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Curvature Effect in Shear Flow: Slowdown of Turbulent Flame Speeds with Markstein Number

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Abstract: It is well-known in the combustion community that curvature effect in general slows down flame propagation speeds because it smooths out wrinkled flames. However, such a folklore has never been justified rigorously. In this paper, as the first theoretical result in this direction, we prove that the turbulent flame speed (an effective burning velocity) is decreasing with respect to the curvature diffusivity (Markstein number) for shear flows in the well-known G-equation model. Our proof involves several novel and rather sophisticated inequalities arising from the nonlinear structure of the equation. On a related fundamental issue, we solve the selection problem of weak solutions or find the "physical fluctuations" when the Markstein number goes to zero and solutions approach those of the inviscid G-equation model. The limiting solution is given by a closed form analytical formula.

1. Introduction

The curvature effect in turbulent combustion was first studied by Markstein [12], which says that if the flame front bends toward the cold region (unburned area, point C in Fig. 1 below), the flame propagation slows down. If the flame front bends toward the hot spot (burned area, point B in Fig. 1), it burns faster.

Below is an empirical linear relation proposed by Markstein [12] to approximate the dependence of the laminar flame speed s_l on the curvature (see also [13,15,16,18], etc):

$$s_l = s_l^0(1 - \tilde{d}\,\kappa). \tag{1.1}$$

Here s_l^0, the mean value, is a positive constant. The parameter $\tilde{d} > 0$ is the so called Markstein length, which is proportional to the flame thickness. The mean curvature along the flame front is κ.

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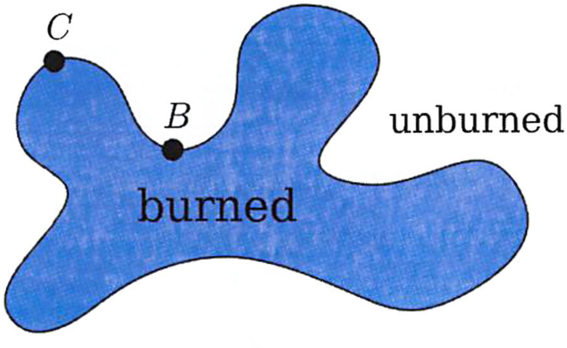

Fig. 1. Curvature effect

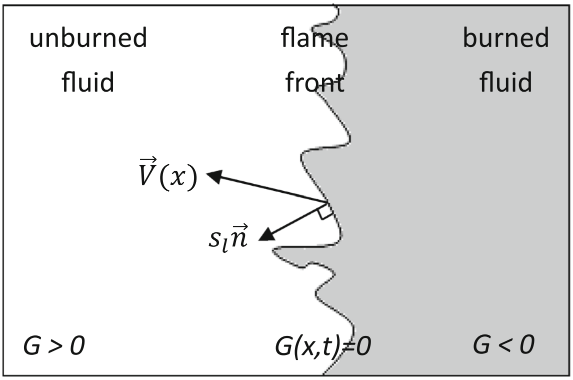

Fig. 2. Level-set formulation of front propagation

In general, κ changes sign along a curved flame front. So a mathematically interesting and physically important question is:

Q1: *How does the "averaged" flame propagation speed depend on the curvature term?*

Of course, we first need to properly define an "averaged speed", which is basically to average fluctuations caused by both the flow and the curvature. The theory of homogenization provides such a rigorous mathematical framework in environments with microscopic structures. In this paper, we employ the popular G-equation model in the combustion community.

Let the flame front be the zero level set of a reference function $G(x,t)$, where the burnt and unburnt regions are $\{G(x,t) < 0\}$ and $\{G(x,t) > 0\}$, respectively (see Fig. 2). The velocity of ambient fluid $V : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be smooth, $\mathbb{Z}^n$-periodic and incompressible (i.e. $div(V) = 0$). The propagation of flame front obeys a simple motion law: $\vec{v}_n = s_l + V(x) \cdot n$, i.e., the normal velocity is the laminar flame speed (s_l) plus the projection of V along the normal direction. This leads to the so-called G-equation, a level-set PDE [14,16]:

$$G_t + V(x) \cdot DG + s_l\,|DG| = 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty).$$

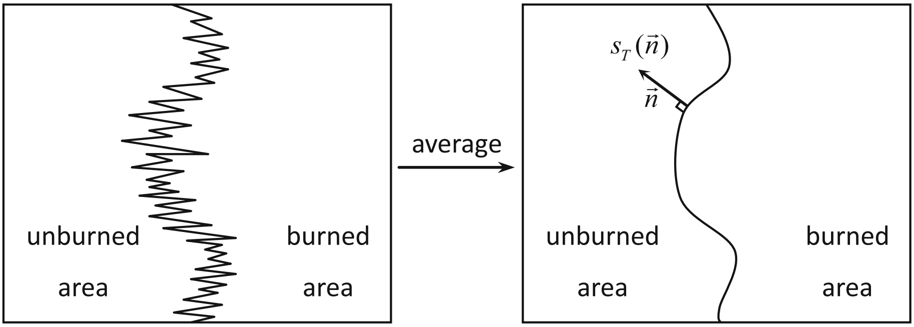

Fig. 3. Average of fluctuations in the homogenization limit

Plugging the expression of the laminar flame speed (1.1) into the G-equation and normalizing the constant $s_l^0 = 1$, we obtain a mean curvature type equation

$$G_t + V(x) \cdot DG + |DG| - \tilde{d}\,|DG|\,\mathrm{div}\left(\frac{DG}{|DG|}\right) = 0. \tag{1.2}$$

