_{g_\epsilon} : C_0^{2,\alpha}(B_1(0)) \to C^{0,\alpha}(B_1(0)) \tag{15.36}$$

is an isomorphism, and there exists a constant C so that if $\Delta_{g_\epsilon} u = f$ and $u = 0$ on the boundary, then

$$\|u\|_{C^{2,\alpha}(B_1(0))} \leq C\|f\|_{C^{0,\alpha}(B_1(0))}. \tag{15.37}$$

Therefore, there exists a solution of the equation

$$\Delta_{g_\epsilon}(h) = -\Delta_{g_\epsilon} x, \tag{15.38}$$

with $h = 0$ on the boundary of $B_1(0)$, where x is any coordinate function in the rescaled coordinates. From (15.37), we have

$$\|h\|_{C^{2,\alpha}(B_1(0))} \leq C\|\Delta_{g_\epsilon} x\|_{C^{0,\alpha}(B_1(0))} \leq \|b^1\|_{C^{0,\alpha}(B(0,1))} \leq C\epsilon^\alpha. \tag{15.39}$$

In particular,

$$|\partial_1 h_\epsilon(0)| \leq \|h_\epsilon\|_{C^1(B(0,1))} \leq \|h_\epsilon\|_{C^{2,\alpha}(B(0,1))} \leq C\epsilon^\alpha. \tag{15.40}$$

So if ϵ is sufficiently small, $\partial_1(x + h_\epsilon)(0) = 1 + \partial_1 h_\epsilon(0) \neq 0$, and we are done.

For higher regularity, we argue as follows. If $g \in C^{k,\alpha}$ then in particular $g \in C^{1,\alpha}$. By the above, we can find $C^{2,\alpha}$ harmonic coordinates $\{x_1, x_2\}$. We then write

$$0 = \Delta x_k = g^{ij}\partial_i\partial_j x_k + b^j. \tag{15.41}$$

That is

$$g^{ij}\partial_i\partial_j x_k = -b^j \in C^{k-1,\alpha}. \tag{15.42}$$

The left hand side is an elliptic operator with $C^{k,\alpha}$ coefficients, so by elliptic regularity arguments, $x_k \in C^{k+2,\alpha}$. If $g \in C^\infty$, the right hand side is also in C^∞, so again by elliptic regularity we see that $x_k \in C^{k,\alpha}$ for any $k \geq 0$, so $x_k \in C^\infty$. For the real analytic case, there is a general result that solutions of elliptic equations with real analytic coefficients are real analytic. $\square$

Unfortunately, the trick in this subsection does not help us to solve the Newlander-Nirenberg problem in higher dimensions. However, the method in the previous section *can* be extended to the higher dimensional case, which we will discuss next.

Remark 15.3. The above methods require that $\mu \in C^{1,\alpha}$. The Beltrami equation can be solved locally for $\mu \in C^{0,\alpha}$ by inverting the $\partial_{\bar{z}}$ operator using the Cauchy-Pompeiu formula. However, this is a bit technical so we will omit. There are many great references for this method, see for example [Spi79], [Ber58], [Ahl66].

The only "hard" analysis we used in the above proof is the following.

Theorem 15.4. *The mapping $\Delta_0 : C_0^{2,\alpha}(B_1(0)) \to C^{0,\alpha}(B_1(0))$ is an isomorphism with bounded inverse, that is, there exists a constant C so that if $\Delta_0 u = f$ and $u = 0$ on the boundary, then*

$$\|u\|_{C^{2,\alpha}(B_1(0))} \leq C\|f\|_{C^{0,\alpha}(B_1(0))}. \tag{15.43}$$

We just indicate how this is proved. Any solution is unique by the maximum principle. There is actually an explicit integral formula for the solution

$$u = \int_{B_1(0)} G(x,y)f(y)dxdy, \tag{15.44}$$

where $G(x,y)$ is the Green's function defined by

$$G(x,y) = \begin{cases} \frac{1}{2\pi}\Big(\log|x-y| - \log\Big(|y||x - y/|y|^2|\Big) & y \neq 0 \\ \frac{1}{2\pi}\log|x| & y = 0 \end{cases}. \tag{15.45}$$

Then one can just directly verify this is a solution if $f \in C^{2,\alpha}(B_1(0))$, and directly verify the estimate (15.43); see [GT01]. Our method above using the inverse function theorem only needed to invert Δ_0; we did not need to use any Schauder Theory for operators with variable coefficients.

16 Lecture 16

16.1 Endomorphisms

Let $End_{\mathbb{R}}(TM)$ denotes the real endomorphisms of the tangent bundle.

Proposition 16.1. *On an almost complex manifold (M,J), the bundle $End_{\mathbb{R}}(TM)$ admit the decomposition*

$$End_{\mathbb{R}}(TM) = End_{+}(TM) \oplus End_{-}(TM) \tag{16.1}$$

where the first factor on the left consists of endomorphisms I commuting with J,

$$IJ = JI \tag{16.2}$$

and the second factor consists of endomorphisms I anti-commuting with J,

$$IJ = -JI \tag{16.3}$$

Proof. Given J, we define

$$I_{+} = \frac{1}{2}(I - JIJ) \tag{16.4}$$

$$I_{-} = \frac{1}{2}(I + JIJ). \tag{16.5}$$

Then

$$I_{+}J = \frac{1}{2}(IJ - JIJ^2) = \frac{1}{2}(IJ + JI),$$

and

$$JI_{+} = \frac{1}{2}(JI - J^2IJ) = \frac{1}{2}(JI + IJ).$$

Next,

$$I_-J = \frac{1}{2}(IJ + JIJ^2) = \frac{1}{2}(IJ - JI),$$

and

$$JI_- = \frac{1}{2}(JI + J^2IJ) = \frac{1}{2}(JI - IJ).$$

Clearly, $I = I_+ + I_-$. To prove it is a direct sum, if $IJ = JI$ and $IJ = -JI$, then $IJ = 0$ which implies that $I = 0$ since J is invertible.

□

We write down the above in a basis. Choose a real basis $\{e_1, \ldots e_{2n}\}$ such that the complex structure J_0 is given by

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{16.6}$$

Then in matrix terms, the proposition is equivalent to the following decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A+D & B-C \\ C-B & A+D \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A-D & B+C \\ B+C & D-A \end{pmatrix}. \tag{16.7}$$

So we have that $End_+(TM) \cong GL(n, \mathbb{C}) \subset GL(2n, \mathbb{C})$ with

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \mapsto \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}, \tag{16.8}$$

and $End_-(TM) \cong \overline{GL}(n, \mathbb{C}) \subset GL(2n, \mathbb{C})$ with

$$\begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mapsto \begin{pmatrix} 0 & A+iB \\ A-iB & 0 \end{pmatrix}. \tag{16.9}$$

16.2 The space of almost complex structures

We define

$$\mathcal{J}(\mathbb{R}^{2n}) \equiv \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n},\ J \in GL(2n, \mathbb{R}), J^2 = -I_{2n}\}. \tag{16.10}$$

We next give some alternative descriptions of this space.

Proposition 16.2. *The space $\mathcal{J}(\mathbb{R}^{2n})$ is the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, and thus*

$$\dim(\mathcal{J}(\mathbb{R}^{2n})) = 2n^2. \tag{16.11}$$

Proof. We note that $GL(2n,\mathbb{R})$ acts on $\mathcal{J}(\mathbb{R}^{2n})$, by the following. If $A \in GL(2n,\mathbb{R})$ and $J \in \mathcal{J}(\mathbb{R}^{2n})$,

$$\Phi_A : J \mapsto AJA^{-1}. \tag{16.12}$$

Obviously,

$$(AJA^{-1})^2 = AJA^{-1}AJA^{-1} = AJ^2A^{-1} = -I, \tag{16.13}$$

and

$$\Phi_{AB}(J) = (AB)J(AB)^{-1} = ABJB^{-1}A^{-1} = \Phi_A\Phi_B(J), \tag{16.14}$$

so is indeed a group action (on the left). Given J and J', there exists bases

$$\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \text{ and } \{e'_1, \ldots, e'_n, J'e'_1, \ldots, J'e'_n\}. \tag{16.15}$$

Define $S \in GL(2n,\mathbb{R})$ by $Se_k = e'_k$ and $S(Je_k) = J'e'_k$. Then $J' = SJS^{-1}$, and the action is therefore transitive. The stabilizer subgroup of J_0 is

$$Stab(J_0) = \{A \in GL(2n,\mathbb{R}) : AJ_0A^{-1} = J_0\}, \tag{16.16}$$

that is, A commutes with J_0. From (16.8) above, this is identified with $GL(n,\mathbb{C})$. $\square$

Given $J \in \mathcal{J}_n$, let $J(t) : (-\epsilon, \epsilon) \to \mathcal{J}_{2n}$ be a smooth path with $J(0) = J$, then differentiation yields

$$-(I_{2n})' = (J \circ J)' = J' \circ J + J \circ J'. \tag{16.17}$$

So letting $J'(0) = I$, we have that

$$IJ + JI = 0. \tag{16.18}$$

Thus we can identify the tangent space at any J as

$$T_J\mathcal{J}_{2n} = \{I \in End(\mathbb{R}^n) \mid IJ + JI = 0\}, \tag{16.19}$$

the space of endomorphisms which anti-commute with J.

16.3 Graph over the reals

Next, we will give another description of $\mathcal{J}(\mathbb{R}^{2n})$. Define

$$\begin{aligned}\mathcal{P}(\mathbb{R}^{2n}) = \{P \subset \mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{C}^{2n} \mid dim_{\mathbb{C}}(P) = n,\\ P \text{ is a complex subspace satisfying } P \cap \overline{P} = \{0\}\}.\end{aligned}$$

If we consider $\mathbb{R}^{2n} \otimes \mathbb{C}$, we note that complex conjugation is a well defined complex anti-linear map $\mathbb{R}^{2n} \otimes \mathbb{C} \to \mathbb{R}^{2n} \otimes \mathbb{C}$.

Proposition 16.3. *The space $\mathcal{P}(\mathbb{R}^{2n})$ can be explicitly identified with $\mathcal{J}(\mathbb{R}^{2n})$ by the following. If $J \in \mathcal{J}(\mathbb{R}^{2n})$ then let*

$$\mathbb{R}^{2n} \otimes \mathbb{C} = T^{1,0}(J) \oplus T^{0,1}(J), \tag{16.20}$$

where

$$T^{0,1}(J) = \{X + iJX, X \in \mathbb{R}^{2n}\} = \{-i\}\text{-eigenspace of } J. \tag{16.21}$$

This an n-dimensional complex subspace of $\mathbb{C}^{2n}$, and letting $T^{1,0}(J) = \overline{T^{0,1}(J)}$, we have $T^{1,0} \cap T^{0,1} = \{0\}$.

For the converse, given $P \in \mathcal{P}(\mathbb{R}^{2n})$, then P may be written as a graph over $\mathbb{R}^{2n} \otimes 1$, that is

$$P = \{X' + iJX' \mid X' \in \mathbb{R}^{2n} \subset \mathbb{C}^{2n}\}, \tag{16.22}$$

with $J \in \mathcal{J}(\mathbb{R}^{2n})$, and

$$\mathbb{R}^{2n} \otimes \mathbb{C} = \overline{P} \oplus P = T^{1,0}(J) \oplus T^{0,1}(J). \tag{16.23}$$

Proof. For the forward direction, we already know this. To see the other direction, consider the projection map Re restricted to P

$$\pi = Re : P \to \mathbb{R}^{2n}. \tag{16.24}$$

We claim this is a real linear isomorphism. Obviously, it is linear over the reals. Let $X \in P$ satisfy $\pi(X) = 0$. Then $Re(X) = 0$, so $X = iX'$ for some real $X' \in \mathbb{R}^{2n}$. But $\overline{X} = -iX' \in P \cap \overline{P}$, so by assumption $X = 0$. Since these spaces are of the same real dimension, π has an inverse, which we denote by J. Clearly then, (16.22) is satisfied. Since P is a complex subspace, given any $X = X' + iJX' \in P$, the vector $iX' = (-JX') + iX'$ must also lie in P, so

$$(-JX') + iX' = X'' + iJX'', \tag{16.25}$$

for some real X'', which yields the two equations

$$JX' = -X'' \tag{16.26}$$

$$X' = JX''. \tag{16.27}$$

applying J to the first equation yields

$$J^2X' = -JX'' = -X'. \tag{16.28}$$

Since this is true for any X', we have $J^2 = -I_{2n}$. □

Remark 16.4. We note that $J \mapsto -J$ corresponds to interchanging $T^{0,1}$ and $T^{1,0}$.

Remark 16.5. If we choose $P = \mathrm{span}_{\mathbb{C}}\{\partial/\partial x^j, j = 1 \dots n\}$. Then P is an n-dimensional complex subspace of $\mathbb{C}^{2n}$, and Re restricted to P is not an isomorphism, for example.

Remark 16.6. The above proposition embeds $\mathcal{J}(\mathbb{R}^{2n})$ as a subset of the complex Grassmannian $G(n, 2n, \mathbb{C})$. These spaces have the same dimension, so it is an *open* subset. Furthermore, the condition that the projection to the real part is an isomorphism is generic, so it is also dense.

17 Lecture 17

17.1 Graphs over $T^{0,1}(J_0)$

Above we viewed $T^{0,1}(J)$ as a graph corresponding to the decomposition $\mathbb{C}^{2n} = \mathbb{R}^{2n} \oplus i\mathbb{R}^{2n}$. In the section we will instead view $T^{0,1}(J)$ as a graph corresponding to the decomposition $\mathbb{C}^{2n} = T^{0,1}(J_0) \oplus T^{1,0}(J_0)$. This corresponds to a mapping

$$\phi : T^{0,1}(J_0) \to T^{1,0}(J_0), \tag{17.1}$$

by writing

$$T^{0,1}(J) = \{v + \phi v \mid v \in T^{0,1}(J_0)\}. \tag{17.2}$$

Note we can view ϕ as an element of

$$Hom(T^{0,1}(J_0), T^{1,0}(J_0)) \cong \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0), \tag{17.3}$$

so we will view ϕ as an element of the latter space. In "coordinates", we can write

$$\phi = \phi_{\bar{j}}^{k} d\bar{z}^j \otimes \frac{\partial}{\partial z^k}, \tag{17.4}$$

and we will view $\phi_{\bar{j}}^{k}$ as an n by n complex matrix. We define $\overline{\phi}$ as a $\mathbb{C}$-linear mapping

$$\overline{\phi} : T^{1,0}(J_0) \to T^{0,1}(J_0), \tag{17.5}$$

by

$$\overline{\phi}(v) = \overline{\phi(\bar{v})}. \tag{17.6}$$

Consider the mapping

$$\phi + \overline{\phi} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}, \tag{17.7}$$

which in matrix form is

$$\phi + \overline{\phi} = \begin{pmatrix} 0 & \phi \\ \overline{\phi} & 0 \end{pmatrix}. \tag{17.8}$$

Recall from (16.9) that this is the complexification of an $\mathbb{R}$-linear mapping

$$I_\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \tag{17.9}$$

satisfying $I_\phi J_0 + J_0 I_\phi = 0$, which is given by

$$I_\phi = \begin{pmatrix} Re(\phi) & Im(\phi) \\ Im(\phi) & -Re(\phi) \end{pmatrix}. \tag{17.10}$$

Proposition 17.1. *If $\phi \in \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0)$, then ϕ determines an almost complex structure if and only if I_ϕ does not have -1 as an eigenvalue. The corresponding almost complex structure is*

$$J_\phi = (Id + I_\phi) J_0 (Id + I_\phi)^{-1}. \tag{17.11}$$

Conversely, given J such that $J_0 + J$ is invertible, then J corresponds to a unique ϕ with $Id + I_\phi$ invertible, which is given by

$$I_\phi = (J_0 + J)^{-1}(J_0 - J). \tag{17.12}$$

Proof. Given

$$\phi \in \Lambda^{0,1}(J_0) \otimes T^{1,0}(J_0) = Hom_{\mathbb{C}}(T^{0,1}(J_0), T^{1,0}(J_0)), \tag{17.13}$$

then

$$T^{0,1}(J_\phi) = \{v + \phi v, v \in T^{0,1}(J_0)\} \tag{17.14}$$

is an n-dimensional complex subspace of $\mathbb{R}^{2n} \otimes \mathbb{C}$. If $X \in T^{0,1}_\phi \cap \overline{T^{0,1}_\phi}$ for a non-zero vector X, then

$$X = v + \phi v = w + \overline{\phi} w, \tag{17.15}$$

where $v \in T^{0,1}(J_0)$ and $w \in T^{1,0}(J_0)$. This yields the equations

$$\overline{\phi} w = v \tag{17.16}$$

$$\phi v = w. \tag{17.17}$$

This is equivalent the matrix $\phi + \overline{\phi}$ having 1 as an eigenvalue with eigenvector (w, v). Since $\phi + \overline{\phi}$ is matrix equivalent to I_ϕ, this is equivalent to I_ϕ having 1 as an eigenvalue. But if $I_\phi V = V$, then

$$I_\phi J_0 V = -J_0 I_\phi V = -J_0 V, \tag{17.18}$$

that is $J_0 V$ is an eigenvalue of I_ϕ with eigenvalue -1. Next, any $\tilde{v} \in T^{0,1}(J_\phi)$ is written as

$$\begin{aligned} \tilde{v} &= v + \phi(v) \\ &= Re(v) + Re(\phi(v)) + i(Im(v) + Im(\phi(v)), \end{aligned} \tag{17.19}$$

for $v \in T^{0,1}(J_0)$. We compute

$$\begin{aligned} Re(\phi(v)) &= \frac{1}{2}\Big(\phi(v) + \overline{\phi(v)}\Big) \\ &= \frac{1}{2}\Big(\phi(v) + \overline{\phi}(\overline{v})\Big) \\ &= (\phi + \overline{\phi})\Big(\frac{v + \overline{v}}{2}\Big) \\ &= I_\phi(Re(v)). \end{aligned} \tag{17.20}$$

Next,

$$\begin{aligned} Im(\phi(v)) &= \frac{1}{2i}\Big(\phi(v) - \overline{\phi(v)}\Big) \\ &= \frac{1}{2i}\Big(\phi(v) - \overline{\phi}(\overline{v})\Big) \\ &= (\phi + \overline{\phi})\Big(\frac{v - \overline{v}}{2i}\Big) \\ &= I_\phi(Im(v)). \end{aligned} \tag{17.21}$$

Next, any element $v \in T^{0,1}(J_0)$ can be written as

$$v = X' + iJ_0X', \tag{17.22}$$

for $X' \in \mathbb{R}^{2n}$, so we have

$$\tilde{v} = (Id + I_\phi)X' + i(Id + I_\phi)(J_0X'). \tag{17.23}$$

But if $\tilde{v} \in T^{0,1}(J_\phi)$, we must have

$$Im(\tilde{v}) = J_\phi Re(\tilde{v}), \tag{17.24}$$

which yields

$$(Id + I_\phi)(J_0X') = J_\phi(Id + I_\phi)X'. \tag{17.25}$$

This implies that

$$J_\phi = (Id + I_\phi)J_0(Id + I_\phi)^{-1}. \tag{17.26}$$

The remainder of the proposition follows by solving this equation for I_ϕ. $\square$

17.2 Complex form of the equations

We next discuss the following characterization of pseduo-holomorphic functions on an almost complex manifold (which holds in any dimension).

Proposition 17.2. *Let (M, J) be almost complex. Then the following are equivalent.*

(i) $f : (M, J) \to (\mathbb{C}, J_0)$ *is pseudo-holomorphic.*

(ii) $\overline{\partial}_J f = 0$.

(iii) $Xf = 0$ *for all vector fields* $X \in \Gamma(T_J^{0,1})$.

Proof. Note that if we take any $X \in \Gamma(TM \otimes \mathbb{C})$

$$(df + iJdf)(X) = Xf + idf(JX) = Xf + iJXf = (X + iJX)f \tag{17.27}$$

so we always have

$$\overline{\partial}_J f(X) = (\Pi_{\Lambda^{0,1}_J} df)(X) = (\Pi_{T^{0,1}_J} X) f. \tag{17.28}$$

If condition (ii) is satisfied, then (iii) follows immediately from (17.28) Conversely, if condition (iii) is satisfied then taking $X \in \Gamma(T^{0,1}_J)$ and using (17.28), then

$$(\Pi_{\Lambda^{0,1}_J} df)(X) = 0 \tag{17.29}$$

for all such X, which implies that condition (ii) is satisfied.

We next show that (ii) is equivalent to (i). If (ii) is satisfied, then

$$J df = i df. \tag{17.30}$$

Recall that if $u : M \to \mathbb{R}$ is a *real-valued* function, and $X \in TM$ is a real tangent vector, then there is a canonical identification

$$du(X) = u_*(X), \tag{17.31}$$

where the right hand side is interpreted as a real number. So then if $f = u + iv$ is a *complex-valued* function, then we have for $X \in TM$,

$$df(X) = u_*(X) + iv_*(X), \tag{17.32}$$

and then we extend this to complex vectors by complex linearity. We plug in a complex tangent vector to (17.30) to get

$$J df(X) = i df(X), \tag{17.33}$$

which is (using the definition of J on 1-forms as the transpose)

$$df(JX) = i df(X). \tag{17.34}$$

Using the above, we then write this as

$$(du + idv)(JX) = i(du + idv)(X) \tag{17.35}$$

which is

$$(u_* + iv_*)(JX) = i(u_* + iv_*)(X). \tag{17.36}$$

This yields the equations

$$u_*(JX) = -v_*(X), \quad v_*(JX) = u_*(X). \tag{17.37}$$

But as a real-valued function, we have

$$f = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{17.38}$$

so we can write f_* in the form

$$f_* = \begin{pmatrix} u_* \\ v_* \end{pmatrix}. \tag{17.39}$$

The equation $f_* J = J_0 f_*$ is then

$$\begin{pmatrix} u_* \\ v_* \end{pmatrix} J = \begin{pmatrix} u_* J \\ v_* J \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_* \\ v_* \end{pmatrix} = \begin{pmatrix} -v_* \\ u_* \end{pmatrix}. \tag{17.40}$$

Therefore (ii) implies (i). Reversing the above argument, we see that (i) implies (ii), and we are done.

□

Next, we let J be a continuous almost complex structure on a open set $U \subset \mathbb{R}^{2n}$ containing the origin. Then $J : U \to \mathcal{J}_n$ is a continuous function. Without loss of generality, we may assume that $J(0) = J_0$. Then $(J_0 + J)(0) = 2J_0$ is invertible, so $J_0 + J$ will also be invertible in some possibly smaller neighborhood $V \subset U$. Then by Proposition 17.1, we obtain a unique

$$\phi^j_{\bar{k}} : V \to Hom(T^{0,1}(J_0), T^{1,0}(J_0)) \cong Mat(n \times n, \mathbb{C}), \tag{17.41}$$

where V is an open subset in $\mathbb{R}^{2n}$.

Proposition 17.3. *If $\phi^j_{\bar{k}}$ defines an almost complex structure on V, then a function $f : V \to \mathbb{C}$ is pseudo-holomorphic if and only if*

$$\frac{\partial}{\partial \overline{z}^j} f + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k} f = 0 \tag{17.42}$$

Proof. By Proposition 17.2, a function f is holomorphic if and only if $Zf = 0$ for all vector fields $Z \in \Gamma(T^{0,1}_\phi)$. A local basis for $T^{0,1}_\phi$ is given by

$$Z_{\bar{j}} = \frac{\partial}{\partial \overline{z}^j} + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k}, \tag{17.43}$$

so we are done. □

Remark 17.4. For $n = 1$, there is only 1 component $\mu = \phi^1_{\bar{1}}$, and the pseduo-holomorphic condition is

$$\frac{\partial}{\partial \overline{z}} f + \mu \frac{\partial}{\partial z} f = 0, \tag{17.44}$$

which is of course the Beltrami equation.

18 Lecture 18

18.1 Integrability

We next interpret the vanishing of the Nijenhuis tensor as an equation on ϕ.

Proposition 18.1. *The almost complex structure J_ϕ is integrable, that is $N(J_\phi) = 0$, if and only if*

$$\frac{\partial}{\partial \bar{z}^l}\phi^j_{\bar{k}} - \frac{\partial}{\partial \bar{z}^k}\phi^j_{\bar{l}} + \phi^m_{\bar{k}}\frac{\partial}{\partial z^m}\phi^j_{\bar{l}} - \phi^m_{\bar{l}}\frac{\partial}{\partial z^m}\phi^j_{\bar{k}} = 0. \tag{18.1}$$

Proof. By Proposition 9.6, the integrability equation is equivalent to $[T^{0,1}_\phi, T^{0,1}_\phi] \subset T^{0,1}_\phi$. Writing

$$\phi = \sum \phi^j_{\bar{k}} d\bar{z}^k \otimes \frac{\partial}{\partial z^j}, \tag{18.2}$$

if J_ϕ is integrable, then we must have

$$\left[\frac{\partial}{\partial \bar{z}^i} + \phi\Big(\frac{\partial}{\partial \bar{z}^i}\Big), \frac{\partial}{\partial \bar{z}^k} + \phi\Big(\frac{\partial}{\partial \bar{z}^k}\Big)\right] \in T^{0,1}_\phi. \tag{18.3}$$

This yields

$$\left[\frac{\partial}{\partial \bar{z}^i}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\right] + \left[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}\right] + \left[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\right] \in T^{0,1}_\phi \tag{18.4}$$

The first two terms are

$$\left[\frac{\partial}{\partial \bar{z}^i}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\right] + \left[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}\right] = \sum_j \left(\frac{\partial \phi^j_{\bar{k}}}{\partial \bar{z}^i} - \frac{\partial \phi^j_{\bar{i}}}{\partial \bar{z}^k}\right)\frac{\partial}{\partial z^j}.$$

The third term is

$$\left[\phi^j_{\bar{i}}\frac{\partial}{\partial z^j}, \phi^l_{\bar{k}}\frac{\partial}{\partial z^l}\right] = \phi^j_{\bar{i}}\Big(\frac{\partial}{\partial z^j}\phi^l_{\bar{k}}\Big)\frac{\partial}{\partial z^l} - \phi^l_{\bar{k}}\Big(\frac{\partial}{\partial z^l}\phi^j_{\bar{i}}\Big)\frac{\partial}{\partial z^j}.$$

Both terms are in $T^{1,0}(J_0)$. For sufficiently small ϕ however, $T^{0,1}_\phi \cap T^{1,0}(J_0) = \{0\}$, and therefore (18.1) holds. The converse holds by reversing this argument. □

We can also see directly that this is related to Proposition17.2 as follows. If there exists a locally defined holomorphic function f, then taking the $\overline{\partial}$-partial of (17.42) yields

$$\frac{\partial^2}{\partial \bar{z}^l \partial \bar{z}^j} f + \frac{\partial}{\partial \bar{z}^l}\Big(\phi^k_{\bar{j}}\frac{\partial}{\partial z^k} f\Big) = 0. \tag{18.5}$$

Intechanging j and l yields

$$\frac{\partial^2}{\partial \bar{z}^j \partial \bar{z}^l} f + \frac{\partial}{\partial \bar{z}^j}\Big(\phi^k_{\bar{l}}\frac{\partial}{\partial z^k} f\Big) \tag{18.6}$$

If f is C^2, then the mixed partials are equal, so subtracting these equations gives

$$\frac{\partial}{\partial \overline{z}^l}\Big(\phi_{\overline{j}}^k \frac{\partial}{\partial z^k} f\Big) - \frac{\partial}{\partial \overline{z}^j}\Big(\phi_{\overline{l}}^k \frac{\partial}{\partial z^k} f\Big) = 0. \tag{18.7}$$

Expanding this out

$$\Big(\frac{\partial}{\partial \overline{z}^l}\phi_{\overline{j}}^k - \frac{\partial}{\partial \overline{z}^j}\phi_{\overline{l}}^k\Big)\frac{\partial}{\partial z^k} f + \phi_{\overline{j}}^k \frac{\partial^2}{\partial \overline{z}^l \partial z^k} f - \phi_{\overline{l}}^k \frac{\partial^2}{\partial \overline{z}^j \partial z^k} = 0 \tag{18.8}$$

The first 2 terms are good. Using (17.42), the last 2 terms are

$$\begin{aligned}
\phi_{\overline{j}}^k \frac{\partial^2}{\partial z^k \partial \overline{z}^l} f - \phi_{\overline{l}}^k \frac{\partial^2}{\partial z^k \partial \overline{z}^j} &= -\phi_{\overline{j}}^k \frac{\partial}{\partial z^k}\Big(\phi_{\overline{l}}^p \frac{\partial}{\partial z^p} f\Big) + \phi_{\overline{l}}^k \frac{\partial}{\partial z^k}\Big(\phi_{\overline{j}}^p \frac{\partial}{\partial z^p} f\Big) && (18.9)\\
&= -\phi_{\overline{j}}^k \Big(\frac{\partial}{\partial z^k}\phi_{\overline{l}}^p\Big)\frac{\partial}{\partial z^p} f + \phi_{\overline{l}}^k \Big(\frac{\partial}{\partial z^k}\phi_{\overline{j}}^p\Big)\frac{\partial}{\partial z^p} f && (18.10)\\
&\quad - \phi_{\overline{j}}^k \phi_{\overline{l}}^p \frac{\partial^2 f}{\partial z^k \partial z^p} + \phi_{\overline{l}}^k \phi_{\overline{j}}^p \frac{\partial^2 f}{\partial z^k \partial z^p}. && (18.11)
\end{aligned}$$

The last 2 terms vanish from symmetry. So we have derived

$$0 = \Big(\frac{\partial}{\partial \overline{z}^l}\phi_{\overline{j}}^k - \frac{\partial}{\partial \overline{z}^j}\phi_{\overline{l}}^k - \phi_{\overline{j}}^p \frac{\partial}{\partial z^p}\phi_{\overline{l}}^k - \phi_{\overline{l}}^p \frac{\partial}{\partial z^p}\phi_{\overline{j}}^p\Big)\frac{\partial}{\partial z^k} f. \tag{18.12}$$

If there exists n holomorphic functions with linearly independent differentials at the origin, then this implies implies the integrability condition (18.1). This latter argument assumes that there exists holomorphic coordinates, but nevertheless still gives the correct formula for the Nijenhuis tensor.