Turbulent combustion usually involves small scales. As a simplified model, we rescale V as $V = V(\frac{x}{\epsilon})$ and write $\tilde{d} = d\epsilon$. Here ϵ denotes the Kolmogorov scale (the small scale in the flow). The diffusivity constant $d > 0$ is called the Markstein number. We would like to point out that the dimensionless Markstein number is $d \cdot \frac{\delta_L}{\epsilon}$ with δ_L denoting the flame thickness [16]. In the thin reaction zone regime, $\delta_L = O(\epsilon)$, see Eq. (2.28) and Fig. 2.8 of [16]. Without loss of generality, let $\frac{\delta_L}{\epsilon} = 1$. Then (1.2) becomes

$$G_t^\epsilon + V(\frac{x}{\epsilon}) \cdot DG^\epsilon + |DG^\epsilon| - d\,\epsilon\,|DG^\epsilon|\,\mathrm{div}\left(\frac{DG^\epsilon}{|DG^\epsilon|}\right) = 0. \tag{1.3}$$

Since $\epsilon \ll 1$, it is natural to look at $\lim_{\epsilon \to 0} G^\epsilon$, i.e., the homogenization limit. If for any $p \in \mathbb{R}^n$, there exists a unique number $\overline{H}_d(p)$ such that the following cell problem has (approximate) $\mathbb{Z}^n$-periodic viscosity solutions in $\mathbb{R}^n$:

$$-d\,|p + Dw|\,\mathrm{div}\left(\frac{p + Dw}{|p + Dw|}\right) + |p + Dw| + V(y) \cdot (p + Dw) = \overline{H}_d(p), \tag{1.4}$$

then standard tools in the homogenization theory imply that

$$\lim_{\epsilon \to 0} G^\epsilon(x, t) = \bar{G}(x, t) \quad \text{locally uniformly in } \mathbb{R} \times [0, +\infty).$$

Here $\bar{G}$ is the unique solution to the following effective equation, which captures the propagation of the mean flame front (see Fig. 3 below).

$$\begin{cases} \bar{G}_t + \overline{H}_d(D\bar{G}) = 0 \\ \bar{G}(x, 0) = G_0(x) \quad \text{initial flame front.} \end{cases} \tag{1.5}$$

Solution to the cell problem (1.4) formally describes fluctuations around the mean flame front, i.e.,

$$G(x, t) = \bar{G}(x, t) + \epsilon w(x, \frac{x}{\epsilon}) + O(\epsilon^2),$$

where for fixed location-time (x, t) and $p = D\bar{G}(x, t)$, $w(x, \cdot)$ is a solution to (1.4) with mean zero, i.e., $\int_0^1 w(x, y)\, dy = 0$. The quantity $\overline{H}_d(p)$, if it exists, can be viewed as the turbulent flame speed ($s_T(p)$) along a given direction p. There is a consensus in combustion literature that the curvature effect slows down flame propagation [17]. Heuristically, this is because the curvature term smooths out the flame front and reduces the total area of chemical reaction [18]. However, this folklore has never been rigorously justified mathematically. If the curvature term is replaced by the full diffusion (i.e., the Laplacian Δ), a dramatic slow-down is proved in [10] for two dimensional cellular flows. So in the G-equation setting, **Question 1** can be formulated as

Q2: *How does $\overline{H}_d(p)$ depend on the Markstein number d? In particular, is it decreasing with respect to d?*

We remark that the decrease of turbulent flame speed with respect to the Markstein number has been experimentally observed (e.g., [5]).

1.1. Slow-down of flame propagation. For general V, we do not even know the existence of $\overline{H}_d(p)$, i.e., the well-posedness of (1.4). In fact, given the counter-example in [3] for a coercive mean curvature type equation, the cell problem (1.4) and the homogenization in our non-coercive setting is very likely not well-posed in general. To avoid this existence issue, as the first step to investigate the above **Question 2**, we consider the shear flow in this paper:

$$V(x) = (v(x_2), 0) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

Here $v : \mathbb{R} \to \mathbb{R}$ is a smooth periodic function. Then for $p = (\gamma, \mu) \in \mathbb{R}^2$, the cell problem (1.4) is reduced to the following ODE:

$$-\frac{d\gamma^2 w''}{\gamma^2 + (\mu + w')^2} + \sqrt{\gamma^2 + (\mu + w')^2} + \gamma v(y) = \overline{H}_d(p) \quad \text{in } \mathbb{R}. \tag{1.6}$$

It is then easy to show that there exists a unique number $\overline{H}_d(p)$ such that the ODE (1.6) has a C^2 periodic solution. Throughout this paper, we denote w as the unique solution satisfying that $w(0) = 0$. To simplify notations, we omit the dependence of w on d. The following is our main result.