18.2 The operator d^c

For an almost complex structure J with $N_J = 0$, we know that

$$d = \partial + \overline{\partial}, \tag{18.13}$$

and

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0. \tag{18.14}$$

We can write these complex operators in the form

$$\overline{\partial} = \frac{1}{2}(d - id^c), \ \partial = \frac{1}{2}(d + id^c). \tag{18.15}$$

for a *real* operator $d^c : \Omega^p \to \Omega^{p+1}$ given by

$$d^c = i(\overline{\partial} - \partial), \tag{18.16}$$

which satisfies

$$d^2 = 0, \ dd^c + d^c d = 0, \ (d^c)^2 = 0. \tag{18.17}$$

We next have an alternative formula for d^c. Recall that $J : TM \to TM$ induces a dual mapping $J : T^*M \to T^*M$, and we extended to $J : \Lambda^r_{\mathbb{C}} \to \Lambda^r_{\mathbb{C}}$ by

$$J\alpha^{p,q} = i^{p-q}\alpha^{p,q}, \tag{18.18}$$

for a α a form of type (p, q). Notice that if $\alpha^r \in \Lambda^r_{\mathbb{C}}$, then

$$J^2\alpha^r = w \cdot \alpha^r, \text{ where } w \cdot \alpha^r = (-1)^r\alpha^r, \tag{18.19}$$

since

$$J^2\alpha^{p,q} = i^{2(p-q)}\alpha^{p,q} = (-1)^{p-q}\alpha^{p,q} = (1)^{p-q+2q}\alpha^{p,q} = (-1)^{p+q}\alpha^{p,q}. \tag{18.20}$$

Proposition 18.2. *For $\alpha \in \Lambda^r$, we have*

$$d^c\alpha = (-1)^{r+1}JdJ\alpha. \tag{18.21}$$

We also have

$$dd^c = 2i\partial\overline{\partial} = (-1)^{r+1}dJdJ\alpha. \tag{18.22}$$

Proof. For $\alpha \in \Lambda^{p,q}$, $p + q = r$, we compute

$$\begin{aligned} JdJ\alpha = i^{p-q}Jd\alpha &= i^{p-q}J(\partial\alpha + \overline{\partial}\alpha) && (18.23)\\ &= i^{p-q}(i^{p+1-q}\partial\alpha + i^{p-q-1}\overline{\partial}\alpha) && (18.24)\\ &= i^{2(p-q)+1}\partial\alpha + i^{2(p-q)-1}\overline{\partial}\alpha && (18.25)\\ &= (-1)^{p+q}(i\partial\alpha - i\overline{\partial}\alpha) = (-1)^{r+1}d^c\alpha. && (18.26) \end{aligned}$$

For (18.22), using (18.14) we have

$$dd^c = (\partial + \overline{\partial})i(\overline{\partial} - \partial) = i(\partial\overline{\partial} + \overline{\partial}^2 - \partial^2 - \overline{\partial}\partial) = 2i\partial\overline{\partial}. \tag{18.27}$$

□

18.3 The analytic case

We assume that $\phi^k_{\bar{j}}$ is analytic. So there exists a power series expansion

$$\phi^k_{\bar{j}} = \sum_{I,J}(\phi^k_{\bar{j}})_{IJ}z^I\bar{z}^J. \tag{18.28}$$

Let group these terms together by homogeneity and write

$$\phi^k_{\bar{j}} = \sum_{m=0}^{\infty}(\phi^k_{\bar{j}})_m \tag{18.29}$$

where

$$(\phi^k_{\bar{j}})_m = \sum_{|I|+|J|=m} (\phi^k_{\bar{j}})_{IJ} z^I \bar{z}^J. \tag{18.30}$$

We may assume that $(\phi^k_{\bar{j}})_0 = 0$. We want to solve the equation

$$\frac{\partial}{\partial \bar{z}^j} f + \phi^k_{\bar{j}} \frac{\partial}{\partial z^k} f = 0. \tag{18.31}$$

Let's do the same for f, we write a formal power series

$$f = \sum_{I,J} f_{IJ} z^I \bar{z}^J, \tag{18.32}$$

and group these terms together by homogeneity and write

$$f = \sum_{m=0}^{\infty} f_m \tag{18.33}$$

where

$$f_m = \sum_{|I|+|J|=m} f_{IJ} z^I \bar{z}^J. \tag{18.34}$$

By subtracting a constant, we can also assume that $f_0 = 0$. Expanding (18.31), we have

$$\overline{\partial}_0(f_1 + f_2 + \cdots) + (\phi_1 + \phi_2 + \cdots)(\partial_0 f_1 + \partial_0 f_2 + \cdots) = 0. \tag{18.35}$$

Grouping terms by homogeneity, we have

$$\begin{aligned}
\overline{\partial}_0 f_1 &= 0 \\
\overline{\partial}_0 f_2 &= -\phi_1 \partial_0 f_1 \\
\overline{\partial}_0 f_3 &= -\phi_1 \partial_0 f_2 - \phi_2 \partial_0 f_1 \\
&\vdots
\end{aligned}$$

and we see that the general term is given by

$$\frac{\partial f_m}{\partial \bar{z}^j} = - \sum_{k+l=m, k\geq 1, l\geq 1} (\phi^p_{\bar{j}})_k \frac{\partial f_l}{\partial z^p} \tag{18.36}$$

Proposition 18.3. *If f_j solves the above system for $j = 1, \ldots, q$, then the expression*

$$H_{q+1} = - \sum_{k+l=q+1, k\geq 1, l\geq 1} (\phi^p_{\bar{j}})_k \frac{\partial f_l}{\partial z^p} d\bar{z}^j \tag{18.37}$$

is a form of type $(0,1)$ with respect to J_0, and satisfies $\overline{\partial}_0 H_{q+1} = 0$.

Proof. We prove this by induction. For $q = 1$, we have $\overline{\partial}_0 f_1 = 0$, so $f_1 = c_j z^j$ is a linear holomorphic function. Then

$$H_2 = -(\phi^p_{\bar{j}})_1 c_p d\overline{z}^j. \tag{18.38}$$

For reasons of homogeneity, the integrability equation (18.1) tells us that

$$\frac{\partial}{\partial \overline{z}^l}(\phi^p_{\bar{j}})_1 - \frac{\partial}{\partial \overline{z}^j}(\phi^p_{\bar{l}})_1 = 0, \tag{18.39}$$

so H_2 clearly satisfies $\overline{\partial} H_2 = 0$. So assume the system is satisfied for $j = 1 \dots q$. Then the function $f = f_1 + \cdots + f_q$ satisfies

$$\overline{\partial}_J f = H_{q+1} + O(|z|^{q+1}) \tag{18.40}$$

For the next step, we use the above fact that the integrability of J implies that the operator $\overline{\partial}_J : \Lambda^{0,1}(J) \to \Lambda^{0,2}(J)$ defined by $\overline{\partial}_J \alpha = \Pi_{\Lambda^{0,2}(J)} d\alpha$ satisfies

$$\overline{\partial}_J \overline{\partial}_J f = 0, \tag{18.41}$$

for any function f. This yields

$$0 = \overline{\partial}_J(H_{q+1} + O(|z|^{q+1}) = \overline{\partial}_J H_{q+1} + O(|z|^q). \tag{18.42}$$

Note that for $\alpha \in \Lambda^{0,1}(J)$, $J\alpha = -i\alpha$, so from Proposition 18.2, we have that

$$\overline{\partial}_J \alpha = \frac{1}{2}(d - id^c)\alpha = \frac{1}{2}(d\alpha - iJdJ\alpha) = \frac{1}{2}(d\alpha - Jd\alpha). \tag{18.43}$$

Expanding this, we obtain

$$\overline{\partial}_J \alpha = \frac{1}{2}(d\alpha - (J - J_0 + J_0)d\alpha) = \frac{1}{2}(d\alpha - J_0 d\alpha) - \frac{1}{2}(J - J_0)d\alpha. \tag{18.44}$$

From Proposition 17.1 above, the correspondence between ϕ and J is analytic, and $\phi = O(|z|)$ implies that $J - J_0 = O(|z|)$ as $z \to 0$. Now we plug in $\alpha = \overline{\partial}_J f$, and by assumption

$$0 = \overline{\partial}_J \overline{\partial}_J f = \overline{\partial}_J(H_{q+1} + O(|z|^{q+1}) = \frac{1}{2}(dH_{q+1} - J_0 dH_{q+1}) + O(|z|^q) \tag{18.45}$$

$$= \overline{\partial}_0 H_{q+1} + O(|z|^q). \tag{18.46}$$

□

Proposition 18.4. *For each $1 \leq p < \infty$, there exists $f = \sum_{j=1}^p f_j$ satisfying $\overline{\partial}_J f = O(|z|^p)$.*

Proof. We prove this by induction. For $p = 1$, we can take $f = c_p z^p$, and then

$$\overline{\partial}_J z^k = \overline{\partial}_0 z^k + \frac{i}{2}(J - J_0)dz^k = 0 + O(|z|). \tag{18.47}$$

Assume that we have found a solution for $j = 1 \dots p$. Let $f = \sum_{j=1}^{p} f_j$, by the induction assumption, we have

$$\overline{\partial}_J f = H_{p+1} + O(|z|^{p+1}), \tag{18.48}$$

and by the above, we need to solve the equation

$$\overline{\partial}_0 f_{p+1} = H_{p+1}. \tag{18.49}$$

From Proposition 19.4, H_{p+1} satisfies $\overline{\partial}_0 H_{p+1} = 0$. Equivalently, we can write

$$H_{p+1} = \alpha_{\bar{j}} d\overline{z}^j, \tag{18.50}$$

where the coefficients satisfy

$$\frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^l} = \frac{\partial \alpha_{\bar{l}}}{\partial \overline{z}^j}, \ j, l = 1, \dots, n. \tag{18.51}$$

Define

$$f_{p+1} = \int_0^1 \sum_{j=1}^{n} \overline{z}^j \alpha_{\bar{j}}(z, t\bar{z}) dt. \tag{18.52}$$

Then we compute

$$\begin{aligned}
\frac{\partial f_{p+1}}{\partial \overline{z}^k} &= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^{n} \overline{z}^j \frac{\partial}{\partial \overline{z}^k} \Big(\alpha_{\bar{j}}(z, t\bar{z}) \Big) \Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^{n} \overline{z}^j \frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^k} (z, t\bar{z}) t \Big) dt \\
&= \int_0^1 \Big(\alpha_{\bar{k}}(z, tz) + \sum_{j=1}^{n} \overline{z}^j \frac{\partial \alpha_{\bar{k}}}{\partial \overline{z}^j} (z, t\bar{z}) t \Big) \\
&= \int_0^1 \frac{d}{dt} \Big(t \alpha_{\bar{k}}(z, t\bar{z}) \Big) dt = \alpha_{\bar{k}}(z, \bar{z}).
\end{aligned} \tag{18.53}$$

□

We will discuss convergence next time.

19 Lecture 19

19.1 Convergence of formal power series solution

Last time, we constructed a formal power series solution. Today, we examine the solution in more detail. We look at the procedure in Proposition 18.4 above. We had H_{p+1} satisfying $\overline{\partial}_0 H_{p+1} = 0$. Writing

$$H_{p+1} = \alpha_{\bar{j}} d\overline{z}^j, \tag{19.1}$$

then the coefficients satisfy

$$\frac{\partial \alpha_{\bar{j}}}{\partial \overline{z}^l} = \frac{\partial \alpha_{\bar{l}}}{\partial \overline{z}^j}, \ j, l = 1, \ldots, n. \tag{19.2}$$

Defining

$$f_{p+1} = \int_0^1 \sum_{j=1}^n \overline{z}^j \alpha_{\bar{j}}(z, t\bar{z}) dt, \tag{19.3}$$

then we showed that $\overline{\partial}_0 f_{p+1} = H_{p+1}$.

Proposition 19.1. *Writing $H_{p+1} = \alpha_{\bar{j}} d\overline{z}^j$, where*

$$\alpha_{\bar{j}} = \sum_{|I|+|J|=p} \alpha_{\bar{j}I\bar{J}} z^I \overline{z}^J, \tag{19.4}$$

and $f_{p+1} = \sum_{|I|+|J|=p+1} f_{I\bar{J}} z^I \overline{z}^J$. Then the coefficients $f_{I\bar{J}}$ are linear functions of the $\alpha_{\bar{j}I\bar{J}}$ with non-negative coefficients.

Proof. We plug (19.4) into (19.3), and compute

$$\begin{aligned} f_{p+1} &= \int_0^1 \sum_{j=1}^n \overline{z}^j \sum_{|I|+|J|=p} \alpha_{\bar{j}I\bar{J}} z^I (t\overline{z})^J dt & (19.5)\\ &= \sum_{j,|I|+|J|=p} \alpha_{\bar{j}I\bar{J}} z^I \overline{z}^J \overline{z}^j \int_0^1 t^{|J|} dt & (19.6)\\ &= \sum_{j,|I|+|J|=p} \frac{1}{|J|+1} \alpha_{\bar{j}I\bar{J}} z^I \overline{z}^J \overline{z}^j, & (19.7) \end{aligned}$$

and we are done. □

Remark 19.2. Our choice above is very important: there is a freedom to add an arbitrary holomorphic homogeneous polynomial to f_{p+1}, and our choice eliminates this ambiguity.

Now we return to solving the equation

$$\frac{\partial}{\partial \overline{z}^j} f + \phi_{\bar{j}}^k \frac{\partial}{\partial z^k} f = 0. \tag{19.8}$$

We will now consider this as an equation in $\mathbb{C}^{2n}$ with coordinates $(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n)$. Note by the transformation $z^j \mapsto -z^j$, we can assume the equation is of the form

$$\frac{\partial}{\partial \overline{z}^j} f - \phi_{\bar{j}}^k \frac{\partial}{\partial z^k} f = 0. \tag{19.9}$$

Remark 19.3. Note that we cannot simply replace ϕ with $-\phi$, since the integrability equation (18.1) is not preserved under this transformation.

We have the homogeneous decompositions

$$f = \sum_{j=1}^{\infty} f_j, \ \ \phi_{\bar{j}}^{k} = \sum_{l=1}^{\infty} (\phi_{\bar{j}}^{k})_l. \tag{19.10}$$

We then found the recursive system

$$\overline{\partial} f_{q+1} = H_{q+1}, \tag{19.11}$$

where

$$H_{q+1} = \sum_{k+l=q+1, k\geq 1, l\geq 1} (\phi_{\bar{j}}^{p})_k \frac{\partial f_l}{\partial z^p} d\bar{z}^j \tag{19.12}$$

satisfies $\overline{\partial}_0 H_{q+1} = 0$. Then we can solve $\overline{\partial} f_{q+1} = H_{q+1}$ uniquely with the above proceudre, where $f_{q+1}(z^1, \cdots, z^n, 0, \cdots, 0) = 0$ for $q > 1$. So once $f_1 = c_j z^j$ is specified, then our procedure gives a unique formal power series solution.

Write

$$(\phi_{\bar{j}}^{k})_p = \sum_{|I|+|J|=p} \phi_{\bar{j}I\bar{J}}^{k} z^I \bar{z}^J. \tag{19.13}$$

Proposition 19.4. *The coefficients $f_{I\bar{J}}$ when $|I| + |J| = p$ are a polynomial function of degree $p-1$ in the $\phi_{\bar{j}K\bar{L}}^{k}$ for $|K| + |L| \leq p-1$, with all coefficients non-negative rational numbers. The polynomials are completely determined by the constants $c_1, \ldots, c_n$.*

Proof. Without loss of generality, assume that $f_1 = z^1$. Then the first nontrivial equation is

$$\frac{\partial f_2}{\partial \bar{z}^j} = (\phi_{\bar{j}}^{p})_1 \frac{\partial f_1}{\partial z^p} = (\phi_{\bar{j}}^{1})_1 = \phi_{\bar{j}k}^{1} z^k + \phi_{\bar{j}\bar{k}}^{1} \bar{z}^k. \tag{19.14}$$

Then

$$f_2 = \phi_{\bar{j}k}^{1} z^k \bar{z}^j + \frac{1}{2} \phi_{\bar{j}\bar{k}}^{1} \bar{z}^j \bar{z}^k, \tag{19.15}$$

so the claim is true for f_2. Then we proceed by induction. So assume the claim is true for $f_1, \ldots, f_p$. Then the equation for f_{p+1} is

$$\frac{\partial f_{p+1}}{\partial \bar{z}^j} = \sum_{k+l=p+1, k\geq 1, l\geq 1} (\phi_{\bar{j}}^{p})_k \frac{\partial f_l}{\partial z^p} \tag{19.16}$$

$$= \sum_{k+l=p+1, k\geq 1, l\geq 1} \sum_{|I|+|J|=k} \phi_{\bar{j}I\bar{J}}^{p} \frac{\partial f_l}{\partial z^p}. \tag{19.17}$$

From Proposition 19.1, we just need to show the claim is true for the coefficients on the right hand side. Since $l \leq p$ in the sum, by induction the claim is true for the coefficients of f_l. The operator $f_l \mapsto \partial f_l / \partial z^p$ obviously preserves non-negativity of the coefficients, so we are done. $\square$

19.2 Cauchy majorant method

By assumption, the series

$$\sum_{I,J} \phi^k_{\bar{j}I\bar{J}} z^I \overline{z}^J. \tag{19.18}$$

converges for any point in the polydisc

$$P(\rho) = \{(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) \mid |z^j| < \rho, |\overline{z}^j| < \rho,\ 1 \leq j \leq n\}, \tag{19.19}$$

with uniform convergence in the polydisc $\overline{P(\rho')}$, for any $\rho' < \rho$. In particular, for any point $(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) \in \overline{P(\rho')}$, there exists a constant $C > 0$ so that

$$|\phi^k_{\bar{j}I\bar{J}} z^I \overline{z}^J| < C \quad \text{(no summation)}. \tag{19.20}$$

Choosing the point $z^j = \rho', \overline{z}^j = \rho'$ for $j = 1, \ldots, n$, this implies that

$$|\phi^k_{\bar{j}I\bar{J}}| < C(\rho')^{-(|I|+|J|)}. \tag{19.21}$$

To simplify notation, let's call ρ' by ρ. Then we define

$$\Phi(w) = C\Big(\frac{1}{1 - w\rho^{-1}} - 1\Big) = \frac{Cw}{\rho - w} \tag{19.22}$$

which is analytic in the disc $\Delta(\rho) = \{w \in \mathbb{C} \mid |w| < \rho\}$. The power series of $\Phi(w)$ is given by

$$\Phi(w) = C \sum_{j=1}^{\infty} \rho^{-j} w^j. \tag{19.23}$$

Next, we let

$$\Phi(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) = \Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n). \tag{19.24}$$

Using the multinomial theorem, we have the expansion

$$\Phi(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) = C \sum_{I,J \neq (0,0)} \rho^{-(|I|+|J|)} \frac{(|I| + |J|)!}{I!J!} z^I \overline{z}^J, \tag{19.25}$$

which converges absolutely in the polydisc

$$P = \{(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) \mid |z^j| < \rho/2n, |\overline{z}^j| < \rho/2n\}. \tag{19.26}$$

That is, the power series coefficients of $\Phi(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n)$ are given by

$$\Phi_{I\bar{J}} = C\rho^{-(|I|+|J|)} \frac{(|I| + |J|)!}{I!J!}, \tag{19.27}$$

with $\Phi_{00} = 0$. Since the multinomial coeffients are at least 1, we have the inequality

$$|\phi^j_{\bar{k}I\bar{J}}| < C\rho^{-(|I|+|J|)} \leq C\rho^{-(|I|+|J|)}\frac{(|I|+|J|)!}{I!J!} = \Phi_{I\bar{J}}. \tag{19.28}$$

Next, we claim that $\Phi^k_{\bar{j}} = \Phi$ determines an integrable almost complex structure. Note we are viewing this as an $n \times n$ matrix will all entries equal. To see this, we use (18.1):

$$\begin{aligned}
&\frac{\partial}{\partial \overline{z}^l}\Phi^j_{\bar{k}} - \frac{\partial}{\partial \overline{z}^k}\Phi^j_{\bar{l}} + \Phi^m_{\bar{k}}\frac{\partial}{\partial z^m}\Phi^j_{\bar{l}} - \Phi^m_{\bar{l}}\frac{\partial}{\partial z^m}\Phi^j_{\bar{k}} \\
&= \frac{\partial}{\partial \overline{z}^l}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) - \frac{\partial}{\partial \overline{z}^k}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) \\
&+ \sum_m \frac{\partial}{\partial z^m}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) - \sum_m \frac{\partial}{\partial z^m}\Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) \\
&= 0 + 0 = 0.
\end{aligned} \tag{19.29}$$

The equation for a holomorphic function with respect to Φ is

$$\frac{\partial F}{\partial \overline{z}^j} = \Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n)\sum_m \frac{\partial F}{\partial z^m}. \tag{19.30}$$

For all $j = 1, \ldots, n$.

Let's assume that we can find a solution F_k of (19.30) satisying the initial conditions

$$F_k(z^1, \ldots, z^n, 0, \ldots 0) = z^k, \tag{19.31}$$

which is analytic in some polydisc $|z| < \rho', |\overline{z}| < \rho'$. Without loss of generality, we can assume that $k = 1$. Then to finish the convergence proof, recall that our formal power series solves

$$f_{I\bar{J}} = P_{I\bar{J}}(\phi^*_*), \tag{19.32}$$

where $P_{I\bar{J}}$ is a polynomial with non-negative coefficients depending only upon ϕ^*_{*KL} for $|K| + |L| < |I| + |J|$. Since F_1 is an analytic solution of the Cauchy-Riemann equations with respect to Φ, and the same initial conditions as f, we must also have

$$F_{I\bar{J}} = P_{I\bar{J}}(\Phi_{KL}), \tag{19.33}$$

where $P_{I\bar{J}}$ is the *same* polynomial since $\Phi(0,0) = 0$ and $F(z^1, \ldots, z^n, 0, \ldots 0) = z^1$ has the same initial conditions as our formal power series solution. We then estimate

$$|f_{I\bar{J}}| = |P_{I\bar{J}}(\phi^*_*)| \leq P_{I\bar{J}}(|\phi^*_*|) \leq P_{I\bar{J}}(\Phi_{K\bar{L}}) = F_{I\bar{J}}. \tag{19.34}$$

The inequalities hold since $P_{I\bar{J}}$ is a polynomial with real non-negative coefficients, and using (19.28). This shows that our power series is majorized by the power series of F, which implies that the power series for f converges in the open polydisc $P(\rho')$, by the comparison test.

19.3 Completion of convergence proof

To finish the convergence proof, we need to find solution of

$$\frac{\partial F}{\partial \overline{z}^j} = \Phi(z^1 + \cdots + z^n + \overline{z}^1 + \cdots + \overline{z}^n) \sum_m \frac{\partial F}{\partial z^m}, \tag{19.35}$$

for all $j = 1, \ldots, n$, satisying the initial conditions

$$F(z^1, \ldots, z^n, 0, \ldots 0) = z^1, \tag{19.36}$$

and which is analytic in some polydisc around the origin in $\mathbb{C}^{2n}$.

Proposition 19.5. *For any choice of $(c_1, \ldots, c_n) \in \mathbb{C}^n$, satisfying $c_1 + \cdots + c_n = 0$, the function $F = \sum c_k z^k$ solves* (19.35).

Proof. The function F obviously makes the left hand side of (19.35) vanish for any $1 \leq j \leq n$. The right hand side of (19.35) is $\Phi \cdot (c_1 + \cdots + c_n) = 0$. $\square$

Next, let's try and find a solution F_+ of the form

$$F_+(z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n) = G(z^1 + \cdots + z^n, \overline{z}^1 + \cdots + \overline{z}^n). \tag{19.37}$$

Let's call $z = z^1 + \cdots + z^n$, $\overline{z} = \overline{z}^1 + \cdots + \overline{z}^n$, and write G as a function of 2 variables $G = G(z, \overline{z})$. Then (19.35) becomes

$$\frac{\partial G}{\partial \overline{z}} = \Phi(z + \overline{z}) n \frac{\partial G}{\partial z}. \tag{19.38}$$

This is just the Beltrami equation:

$$\frac{\partial G}{\partial \overline{z}} = \frac{nC(z + \overline{z})}{\rho - z - \overline{z}} \frac{\partial G}{\partial z}. \tag{19.39}$$

But we have already found an analytic solution G for this equation, it is done in Proposition 11.1 (only the constant C has changed to nC), which satisfies the initial condition $G(z, 0) = 0$. So the corresponding solution of the n-dimensional problem satisfies

$$F_+(z^1, \ldots, z^n, 0, \ldots 0) = G(z^1 + \cdots + z^n, 0) = z^1 + \cdots + z^n. \tag{19.40}$$

Using Proposition 19.5, we see that the function

$$F = \frac{1}{n}\Big(F_+ + (z^1 - z^2) + (z^1 - z^3) + \cdots + (z^1 - z^n)\Big) \tag{19.41}$$

is holomorphic with respect to Φ, is analytic in some polydisc $P(\rho')$, and satisfies the initial conditions (19.36). This finishes the proof.

20 Lecture 20

20.1 Reduction to the analytic case

In the subsection, we will discuss a method of Malgrange, which transforms the C^2 case into the analytic case [Mal69, Nir73]. In the z-coordinates, our holomorphic equation is

$$\frac{\partial w}{\partial \overline{z}^j} + \phi_{\bar{j}}^k \frac{\partial w}{\partial z^k} = 0 \tag{20.1}$$

We now view w as a vector-valued function to $\mathbb{C}^n$. We want to change coordinates $\xi = \xi(z, \overline{z})$ so that such that our holomorphic equation transform into another holomorphic equation with analytic coefficients. Write

$$w(z, \overline{z}) = W(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})) \tag{20.2}$$
$$\phi_{\bar{j}}^k(z, \overline{z}) = U_{\bar{j}}^k(\xi(z, \overline{z}), \bar{\xi}(z, \overline{z})). \tag{20.3}$$

Then

$$\frac{\partial w}{\partial \overline{z}^j} = \frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial \overline{z}^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial \overline{z}^j} \tag{20.4}$$
$$\frac{\partial w}{\partial z^j} = \frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial z^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial z^j}. \tag{20.5}$$

So the holomorphic equations become

$$\frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial \overline{z}^j} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial \overline{z}^j} + U_{\bar{j}}^k(\xi(z,\bar{z}), \bar{\xi}(z,\overline{z}))\Big(\frac{\partial W}{\partial \xi^l}\frac{\partial \xi^l}{\partial z^k} + \frac{\partial W}{\partial \bar{\xi}^l}\frac{\partial \bar{\xi}^l}{\partial z^k}\Big) = 0. \tag{20.6}$$

By inverting the matrix coefficients, this transforms into another holomorphic system of the form

$$\frac{\partial W}{\partial \bar{\xi}^j} + \tilde{U}_{\bar{j}}^k(\xi, \bar{\xi})\frac{\partial W}{\partial \xi^k} = 0 \tag{20.7}$$

where $\tilde{U}$ is of the form

$$\tilde{U}_{\bar{j}}^k = \Big(\Big(\frac{\partial \bar{\xi}^*}{\partial \overline{z}^*} + U_*^p \frac{\partial \bar{\xi}^*}{\partial z^p}\Big)^{-1}\Big)_{\bar{j}}^{\bar{q}}\Big(\frac{\partial \xi^k}{\partial \overline{z}^q} + U_{\bar{q}}^p \frac{\partial \xi^k}{\partial z^p}\Big). \tag{20.8}$$

Let us try to find coordinates so that

$$\sum_j \frac{\partial}{\partial \xi^j}\tilde{U}_{\bar{j}}^k(\xi, \bar{\xi}) = 0. \tag{20.9}$$

To find the coordinate system ξ, we must write out (20.9), and this becomes a second order system for ξ as a function of the original z coordinates. From the chain rule, we have

$$\frac{\partial}{\partial \xi^j} = \frac{\partial z^l}{\partial \xi^j}\frac{\partial}{\partial z^l} + \frac{\partial \overline{z}^l}{\partial \xi^j}\frac{\partial}{\partial \overline{z}^l}, \tag{20.10}$$

so (20.9) becomes

$$\sum_j \Big(\frac{\partial z^l}{\partial \xi^j}\frac{\partial}{\partial z^l} + \frac{\partial \overline{z}^l}{\partial \xi^j}\frac{\partial}{\partial \overline{z}^l}\Big)\Big(\Big(\frac{\partial \bar{\xi}^*}{\partial \overline{z}^*} + U^p_*\frac{\partial \bar{\xi}^*}{\partial z^p}\Big)^{-1}\Big)^{\bar{q}}_{\bar{j}}\Big(\frac{\partial \xi^k}{\partial \overline{z}^q} + U^p_{\bar{q}}\frac{\partial \xi^k}{\partial z^p}\Big). \tag{20.11}$$

The inverse function theorem says that

$$\begin{pmatrix} \dfrac{\partial z^*}{\partial \xi^*} & \dfrac{\partial z^*}{\partial \bar{\xi}^*} \\ \dfrac{\partial \overline{z}^*}{\partial \xi^*} & \dfrac{\partial \overline{z}^*}{\partial \bar{\xi}^*} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \xi^*}{\partial z^*} & \dfrac{\partial \xi^*}{\partial \overline{z}^*} \\ \dfrac{\partial \bar{\xi}^*}{\partial z^*} & \dfrac{\partial \bar{\xi}^*}{\partial \overline{z}^*} \end{pmatrix}^{-1}. \tag{20.12}$$

Making this substitution, and replacing $U^p_{\bar{q}}(\xi, \bar{\xi}) = \phi^p_{\bar{q}}(z, \overline{z})$, we see that the equation (20.11) is a *quasilinear* system of the form

$$F(D^2\xi, D\xi, \xi, z, \overline{z}) = 0. \tag{20.13}$$

We recall the definition of the linearization.

Definition 20.1. The linearization of F at a function ξ is given by

$$F'_\xi(h) = \frac{d}{dt}F(D^2(\xi + th), D(\xi + th), \xi + th, z, \overline{z})\Big|_{t=0}. \tag{20.14}$$

Proposition 20.2. *Assuming $\phi \in C^1$, then the linearization of F at $\xi = z$ is*

$$F'_z(h) = -\frac{1}{4}\Delta h + (\phi + \phi^2) * \nabla^2 h + (\nabla\phi + \phi * \nabla\phi) * \nabla h. \tag{20.15}$$

If $\phi(0) = 0$, then we have

$$F'_z(h)(0) = \frac{1}{4}\Delta h + \nabla\phi * \nabla h. \tag{20.16}$$

If ϕ has sufficiently small $C^{1,\alpha}$, norm then F'_z is an elliptic operator with Hölder coefficients bounded in C^α.

Proof. We use the following formula: if $A(t)$ is a path of matrices, then

$$\frac{d}{dt}A(t)^{-1} = -A^{-1} \circ \frac{d}{dt}A \circ A^{-1}. \tag{20.17}$$

Let look at each term in (20.11)-(20.12). First, for $\xi = z$, we have

$$\begin{pmatrix} \dfrac{\partial \xi^*}{\partial z^*} & \dfrac{\partial \xi^*}{\partial \overline{z}^*} \\ \dfrac{\partial \bar{\xi}^*}{\partial z^*} & \dfrac{\partial \bar{\xi}^*}{\partial \overline{z}^*} \end{pmatrix}^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}. \tag{20.18}$$

Also,

$$\frac{d}{dt}\begin{pmatrix} \dfrac{\partial (z+th)^*}{\partial z^*} & \dfrac{\partial (z+th)^*}{\partial \overline{z}^*} \\ \dfrac{\partial \overline{(z+th)}^*}{\partial z^*} & \dfrac{\partial \overline{(z+th)}^*}{\partial \overline{z}^*} \end{pmatrix}^{-1} \Bigg|_{t=0} = -\begin{pmatrix} \dfrac{\partial h^*}{\partial z^*} & \dfrac{\partial h^*}{\partial \overline{z}^*} \\ \dfrac{\partial \bar{h}^*}{\partial z^*} & \dfrac{\partial \bar{h}^*}{\partial \overline{z}^*} \end{pmatrix}. \tag{20.19}$$

Next, we look at the last matrix. At $\xi = z$, we have

$$\frac{\partial \xi^k}{\partial \overline{z}^q} + U^p_{\bar{q}} \frac{\partial \xi^k}{\partial z^p} = U^p_{\bar{q}} \delta^k_p = U^k_{\bar{q}}. \tag{20.20}$$

The linearization of this term is

$$\frac{d}{dt}\Big(\frac{\partial (z+th)^k}{\partial \overline{z}^q} + U^p_{\bar{q}} \frac{\partial (z+th)^k}{\partial z^p}\Big)\Big|_{t=0} = \frac{\partial h^k}{\partial \overline{z}^q} + U^p_{\bar{q}} \frac{\partial h^k}{\partial z^p}. \tag{20.21}$$

Next, we look at the middle matrix. At $\xi = z$, we have

$$\Big(\Big(\frac{\partial \bar{\xi}^*}{\partial \overline{z}^*} + U^p_* \frac{\partial \bar{\xi}^*}{\partial z^p}\Big)^{-1}\Big)^{\bar{q}}_{\bar{j}} = (I_n)^{\bar{q}}_{j}. \tag{20.22}$$

The linearization is

$$\frac{d}{dt}\Big(\Big(\frac{\partial \overline{(z+th)}^*}{\partial \overline{z}^*} + U^p_* \frac{\partial \overline{(z+th)}^*}{\partial z^p}\Big)^{-1}\Big)^{\bar{q}}_{\bar{j}}\Big|_{t=0} = -\Big(\frac{\partial \bar{h}^{\bar{q}}}{\partial \overline{z}^j} + U^p_{\bar{j}} \frac{\partial \bar{h}^{\bar{q}}}{\partial z^p}\Big) \tag{20.23}$$

Putting everything together, we obtain

$$F'_z(h) = -\sum_j \Big(\frac{\partial h^l}{\partial z^j}\frac{\partial}{\partial z^l} + \frac{\partial \bar{h}^l}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{z}^l}\Big)\Big(\delta^{\bar{q}}_{\bar{j}} U^k_{\bar{q}}\Big) - \sum_j \frac{\partial}{\partial z^j}\Big(\frac{\partial \bar{h}^{\bar{q}}}{\partial \overline{z}^j} + U^p_{\bar{j}}\frac{\partial \bar{h}^{\bar{q}}}{\partial z^p}\Big)U^k_{\bar{q}} \tag{20.24}$$

$$-\sum_j \frac{\partial}{\partial z^j}\delta^{\bar{q}}_{\bar{j}}\Big(\frac{\partial h^k}{\partial \overline{z}^q} + U^p_{\bar{q}}\frac{\partial h^k}{\partial z^p}\Big) \tag{20.25}$$

$$= -\sum_j \Big(\frac{\partial h^l}{\partial z^j}\frac{\partial}{\partial z^l} + \frac{\partial \bar{h}^l}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{z}^l}\Big)U^k_{\bar{j}} - \sum_j \frac{\partial}{\partial z^j}\Big(\frac{\partial \bar{h}^{\bar{q}}}{\partial \overline{z}^j} + U^p_{\bar{j}}\frac{\partial \bar{h}^{\bar{q}}}{\partial z^p}\Big)U^k_{\bar{q}} \tag{20.26}$$

$$-\sum_j \frac{\partial}{\partial z^j}\Big(\frac{\partial h^k}{\partial \overline{z}^j} + U^p_{\bar{j}}\frac{\partial h^k}{\partial z^p}\Big). \tag{20.27}$$

This is of the form

$$F'_z(h) = -\frac{1}{4}\Delta h + \nabla h * \nabla U + \nabla^2 h * U + \nabla h * U * \nabla U + \nabla^2 h * U^2. \tag{20.28}$$

At the origin, all of the terms with U vanish by assumption, so we have

$$F'_z(h)(0) = -\frac{1}{4}\Delta h + \nabla h * \nabla U \tag{20.29}$$

□

From the above discussion on coordinate changes, let $\xi = \epsilon^{-1}z$. If $\xi \in B_1(0)$ then $z \in B_\epsilon(0)$. With $\phi^k_{\bar{j}}(z,\overline{z}) = U^k_{\bar{j}}(\epsilon^{-1}z, \epsilon^{-1}\overline{z})$, then we have

$$\tilde{U}^k_{\bar{j}}(\xi,\bar{\xi}) = \epsilon \cdot \epsilon^{-1} \cdot U^k_{\bar{j}}(\xi,\bar{\xi}) = U^k_{\bar{j}}(\xi,\bar{\xi}). \tag{20.30}$$

Note that

$$\|\tilde{U}\|_{C^0(B_1(0))} = \|\phi\|_{C^0(B_\epsilon(0))}, \tag{20.31}$$

But since $\phi \in C^1(B_\epsilon(0)) \subset C^{1,\alpha}(B_\epsilon(0))$, by the mean value theorem we have

$$|\phi(x) - \phi(y)| \leq C|x-y|. \tag{20.32}$$

Letting $y = 0$, since $\phi(y) = 0$ by assumption, we have

$$|\phi(x)| \leq C|x|. \tag{20.33}$$

Therefore, we have

$$\|\tilde{U}\|_{C^0(B_1(0))} \leq C\epsilon. \tag{20.34}$$

Next, with a slight abuse of notation, we have

$$\begin{aligned}
\|\nabla\tilde{U}\|_{C^0(B_1(0))} &= \sup_{\xi\in B_1(0)} \Big|\frac{\partial\tilde{U}(\xi,\bar{\xi})}{\partial\xi}\Big| + \Big|\frac{\partial\tilde{U}(\xi,\bar{\xi})}{\partial\bar{\xi}}\Big| && (20.35)\\
&= \sup_{\xi\in B_1(0)} \Big|\frac{\partial\phi(\epsilon\xi,\epsilon\bar{\xi})}{\partial\xi}\Big| + \Big|\frac{\partial\phi(\epsilon\xi,\epsilon\bar{\xi})}{\partial\bar{\xi}}\Big| && (20.36)\\
&= \epsilon \sup_{z\in B_\epsilon(0)} \Big|\frac{\partial\phi(z,\overline{z})}{\partial z}\Big| + \Big|\frac{\partial\phi(z,\overline{z})}{\partial\overline{z}}\Big| && (20.37)\\
&= \epsilon \cdot \|\nabla\phi\|_{C^0(B_\epsilon(0))}. && (20.38)
\end{aligned}$$

Also, we compute

$$\begin{aligned}
\sup_{x,y\in B_1(0),x\neq y} \frac{|\nabla\tilde{U}(x) - \nabla\tilde{U}(y)|}{|x-y|^\alpha} &= \sup_{x,y\in B_\epsilon(0),x\neq y} \frac{\epsilon\nabla_z\phi(x,\bar{x}) - \epsilon\nabla_z\phi(y,\bar{y})}{\epsilon^{-\alpha}|x-y|^\alpha} && (20.39)\\
&= \epsilon^{1+\alpha} \sup_{x,y\in B_\epsilon(0),x\neq y} \frac{\nabla_z\phi(x,\bar{x}) - \nabla_z\phi(y,\bar{y})}{|x-y|^\alpha}. && (20.40)
\end{aligned}$$

Since the $C^{1,\alpha}$ norm is the sum of these 3 parts, we can assume without loss of generality that

$$\|\phi\|_{C^{1,\alpha}(B_1(0))} < \epsilon. \tag{20.41}$$

for any $\epsilon > 0$.

Proposition 20.3. *For ϵ sufficiently small, the linearized operator*

$$F'_z : C^{2,\alpha}_0(B_1(0)) \to C^{0,\alpha}(B_1(0)) \tag{20.42}$$

is invertible with bounded inverse (independent of ϵ), where the domain satisfies Dirichlet boundary conditions.

Proof. The leading term in F_z' is just the vector Laplacian, which is completely uncoupled. So by standard elliptic elliptic theory for the Laplacian (on functions), we know that the leading term is invertible, with bounded inverse (with Dirichlet boundary conditions on each component). We next show that for ϵ sufficiently small, F_z' will be an arbitrarily small perturbation of the Laplacian in operator norm. To see this, we write

$$F_z'(h) = -\frac{1}{4}\Delta h + Qh, \tag{20.43}$$

where Q are the lower order terms. Let $\mathcal{B}_1 = C^{2,\alpha}(B_1(0))_0$ and $\mathcal{B}_2 = C^{0,\alpha}(B_1(0))$. Recall that for the Hölder norms, we have

$$\|fg\|_{C^{k,\alpha}} \leq \|f\|_{C^{k,\alpha}} \cdot \|g\|_{C^{k,\alpha}}, \tag{20.44}$$

so we estimate

$$\begin{aligned}
\|Qh\|_{\mathcal{B}_2} &= \|(\phi + \phi^2) * \nabla^2 h + (\nabla\phi + \phi * \nabla\phi) * \nabla h\|_{\mathcal{B}_2} && (20.45)\\
&\leq (\|\phi\|_{\mathcal{B}_2} + \|\phi\|_{\mathcal{B}_2}^2) \cdot \|\nabla^2 h\|_{\mathcal{B}_2} + (\|\nabla\phi\|_{\mathcal{B}_2} + \|\phi\|_{\mathcal{B}_2}\|\nabla\phi\|_{\mathcal{B}_2}) \cdot \|\nabla h\|_{\mathcal{B}_2} && (20.46)\\
&\leq (\epsilon + \epsilon^2) \cdot \|h\|_{\mathcal{B}_1}. && (20.47)
\end{aligned}$$

So the operator norm of Q is estimated

$$\sup_{0 \neq h \in \mathcal{B}_1} \frac{\|Qh\|_{\mathcal{B}_2}}{\|h\|_{\mathcal{B}_1}} \leq \epsilon + \epsilon^2. \tag{20.48}$$

So by the above inverse function theorem, Lemma 15.1, F_z' is also invertible with bounded inverse if ϵ is sufficiently small. □

Remark 20.4. Note that the above proof reduced everything to invertibility of the Laplacian on functions, we did not need to quote any results about elliptic systems of PDEs.

Proposition 20.5. *If ϵ is sufficiently small then*

$$\|F(z)\|_{\mathcal{B}_2} < C\epsilon. \tag{20.49}$$

Proof. From the above computations, we have

$$F(z) = \sum_j \frac{\partial}{\partial z^j} \phi_{\bar{j}}^k, \tag{20.50}$$

so we have

$$\|F(z)\|_{C^{0,\alpha}(B_1(0))} \leq C\|\nabla\phi\|_{C^{0,\alpha}(B_1(0))} \leq C\|\phi\|_{C^{1,\alpha}(B_1(0))} \leq C\epsilon. \tag{20.51}$$

□

21 Lecture 21

21.1 Inverse function theorem

To use the inverse function theorem, Lemma 13.3, it remains to verify the estimate on the non-linear terms. Recall $\mathcal{B}_1 = C^{2,\alpha}_0(B_1(0), \mathbb{R}^{2n})$ and $\mathcal{B}_2 = C^{0,\alpha}(B_1(0), \mathbb{R}^{2n})$. Note that F and ξ are vector-valued, but for simplicity of notation in the following discussion, we will assume they are scalar-valued. Let write our nonlinear operator as $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ as

$$\mathcal{F}(\xi) = F(D^2\xi, D\xi, \xi, z, \overline{z}), \tag{21.1}$$

where $F : \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R} \times B_1(0) \to \mathbb{R}$. Write these variables as (r_{ij}, p_i, u, x). From (20.11), we have that for any fixed $x \in B_1(0)$, F is analytic in the r_{ij}, p_i, and u variables, and

$$F, \nabla_{r,p,u} F, \nabla^2_{r,p,u} F \in C^{0,\alpha}(\mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R} \times B_1(0)). \tag{21.2}$$

Note that we are slightly abusing notation, since this is only true on the subset for which the inverse matrix in (20.12) exists. Define

$$H : C^{2,\alpha}(B_1(0), \mathbb{R}) \to C^{0,\alpha}(B_1(0), \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R}) \tag{21.3}$$

by

$$u \mapsto (\nabla^2 u, \nabla u, u). \tag{21.4}$$

Then we can write

$$\mathcal{F}(u) = G \circ H(u), \tag{21.5}$$

where $G : C^{0,\alpha}(B_1(0), \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R}) \to C^{0,\alpha}(B_1(0), \mathbb{R})$ is defined by

$$G(r_{ij}(x), p_i(x), u(x)) = F(r_{ij}(x), p_i(x), u(x), x). \tag{21.6}$$

We want to show the estimate on the nonlinear terms

$$\|\mathcal{F}(u_2) - \mathcal{F}(u_1) - \mathcal{F}'_z(u_2 - u_1)\|_{\mathcal{B}_2} \leq C(\|u_1\|_{\mathcal{B}_1} + \|u_2\|_{\mathcal{B}_1}) \cdot \|u_1 - u_2\|_{\mathcal{B}_1}. \tag{21.7}$$

From the chain rule, we have

$$\mathcal{F}'_z(h) = G'_{H(z)} \circ H'_z(h). \tag{21.8}$$

But H is a bounded linear operator, because

$$\|H(u)\|_{C^{0,\alpha}} = \|(\nabla^2 u, \nabla u, u)\|_{C^{0,\alpha}} \leq \|u\|_{C^{2,\alpha}}. \tag{21.9}$$

So if we show for $a_1, a_2 \in C^{0,\alpha}(B_1(0), \mathbb{R}^{4n^2} \times \mathbb{R}^{2n} \times \mathbb{R})$ that

$$\|G(a_2) - G(a_1) - G'_{H(z)}(a_2 - a_1)\|_{\mathcal{B}_2} \leq C(\|a_1\|_{C^{0,\alpha}} + \|a_2\|_{C^{0,\alpha}}) \cdot \|a_2 - a_1\|_{C^{0,\alpha}}, \tag{21.10}$$

then since H is linear,

$$\begin{aligned}\|\mathcal{F}(u_2) - \mathcal{F}(u_1) - \mathcal{F}'_z(u_2 - u_1)\|_{\mathcal{B}_2} &= \|G \circ H(u_2) - G \circ H(u_1) - G'_{H(z)} \circ H'_z(u_2 - u_1)\|_{\mathcal{B}_2} \\ &= \|G \circ H(u_2) - G \circ H(u_1) - G'_{H(z)} \circ (Hu_2 - Hu_1)\|_{\mathcal{B}_2} \\ &\leq C(\|H(u_2)\|_{\mathcal{B}_2} + \|H(u_1)\|_{\mathcal{B}_2}) \cdot \|H(u_2) - H(u_1)\|_{\mathcal{B}_2} \\ &\leq C'(\|u_2\|_{\mathcal{B}_2} + \|u_1\|_{\mathcal{B}_2}) \cdot \|u_2 - u_1\|_{\mathcal{B}_2}.\end{aligned} \tag{21.11}$$

So we just need to show an estimate on G. Again, the fact that the domain of G is vector-valued functions doesn't matter, so for simplicity, we just assume that we have $G : C^{0,\alpha}(B_1(0), \mathbb{R}) \to C^{0,\alpha}(B_1(0), \mathbb{R})$ where $G(u(x)) = F(u(x), x)$, and $F : \mathbb{R} \times B_1(0) \to \mathbb{R}$, with

$$F, F_u, F_{uu} \in C^{0,\alpha}(\mathbb{R} \times B_1(0)). \tag{21.12}$$

The linearized operator of G at a function u_0 is simply

$$G'_{u_0}(h) = \frac{d}{dt} F((u_0 + th)(x), x)|_{t=0} = F_u(u_0(x), x)h. \tag{21.13}$$

By considering the function $G(u_0 + u)$ instead, which satisfies the same properties as the original G, we can assume that $u_0 = 0$. We let

$$f(t) = G((1 - t)u_1 + tu_2). \tag{21.14}$$

The fundamental theorem of calculus says

$$f(1) - f(0) = \int_0^1 f'(t)dt, \tag{21.15}$$

which is

$$G(u_2) - G(u_1) = \int_0^1 G'_{(1-t)u_1+tu_2}(u_2 - u_1)dt. \tag{21.16}$$

We rewrite this as

$$G(u_2) - G(u_1) - G'_0(u_2 - u_1) = \int_0^1 \Big(G'_{(1-t)u_1+tu_2} - G'_0\Big)(u_2 - u_1)dt. \tag{21.17}$$

So we need to prove the estimate

$$\|(G'_u - G'_0)h\|_{C^{0,\alpha}(B_1(0))} \leq C\|h\|_{C^{0,\alpha}(B_1(0))} \cdot \|u\|_{C^{0,\alpha}(B_1(0))}. \tag{21.18}$$

But we have

$$(G'_u - G'_0)h = (F_u(u(x), x) - F_u(0, x))h, \tag{21.19}$$

which implies

$$\|(G'_u - G'_0)h\|_{C^{0,\alpha}(B_1(0))} \leq \|F_u(u(x),x) - F_u(0,x)\|_{C^{0,\alpha}(B_1(0))}\|h\|_{C^{0,\alpha}(B_1(0))}, \tag{21.20}$$

and we just need to prove the estimate

$$\|F_u(u(x),x) - F_u(0,x)\|_{C^{0,\alpha}(B_1(0))} \leq C\|u\|_{C^{0,\alpha}(B_1(0))}. \tag{21.21}$$

Now we let $f(t) = F_u(tu(x),x)$. The fundamental theorem of calculus gives

$$F_u(u(x),x) - F_u(0,x) = \int_0^1 F_{uu}(tu(x),x)u(x)dt. \tag{21.22}$$

First, we estimate the C^0-norm

$$\begin{aligned}\|F_{uu}(tu(x),x)u(x)\|_{C^0(B_1(0))} &\leq \|F_{uu}(tu(x),x)\|_{C^0(B_1(0))}\|u(x)\|_{C^0(B_1(0))}\\ &\leq C\|u(x)\|_{C^0(B_1(0))},\end{aligned} \tag{21.23}$$

as long as u is small enough so that $u(x)$ is in the domain of definition of F_{uu}.

Next, we estimate the C^α semi norm. Note that

$$\begin{aligned}(f\cdot g)(x) - (f\cdot g)(y) &= f(x)g(x) - f(y)g(y)\\ &= f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)\\ &\leq (f(x)-f(y))g(x) + f(y)(g(x)-g(y)),\end{aligned} \tag{21.24}$$

which implies the estimate

$$[fg]_\alpha \leq [f]_\alpha\|g\|_{C^0} + \|f\|_{C^0}[g]_\alpha. \tag{21.25}$$

So we estimate

$$[F_{uu}(tu(x),x)u(x)]_\alpha \leq [F_{uu}(tu(x),x)]_\alpha\|u\|_{C^0} + \|F_{uu}(tu(x),x)\|_{C^0}[u]_\alpha. \tag{21.26}$$

We have that

$$|F_{uu}(tu(x),x) - F_{uu}(tu(y),y))| \leq [F_{uu}]_\alpha(|tu(x)-tu(y)| + |x-y|)^\alpha \tag{21.27}$$

$$\leq [F_{uu}]_\alpha(t[u]_\alpha|x-y|^\alpha + |x-y|)^\alpha \leq C[F_{uu}]_\alpha, \tag{21.28}$$

as long as $[u]_\alpha$ is bounded. Putting all this together, we have

$$\begin{aligned}\|F_u(u(x),x) - F_u(0,x)\|_{C^{0,\alpha}(B_1(0))} &\leq C\|F_{uu}(tu(x),x)u(x)\|_{C^{0,\alpha}(B_1(0))}\\ &\leq C\|u\|_{C^{0,\alpha}(B_1(0))}.\end{aligned} \tag{21.29}$$

Returning to our problem, we have

$$\mathcal{F}(h) = F(z+h), \tag{21.30}$$

satisfying

$$\|\mathcal{F}(0)\|_{\mathcal{B}_2} \leq C\epsilon, \tag{21.31}$$

and $\mathcal{F}'_0 : \mathcal{B}_1 \to \mathcal{B}_2$ is invertible with bounded inverse independent of ϵ, for sufficiently small ϵ. Letting

$$\mathcal{F}(h) = \mathcal{F}(0) + \mathcal{F}'_0(h) + \mathcal{Q}(h), \tag{21.32}$$

we have proved that there exists a constant C so that

$$\|\mathcal{Q}(x) - \mathcal{Q}(y)\|_{\mathcal{B}_2} \leq C_2 \cdot (\|x\|_{\mathcal{B}_1} + \|y\|_{\mathcal{B}_1}) \cdot \|x - y\|_{\mathcal{B}_1} \tag{21.33}$$

So by Lemma 13.3, there exists a solution to $\mathcal{F}(h) = 0$ satisfying

$$\|h\|_{\mathcal{B}_1} \leq C\|\mathcal{F}(0)\|_{\mathcal{B}_2} \leq C\epsilon. \tag{21.34}$$

Then the vector-valued function $z^j + h^j$ satisfies

$$\nabla_i(z^j + h_j) = \delta_i^j + \nabla_i h_j, \tag{21.35}$$

so for ϵ sufficiently small the Jacobian at 0 is invertible, and we have therefore found a coordinate system.

21.2 Analyticity

We begin this subsection with the following observation.

Proposition 21.1. *Let (M_2, J) be an almost complex manifold, and $f : M_1 \to M_2$ a C^1 diffeomorphism. Then $N_{f^*J} = f^* N_J$. Consequently, the equations of integrability are independent of the coordinate system.*

Proof. Recall the definition of the Nijenhuis tensor

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\}. \tag{21.36}$$

Let $X, Y \in \Gamma(TM_2)$. Then $(f^*J)(X) = f_*^{-1} J f_* X$. Also, for a diffeomorphism, we have $f_*[X, Y] = [f_*X, f_*Y]$; see [**?**]. Therefore

$$\begin{aligned}
N_{f^*J}(X, Y) &= 2\{[f_*^{-1}Jf_*X, f_*^{-1}Jf_*Y] - [X, Y] \\
&\qquad - f_*^{-1}Jf_*[X, f_*^{-1}Jf_*Y] - f_*^{-1}Jf_*[f_*^{-1}Jf_*X, Y]\} \\
&= 2\{f_*^{-1}[Jf_*X, Jf_*Y] - f_*^{-1}[f_*X, f_*Y] - f_*^{-1}J[f_*X, Jf_*Y] - f_*^{-1}J[Jf_*X, f_*Y]\} \\
&= 2f_*^{-1}\{[Jf_*X, Jf_*Y] - [f_*X, f_*Y] - J[f_*X, Jf_*Y] - J[Jf_*X, f_*Y]\} \\
&= f_*^{-1}N_J(f_*X, f_*Y) = f^*N_J.
\end{aligned} \tag{21.37}$$

□

By assumption, the complex structure J corresponding to ϕ is integrable, so by Proposition 18.1, we have

$$\frac{\partial}{\partial \bar{z}^l}\phi^j_{\bar{k}} - \frac{\partial}{\partial \bar{z}^k}\phi^j_{\bar{l}} + \phi^m_{\bar{k}}\frac{\partial}{\partial z^m}\phi^j_{\bar{l}} - \phi^m_{\bar{l}}\frac{\partial}{\partial z^m}\phi^j_{\bar{k}} = 0. \tag{21.38}$$

Let $f : U \to V$ be the change of coordinates mapping from the ξ-coordinates to the z-coordinates. Clearly f^*J is associated to $f^*\phi$, and the above derivation shows that the components of $f^*\phi$ in the ξ-coordinates are given by $\tilde{U}$. By Proposition 21.1, the integrability condition is independent of coordinates, so we have that

$$\frac{\partial}{\partial \bar{\xi}^l}\tilde{U}^j_{\bar{k}} - \frac{\partial}{\partial \bar{\xi}^k}\tilde{U}^j_{\bar{l}} + \tilde{U}^m_{\bar{k}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{l}} - \tilde{U}^m_{\bar{l}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{k}} = 0. \tag{21.39}$$

Above, we have found the coordinates ξ so that

$$\sum_j \frac{\partial}{\partial \xi^j}\tilde{U}^k_{\bar{j}}(\xi, \bar{\xi}) = 0. \tag{21.40}$$

Now we view the coupled system (21.39)-(21.40) as an equation for $\tilde{U}^k_{\bar{j}}$ in the new ξ-coordinates.

Proposition 21.2. *If $\|\tilde{U}\|_{C^0}$ is sufficiently small, then the system* (21.39)-(21.40) *is an overdetermined elliptic first-order system with analytic coefficients.*

Proof. We need to linearize at $\tilde{U}$, but under the assumptions, it is clearly equivalent to proving ellipticity for the system

$$\frac{\partial}{\partial \overline{z}^l}\phi^j_{\bar{k}} - \frac{\partial}{\partial \overline{z}^k}\phi^j_{\bar{l}} = 0 \tag{21.41}$$

$$\sum_j \frac{\partial}{\partial z^j}\phi^k_{\bar{j}} = 0. \tag{21.42}$$

For $(\xi, \bar{\xi})$ a complex cotangent vector, the symbol is

$$\phi \mapsto \left(\bar{\xi}_l\phi^j_{\bar{k}} - \bar{\xi}_{\bar{k}}\phi^j_{\bar{l}}, \sum_j \xi_j\phi^k_{\bar{j}}\right) \tag{21.43}$$

If the right hand side vanishes, then we have

$$0 = \sum_k \xi_k\bar{\xi}_l\phi^j_{\bar{k}} - \sum_k \xi_k\bar{\xi}_{\bar{k}}\phi^j_{\bar{l}} = -|\xi|^2\phi^j_{\bar{l}}. \tag{21.44}$$

So if $\xi \neq 0$, then the symbol mapping is injective. □

Remark 21.3. In other words, applying $\partial/\partial\xi^k$ to the first equations, and using the second equation yields

$$-\frac{\partial}{\partial \xi^k}\frac{\partial}{\partial \bar{\xi}^k}\tilde{U}^j_{\bar{l}} + \frac{\partial}{\partial \xi^k}\left(\tilde{U}^m_{\bar{k}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{l}} - \tilde{U}^m_{\bar{l}}\frac{\partial}{\partial \xi^m}\tilde{U}^j_{\bar{k}}\right) = 0, \tag{21.45}$$

which is a determined second-order elliptic system, if $\|\tilde{U}\|_{C^0}$ is sufficiently small.

A classical result implies that $\tilde{U}^k_{\bar{j}}$ are then analytic functions in the ξ coordinates; see [**?**]. So we have proved:

Theorem 21.4. *If (M, J) satisfies $J \in C^{1,\alpha}$ and J is an integrable complex structure, then there exists a coordinate system defined in a neighborhood of any point such that J is real analytic in these coordinates.*

Remark 21.5. By more analysis of the Malgrange system, Hill-Taylor have reduced the regularity assumption; [**?**]. For example, it suffices to assume $J \in W^{1,p}$, for $p > 2n$.

22 Lecture 22

This lecture was about presheaves, sheaves, morphisms of sheaves, and exact sequences of sheaves.

23 Lecture 23

This lecture was about Čech cohomology, good covers, Dolbeault isomorphism.

24 Lecture 24

This lecture was about short exact sequences of sheaves and the resulting long exact sequence in cohomology, without any assumption on existence of a good cover. We also showed that the Čech cohomology is equivalent to the cohomology of a acyclic resolution. Discussion of the exponential sequence.

25 Lecture 25

25.1 Kähler metrics

We next consider (M, J, g) where g is a Riemannian metric, and J is an almost complex structure. We assume that g and J are compatible, that is,

$$g(X, Y) = g(JX, JY). \tag{25.1}$$

The metric g is called an almost-Hermitian metric. If J is also integrable, then g is called Hermitian.

To an almost Hermitian metric (M, J, g) we associate a 2-form

$$\omega(X, Y) = g(JX, Y). \tag{25.2}$$

This is indeed a 2-form since

$$\omega(Y, X) = g(JY, X) = g(J^2Y, JX) = -g(JX, Y) = -\omega(X, Y). \tag{25.3}$$

Since

$$\omega(JX, JY) = \omega(X, Y), \tag{25.4}$$

this form is a real form of type $(1, 1)$, and is called the *Kähler form* or *fundamental* 2*-form*.

In Euclidean space, this form is

$$\omega_{Euc} = \frac{i}{2}\sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{25.5}$$

Definition 25.1. An almost Hermitian manifold (M, g, J) is *Kähler* if J is integrable and $d\omega = 0$.

Proposition 25.2. *An almost Hermitian manifold (M, g, J) is Kähler if and only if $\nabla J = 0$.*

Proof. This follows from the identity

$$2g((\nabla_X J)Y, Z) = -d\omega(X, JY, JZ) + d\omega(X, Y, Z) + \frac{1}{2}g(N(Y, Z), JX), \tag{25.6}$$

which is true on any almost Hermitian manifold.

If (M, g, J) is Kähler, then the right hand sides vanishes, so J is parallel.

Conversely, if $\nabla J = 0$. Then since $\omega(X, Y) = g(JX, Y)$, it follows that ω is parallel. Then we recall that the exterior derivative $d : \Omega^p \to \Omega^{p+1}$ can be written in terms of covariant differentiation.

$$d\omega(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^j (\nabla_{X_j}\omega)(X_0, \ldots, \hat{X}_j, \ldots, X_p), \tag{25.7}$$

which follows immediately from the usual formula for the exterior derivative, and using normal coordinates around a point. This shows that a parallel form is closed, so then (25.6) implies that the Nijenhuis tensor vanishes. □

25.2 Complex tensor notation

We extend g by complex linearity to a symmetric inner product on $TM \otimes \mathbb{C}$. Choosing any real basis of the form $\{X_1, JX_1, \ldots, X_n, JX_n\}$, let us abbreviate

$$Z_\alpha = \frac{1}{2}\Big(X_\alpha - iJX_\alpha\Big) \tag{25.8}$$

$$Z_{\overline{\alpha}} = \frac{1}{2}\Big(X_\alpha + iJX_\alpha\Big), \tag{25.9}$$

and define

$$g_{\alpha\beta} = g(Z_\alpha, Z_\beta) \tag{25.10}$$

$$g_{\overline{\alpha}\overline{\beta}} = g(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) \tag{25.11}$$

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) \tag{25.12}$$

$$g_{\overline{\alpha}\beta} = g(Z_{\overline{\alpha}}, Z_\beta). \tag{25.13}$$

Notice that

$$\begin{aligned} g_{\alpha\beta} = g(Z_\alpha, Z_\beta) &= \frac{1}{4}g(X_\alpha - iJX_\alpha, X_\beta - iJX_\beta) \\ &= \frac{1}{4}\Big(g(X_\alpha, X_\beta) - g(JX_\alpha, JX_\beta) - i(g(X_\alpha, JX_\beta) + g(JX_\alpha, X_\beta))\Big) \\ &= 0, \end{aligned}$$

since g is J-invariant, and $J^2 = -Id$. Similarly,

$$g_{\overline{\alpha}\overline{\beta}} = 0, \tag{25.14}$$

Also, from symmetry of g, we have

$$g_{\alpha\overline{\beta}} = g(Z_\alpha, Z_{\overline{\beta}}) = g(Z_{\overline{\beta}}, Z_\alpha) = g_{\overline{\beta}\alpha}. \tag{25.15}$$

However, applying conjugation, since g is real we have

$$\overline{g_{\alpha\overline{\beta}}} = \overline{g(Z_\alpha, Z_{\overline{\beta}})} = g(Z_{\overline{\alpha}}, Z_\beta) = g(Z_\beta, Z_{\overline{\alpha}}) = g_{\beta\overline{\alpha}}, \tag{25.16}$$

which says that $g_{\alpha\overline{\beta}}$ is a Hermitian matrix.

We repeat the above for the fundamental 2-form ω, and define

$$\omega_{\alpha\beta} = \omega(Z_\alpha, Z_\beta) = i g_{\alpha\beta} = 0 \tag{25.17}$$
$$\omega_{\overline{\alpha}\overline{\beta}} = \omega(Z_{\overline{\alpha}}, Z_{\overline{\beta}}) = -i g_{\overline{\alpha}\overline{\beta}} = 0 \tag{25.18}$$
$$\omega_{\alpha\overline{\beta}} = \omega(Z_\alpha, Z_{\overline{\beta}}) = i g_{\alpha\overline{\beta}} \tag{25.19}$$
$$\omega_{\overline{\alpha}\beta} = \omega(Z_{\overline{\alpha}}, Z_\beta) = -i g_{\overline{\alpha}\beta}. \tag{25.20}$$

The first 2 equations are just a restatement that ω is of type $(1,1)$. Also, note that

$$\omega_{\alpha\overline{\beta}} = i g_{\alpha\overline{\beta}}, \tag{25.21}$$

defines a skew-Hermitian matrix.

On a complex manifold, the fundamental 2-form in holomorphic coordinates takes the form

$$\omega = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta. \tag{25.22}$$

Remark 25.3. Note that for the Euclidean metric, we have $g_{\alpha\overline{\beta}} = \frac{1}{2}\delta_{\alpha\beta}$, so

$$\omega_{Euc} = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\overline{z}^j. \tag{25.23}$$

Proposition 25.4. *(M, g, J) is Kähler if and only if in any local holomorphic coordinate system,*

$$\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} = \frac{\partial g_{k\overline{\beta}}}{\partial z^\alpha}, \tag{25.24}$$

Proof. If (M, g, J) is Kähler, then

$$\begin{aligned}
0 = d\omega &= i \sum_{\alpha,\beta=1}^{n} (dg_{\alpha\overline{\beta}}) \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta=1}^{n} (\partial g_{\alpha\overline{\beta}} + \overline{\partial} g_{\alpha\overline{\beta}}) \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta=1}^{n} \Big\{ \sum_{k} \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} dz^k \Big) + \sum_{k} \Big(\frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^k} d\overline{z}^k \Big) \Big\} \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} \\
&= i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z^k} dz^k \wedge dz^{\alpha} \wedge d\overline{z}^{\beta} + i \sum_{\alpha,\beta,k=1}^{n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^{\alpha} \wedge d\overline{z}^{\beta}.
\end{aligned} \tag{25.25}$$

However, the first term is a form of type $(2, 1)$, and the second term is a form of type $(1, 2)$ so both sums must vanish, which is equivalent to (25.24). The converse follows by reversing the above calculation. □

We also see that the Kähler condition on a Hermitian manifold is equivalent to $\overline{\partial}\omega = 0$, which is also equivalent to $\partial\omega = 0$, since ω is real.

25.3 Existence of local Kähler potential

We will prove the following very special property of Kähler metrics.

Proposition 25.5. *If (M, g, J) is Kähler then for each $p \in M$, there exists an open neighborhood U of p and a function $u : U \to \mathbb{R}$ such that $\omega = i\partial\overline{\partial}u$.*

Proof. Choose local homorphic coordinates z^j around p. Then in a ball B in these coordinates, since ω is a real closed 2-form, from the usual Poincaré lemma, there exists a real 1-form α such that $\omega = d\alpha$ in B. Next, write $\alpha = \alpha^{1,0} + \alpha^{0,1}$ where $\alpha^{1,0}$ is a 1-form of type $(1, 0)$, and $\alpha^{0,1}$ is a 1-form of type $(0, 1)$. Since α is real, $\overline{\alpha^{1,0}} = \alpha^{0,1}$. Next,

$$\begin{aligned}
\omega = d\alpha &= \partial\alpha + \overline{\partial}\alpha \\
&= \partial\alpha^{1,0} + \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0} + \overline{\partial}\alpha^{0,1}
\end{aligned} \tag{25.26}$$

The first and last terms on the right hand side are forms of type $(2, 0)$ and $(0, 2)$, respectively. Since ω is of type $(1, 1)$, we must have $\overline{\partial}\alpha^{0,1} = 0$. Since we are in a ball in $\mathbb{C}^n$, the $\overline{\partial}-$Poincaré Lemma says that there exists a function $f : B \to \mathbb{C}$ such that $\alpha^{0,1} = \overline{\partial} f$ in B. Substituting this into (25.26), we obtain

$$\omega = \partial\overline{\partial} f + \overline{\partial}\partial\overline{f} = i\partial\overline{\partial}(2Im(f)). \tag{25.27}$$

□

Proposition 25.6. *(M, g, J) is Kähler if and only if for each $p \in M$, there exists a holomorphic coordinate system around p such that*

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} (\delta_{jk} + O(|z|^2)_{jk}) dz^j \wedge d\overline{z}^k, \tag{25.28}$$

as $|z| \to 0$.

Proof. If this is true then $d\omega(p) = 0$ for any point p, so $d\omega \equiv 0$. Conversely, we can assume that $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$. From Proposition 25.5, we can find $u : B \to \mathbb{R}$ so that

$$u = c_0 + Re(c_{1j}z^j) + Re(c_{2ij}z^i z^j + c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3), \tag{25.29}$$

and $\omega = i\partial\overline{\partial}u$. But the first terms on the left hand side are in the kernel of the $\partial\overline{\partial}$-operator, so by subtracting these terms, we can assume that

$$u = Re(c_{2j\overline{k}}z^j\overline{z}^k) + O(|z|^3). \tag{25.30}$$

Then since $\omega(p) = \frac{i}{2}\sum_j dz^j \wedge d\overline{z}^j$, we have that

$$u = \frac{1}{2}|z|^2 + Re\{a_{jkl}z^j z^k z^l + b_{jkl}\overline{z}^j z^k z^l)\} + O(|z|^4). \tag{25.31}$$

Consider the coordinate change

$$z^k = w^k + \sum c_{klm}w^l w^m. \tag{25.32}$$

This will eliminate the b_{jkl} terms in the expansion of u, and the remaining cubic terms are annihilated by the $\partial\overline{\partial}$-operator, so by subtracting those terms, we can arrange that

$$u = \frac{1}{2}|w|^2 + O(|w|^4), \tag{25.33}$$

and (25.28) follows. □

26 Lecture 26

26.1 L^2 adjoints

For the real operator $d : \Lambda^p \to \Lambda^{p+1}$, the formal L^2-adjoint d^* is defined by

$$\int_M \langle d^*\alpha, \beta\rangle dV = \int_M \langle \alpha, d\beta\rangle dV, \tag{26.1}$$

where $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$, and where $\langle\cdot,\cdot\rangle = g(\cdot,\cdot)$, and dV is the oriented Riemannian volume element.

The Riemannian inner product on forms extends by complex linearity to an inner product on complex valued forms. For α and β be sections of $\Lambda^k_{\mathbb{C}}$, we define the Hermitian inner product of α and β to be

$$(\alpha, \beta) = g(\alpha, \overline{\beta}). \tag{26.2}$$

The formula (26.1) holds for complex valued forms. Replacing β with $\overline{\beta}$, we have

$$\int_M \langle d^*\alpha, \overline{\beta}\rangle dV = \int_M \langle \alpha, d\overline{\beta}\rangle dV. \tag{26.3}$$

But since d is a real operator, $d\overline{\beta} = \overline{d\beta}$, so we can write this as

$$\int_M (d^*\alpha, \beta) dV = \int_M (\alpha, d\beta) dV. \tag{26.4}$$

That is, d^* is the L^2 adjoint of d with respect to the Hermitian inner product.

We next want to compute the formal L^2 adjoints of other operators. For

$$\Gamma(\Lambda^{p,q}) \xrightarrow{\overline{\partial}} \Gamma(\Lambda^{p,q+1}), \tag{26.5}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{p,q+1}) \xrightarrow{\overline{\partial}^*} \Gamma(\Lambda^{p,q}), \tag{26.6}$$

is defined as follows. For $\alpha \in \Gamma(\Lambda^{p,q+1})$ and $\beta \in \Gamma(\Lambda^{p,q})$, we have

$$\int_M (\alpha, \overline{\partial}\beta) dV = \int_M (\overline{\partial}^* \alpha, \beta) dV, \tag{26.7}$$

where dV denotes the Riemannian volume element. For

$$\Gamma(\Lambda^{p,q}) \xrightarrow{\partial} \Gamma(\Lambda^{p+1,q}), \tag{26.8}$$

the L^2-Hermitian adjoint

$$\Gamma(\Lambda^{p+1,q}) \xrightarrow{\overline{\partial}^*} \Gamma(\Lambda^{p,q}), \tag{26.9}$$

is defined similarly.

The Hodge Laplacian is $\Delta_H : \Lambda^p \to \Lambda^p$ defined by

$$\Delta_H = d^* d + d d^*. \tag{26.10}$$

We also have the following Laplacians on (p,q)-forms

$$\Delta_\partial : \Lambda^{p,q} \to \Lambda^{p,q} \tag{26.11}$$
$$\Delta_{\overline{\partial}} : \Lambda^{p,q} \to \Lambda^{p,q}. \tag{26.12}$$

are defined by

$$\Delta_\partial = \partial^* \partial + \partial \partial^* \tag{26.13}$$
$$\Delta_{\overline{\partial}} = \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*. \tag{26.14}$$

Remark 26.1. By definition, Δ_∂ and $\Delta_{\overline{\partial}}$ preserve the type, but we do not know whether Δ_H maps $\Lambda^{p,q}$ to $\Lambda^{p,q}$ i.e., there is no obvious reason why it should preserve the type.

26.2 Hodge star operator

For a real oriented Riemannian manifold of dimension n, the Hodge star operator is a mapping

$$* : \Lambda^p \to \Lambda^{n-p} \tag{26.15}$$

defined by

$$\alpha \wedge *\beta = g_{\Lambda^p}(\alpha, \beta) dV_g, \tag{26.16}$$

for $\alpha, \beta \in \Lambda^p$, where dV_g is the oriented Riemannian volume element. Note that

$$*^2 = (-1)^{p(n-p)} Id_{\Lambda^p}. \tag{26.17}$$

The Hodge star operator yields an explicit formula for d^*.

Proposition 26.2. *On a Riemannian manifold (M, g), for $\alpha \in \Omega^p(M)$, we have*

$$d^*\alpha = (-1)^{n(p+1)+1} * d * \omega. \tag{26.18}$$

Proof. For $\alpha \in \Omega^p(M)$, and $\beta \in \Omega^{p-1}(M)$, we compute

$$\begin{aligned}
\int_M \langle \alpha, d\beta \rangle dV &= \int_M d\beta \wedge *\alpha \\
&= \int_M \Big(d(\beta \wedge *\alpha) + (-1)^p \beta \wedge d * \alpha \Big) \\
&= \int_M (-1)^{p+(n-p+1)(p-1)} \beta \wedge * * d * \alpha \\
&= \int_M \langle \beta, (-1)^{n(p+1)+1} * d * \alpha \rangle dV \\
&= \int_M \langle \beta, d^*\alpha \rangle dV.
\end{aligned} \tag{26.19}$$

□

If M is a complex manifold of complex dimension $m = n/2$, and g is a Hermitian metric, then the Hodge star extends to the complexification

$$* : \Lambda^p \otimes \mathbb{C} \to \Lambda^{2m-p} \otimes \mathbb{C}. \tag{26.20}$$

Proposition 26.3. *We have*

$$* : \Lambda^{p,q} \to \Lambda^{n-q,n-p}. \tag{26.21}$$

Proof. This is easily seen to hold on $\mathbb{C}^n$, therefore it holds any any point of a Hermitian manifold (it is not a differential operator). □

Therefore the operator

$$\bar{*} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}, \tag{26.22}$$

defined by

$$\bar{*}\alpha = \overline{*\alpha} \tag{26.23}$$

is a $\mathbb{C}$-antilinear mapping and satisfies

$$\alpha \wedge \bar{*}\beta = g_{\Lambda^p}(\alpha, \overline{\beta})dV_g. \tag{26.24}$$

for $\alpha, \beta \in \Lambda^p \otimes \mathbb{C}$.

Proposition 26.4. *The L^2-adjoints of $d, \overline{\partial}, \overline{\partial}^*$ are given by*

$$d^* = -\bar{*}\ d\ \bar{*} \tag{26.25}$$
$$\partial^* = -\bar{*}\ \partial\ \bar{*} \tag{26.26}$$
$$\overline{\partial}^* = -\bar{*}\ \overline{\partial}\ \bar{*}, \tag{26.27}$$

Proof. The dimension of an almost complex manifold is even, so know that $d^* = - * d *$. Taking a conjugate of this equation yields the first formula. Apply the first formula to $d = \partial + \overline{\partial}$, we have

$$\partial^* + \overline{\partial}^* = d^* = -\bar{*}\ d\ \bar{*} = -\bar{*}\ \partial\ \bar{*} - \bar{*}\ \overline{\partial}\ \bar{*} \tag{26.28}$$

Considering the degrees of the operators on the right hand side yields the last 2 formulas. □

Corollary 26.5. *On a Hermitian manifold, we have*

$$\Delta_{\overline{\partial}}\bar{*} = \bar{*}\Delta_{\overline{\partial}} \tag{26.29}$$

Proof. We compute on $\Lambda^k_{\mathbb{C}}$,

$$\Delta_{\overline{\partial}}\bar{*} = (\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*)\bar{*} = (-\bar{*}\overline{\partial}\bar{*}\overline{\partial} - \overline{\partial}\bar{*}\overline{\partial}\bar{*})\bar{*} = -\bar{*}\overline{\partial}\bar{*}\overline{\partial}\bar{*} + (-1)^{k+1}\overline{\partial}\bar{*}\overline{\partial} \tag{26.30}$$

On the other hand,

$$\bar{*}\Delta_{\overline{\partial}} = \bar{*}(-\bar{*}\overline{\partial}\bar{*}\overline{\partial} - \overline{\partial}\bar{*}\overline{\partial}\bar{*}) = (-1)^{k+1}\overline{\partial}\bar{*}\overline{\partial} - \bar{*}\overline{\partial}\bar{*}\overline{\partial}\bar{*}. \tag{26.31}$$

□

27 Lecture 27

27.1 Serre duality

Letting

$$\mathbb{H}^{p,q}(M, g) = \{\alpha \in \Lambda^{p,q} | \Delta_{\overline{\partial}}\alpha = 0\}, \tag{27.1}$$

Hodge theory tells us that

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathbb{H}^{p,q}(M,g), \tag{27.2}$$

is finite-dimensional, and that

$$\begin{aligned} \Lambda^{p,q} &= \mathbb{H}^{p,q}(M,g) \oplus Im(\Delta_{\overline{\partial}}) && (27.3)\\ &= \mathbb{H}^{p,q}(M,g) \oplus Im(\overline{\partial}) \oplus Im(\overline{\partial}^*), && (27.4) \end{aligned}$$

with this being an orthogonal direct sum in L^2.

Corollary 27.1. *Let (M,J) be a compact complex manifold of complex dimension n. Then*

$$H^{p,q}_{\overline{\partial}}(M) \cong (H^{n-p,n-q}_{\overline{\partial}}(M))^*, \tag{27.5}$$

and therefore

$$b^{p,q}(M) = b^{n-p,n-q}(M) \tag{27.6}$$

Proof. From Corollary 26.5, the mapping $\overline{*}$ preserves the space of harmonic forms, and is invertible. The result then follows from Hodge theory. The dual appears since the operator $\overline{*}$ is $\mathbb{C}$-antilinear. ☐

27.2 The Laplacian on a Kähler manifold

Let L denote the mapping

$$L : \Lambda^{p,q} \to \Lambda^{p+1,q+1} \tag{27.7}$$

given by $L(\alpha) = \omega \wedge \alpha$, where ω is the Kähler form. Define

$$\Lambda \equiv L^* : \Lambda^{p,q} \to \Lambda^{p-1,q-1}. \tag{27.8}$$

Proposition 27.2. *If $(M, J.g)$ is Kähler then*

$$\begin{aligned} &[\Lambda, \partial] = i\overline{\partial}^*,\ [\Lambda, \overline{\partial}] = -i\partial^*,\ [\Lambda, d] = -(d^c)^* && (27.9)\\ &[L, \partial^*] = i\overline{\partial},\ [L, \overline{\partial}^*] = -i\partial,\ [L, d^*] = -d^c. && (27.10) \end{aligned}$$

Proof. Note that the second identity is the conjugate of the first. Therefore, if the first identity is true,

$$[\Lambda, d] = [\Lambda, \partial + \overline{\partial}] = [\Lambda, \partial] + [\Lambda, \overline{\partial}] = i\overline{\partial}^* - i\partial^* = (-i(\overline{\partial} - \partial))^* = -(d^c)^*, \tag{27.11}$$

then the third identity follows. The last three identities are just the adjoints of the first three.

So to prove all of these identities, we only need to prove the first. To prove the first identity, one proves this for $\mathbb{C}^n$ with the standard Kähler form. The proof is a 2 page calculation, and is left as an exercise. Then for an arbitrary Kähler manifold, the identity follows by using Kähler normal coordinates at any point, and the fact that the identity only depends on the metric and its first derivatives at the point. ☐

On a Kähler manifold, we have the following very special occurrence.

Proposition 27.3. *For $\alpha \in \Gamma(\Lambda^{p,q})$, if (M, J, g) is Kähler, then*

$$\Delta_H \alpha = 2\Delta_\partial \alpha = 2\Delta_{\overline{\partial}} \alpha. \tag{27.12}$$

Proof. We first show that

$$\Delta_H = \Delta_\partial + \Delta_{\overline{\partial}}. \tag{27.13}$$

To see this

$$\begin{aligned} \Delta_H = dd^* + d^*d = &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\ &= \partial\partial^* + \partial^*\partial + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} + \partial\overline{\partial}^* + \overline{\partial}^*\partial + \overline{\partial}\partial^* + \partial^*\overline{\partial} \\ &= \Delta_\partial + \Delta_{\overline{\partial}} + \partial\overline{\partial}^* + \overline{\partial}^*\partial + \overline{\partial}\partial^* + \partial^*\overline{\partial}. \end{aligned} \tag{27.14}$$

Using Proposition 27.2,

$$\begin{aligned} i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) &= \partial[\Lambda, \partial] + [\Lambda, \partial]\partial \\ &= \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial \\ &= \partial\Lambda\partial - \partial\Lambda\partial = 0. \end{aligned} \tag{27.15}$$

The sum of the last two terms in (27.14) also vanishes, just by taking the conjugate of the above computation, and (27.13) follows.

To finish the proof, we show that

$$\Delta_\partial = \Delta_{\overline{\partial}} \tag{27.16}$$

To see this, we again use Proposition 27.2, to compute

$$\begin{aligned} i\Delta_\partial = i\partial\partial^* + i\partial^*\partial &= \partial(-[\Lambda, \overline{\partial}]) - [\Lambda, \overline{\partial}]\partial \\ &= \partial\overline{\partial}\Lambda - \partial\Lambda\overline{\partial} - \Lambda\overline{\partial}\partial + \overline{\partial}\Lambda\partial. \end{aligned} \tag{27.17}$$

Also, we compute

$$\begin{aligned} i\Delta_{\overline{\partial}} = i\overline{\partial}\overline{\partial}^* + i\overline{\partial}^*\overline{\partial} &= \overline{\partial}([\Lambda, \partial]) + [\Lambda, \partial]\overline{\partial} \\ &= \overline{\partial}\Lambda\partial - \overline{\partial}\partial\Lambda + \Lambda\partial\overline{\partial} - \partial\Lambda\overline{\partial} \\ &= \overline{\partial}\Lambda\partial + \partial\overline{\partial}\Lambda - \Lambda\overline{\partial}\partial - \partial\Lambda\overline{\partial}, \end{aligned} \tag{27.18}$$

from which (27.16) follows.

∎

Using Hodge theory, we get the following structure on the cohomology of a Kähler manifold.

Proposition 27.4. *If (M, J, g) is a compact Kähler manifold, then*

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M), \tag{27.19}$$

and

$$H^{p,q}_{\overline{\partial}}(M) \cong H^{q,p}_{\overline{\partial}}(M)^*. \tag{27.20}$$

Consequently,

$$b^k(M) = \sum_{p+q=k} b^{p,q}(M) \tag{27.21}$$

$$b^{p,q}(M) = b^{q,p}(M). \tag{27.22}$$

Proof. This follows because if a harmonic k-form is decomposed as

$$\phi = \phi^{p,0} + \phi^{p-1,1} + \cdots + \phi^{1,p-1} + \phi^{0,p}, \tag{27.23}$$

then

$$0 = \Delta_H \phi = 2\Delta_{\overline{\partial}}\phi^{p,0} + 2\Delta_{\overline{\partial}}\phi^{p-1,1} + \cdots + 2\Delta_{\overline{\partial}}\phi^{1,p-1} + 2\Delta_{\overline{\partial}}\phi^{0,p}, \tag{27.24}$$

therefore

$$\Delta_{\overline{\partial}}\phi^{p-k,k} = 0, \tag{27.25}$$

for $k = 0 \dots p$.

Next,

$$\overline{\Delta_{\overline{\partial}}\phi} = \Delta_{\partial}\overline{\phi}, \tag{27.26}$$

so conjugation sends harmonic forms to harmonic forms. □

This yields a topologicial obstruction for a complex manifold to admit a Kähler metric:

Corollary 27.5. *If (M, J, g) is a compact Kähler manifold, then the odd Betti numbers of M are even.*

Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$

$$(z_1, z_2) \to 2^k(z_1, z_2). \tag{27.27}$$

This is a free and properly discontinuous action, so the quotient $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ is a manifold, which is called a primary Hopf surface. A primary Hopf surface is diffeomorphic to $S^1 \times S^3$, which has $b^1 = 1$, therefore it does not admit any Kähler metric.

27.3 Lefschetz decomposition

We will not prove this completely here, but just motivate by the following brief discussion.

Proposition 27.6. *On a Kähler manifold, we have*

$$[L, \Delta_H] = 0, \ [\Lambda, \Delta_H] = 0. \tag{27.28}$$

Proof. Since Δ_H is self-adjoint, these identities are equivalent. Next, we have

$$[L, d] = 0. \tag{27.29}$$

To see this, for any α,

$$d(L\alpha) = d(\omega \wedge \alpha) = \omega \wedge d\alpha = L(d\alpha), \tag{27.30}$$

since the Kähler form ω is closed. By taking adjoints, we have

$$[\Lambda, d^*] = 0. \tag{27.31}$$

Then we use Proposition 27.2 to compute

$$\begin{aligned} \Lambda\Delta_H &= \Lambda dd^* + \Lambda d^* d \\ &= d\Lambda d^* - (d^c)^* d^* + d^*\Lambda d \\ &= dd^*\Lambda - (d^c)^* d^* + d^*(d\Lambda - (d^c)^*) \\ &= \Delta_H \Lambda - (d^c d + dd^c)^*. \end{aligned} \tag{27.32}$$

But the operators d and d^c anti-commute, so we are done. $\square$

This proposition implies that the operators L and Λ map harmonic forms to harmonic forms. This yields an extra decomposition on cohomology called the Lefschetz decomposition, which we do not have time to discuss further here.

27.4 The Hodge diamond

The following picture is called the Hodge diamond:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & & \vdots & & & \\ h^{n,0} & \cdots & & \vdots & & \cdots & h^{0,n}. \\ & & & \vdots & & & \\ & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & \\ & & h^{n,n-1} & & h^{n-1,n} & & \\ & & & h^{n,n} & & & \end{array} \tag{27.33}$$

Reflection about the center vertical is conjugation. Reflection about the center horizontal is Hodge star. The composition of these two operations, or rotation by π, is Serre duality.

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