Theorem 1.1. *Assume that $v = v(y)$ is not a constant function. Then*

(1) $\overline{H}_d(0, \pm\mu) = |\mu|$;

(2) **(Major Part).** *If $\gamma \neq 0$,*

$$\frac{\partial \overline{H}_d(p)}{\partial d} < 0.$$

So $\overline{H}_d$ is strictly decreasing with respect to the Markstein number $d > 0$.

(3) $\lim_{d\to 0^+} \overline{H}_d = \overline{H}_0$. *Here $\overline{H}_0(p)$ is the unique number (effective Hamiltonian) such that the following inviscid equation admits periodic viscosity solutions*

$$\sqrt{\gamma^2 + (\mu + w_0')^2} + \gamma v(y) = \overline{H}_0(p) \quad \textit{in } \mathbb{R}.$$

(4) $\lim_{d\to+\infty} \overline{H}_d = |p| + \gamma \int_0^1 v(y)\, dy$ *and* $\lim_{d\to+\infty} w = 0$ *uniformly in* $\mathbb{R}$.

Proofs for (1), (3) and (4) are simple. The real challenge is to prove the major part (2). A key step in our proof is to establish a highly sophisticated class of inequalities, see Lemma 2.3 (the discrete version) and Theorem 2.1 (a specific continuous version). Some calculations in high dimensions will be presented in Sect. 2.2 when the ambient fluid is near rest.

It might be tempting to think that there exists an explicit formula of $\overline{H}_d(p)$ since (1.6) is "just" an ODE. However, this is not the case. For example, let us look at a simpler cell problem associated with the 1-d viscous Hamilton–Jacobi equation arising from large deviations and quantum mechanics:

$$-d\,w'' + |p + w'|^2 + G(y) = \overline{H}(p, d) \quad \text{in } \mathbb{R}.$$

Here the potential G is a smooth periodic function and $\overline{H}(p, d)$ is the unique number such that the above equation has C^2 solutions. The viscous effective Hamiltonian $\overline{H}(p, d)$ actually determines the spectrum of the 1-d Schrödinger operator ($Lu = -du'' + Gu$) and it is closely related to the inverse scattering solution of the KdV equation [11]. We want to remark that the strict decreasing of $\overline{H}(p, d)$ with respect to d can be easily established in any dimension. See (2.13) in Remark 2.1.

1.2. Selection of physical fluctuations as $d \to 0^+$. To have a more complete picture, it is also interesting to ask what is the limit of solutions of (1.6) as $d \to 0^+$ (the vanishing curvature limit). When $d = 0$, equation (1.3) becomes the inviscid G-equation

$$G_t^\epsilon + V(\frac{x}{\epsilon}) \cdot DG^\epsilon + |DG^\epsilon| = 0.$$

It is proved in [4, 19] independently that there exists a unique $\overline{H}_0(p)$ such that the corresponding cell problem

$$|p + Dw| + V(y) \cdot (p + Dw) = \overline{H}_0(p) \quad \text{in } \mathbb{R}^n \tag{1.7}$$

admits a periodic (approximate) viscosity solution. This implies that

$$\lim_{\epsilon \to 0} G^\epsilon(x, t) = \bar{G}(x, t) \quad \text{locally uniformly in } \mathbb{R} \times [0, +\infty).$$

As in the curvature case, here $\bar{G}$ is the unique solution to the following effective equation, which captures the propagation of the mean flame front:

$$\begin{cases} \bar{G}_t + \overline{H}_0(D\bar{G}) = 0 \\ \bar{G}(x, 0) = G_0(x) \quad \text{initial flame front.} \end{cases}$$

The formal two-scale expansion says that

$$G_\epsilon(x, t) = \bar{G}(x, t) + \epsilon w(x, \frac{x}{\epsilon}) + O(\epsilon^2),$$

where the fluctuation $w(x, \cdot)$ is a solution to (1.7) with $p = D\bar{G}(x, t)$ for fixed (x, t). Nevertheless, solutions to (1.7) are in general not unique even up to a constant. This motivates

Q3: *which solution to* (1.7) *is the physical solution that captures the fluctuation of flame front?*

One natural approach is to look at the limit of solutions to (1.4) (if it exists uniquely) as $d \to 0$. The limit is, however, very challenging and unknown in general. In this paper, we identify the limit for the Eq. (1.6) under some non-degeneracy conditions.

It is easy to show that as $d \to 0^+$, the solution w to (1.6), up to a subsequence, converges to a periodic viscosity solution w_0 of

$$\sqrt{\gamma^2 + (\mu + w_0')^2} + \gamma v(y) = \overline{H}_0(p) \quad \text{in } \mathbb{R}. \tag{1.8}$$

When $\gamma = 0$, $w = w_0 \equiv 0$. Without loss of generality, we set $\gamma = 1$ in this section and denote

$$\overline{H}_0(\mu) = \overline{H}_0(p).$$

Without loss of generality, in this section, we also assume that

$$\max_{\mathbb{R}} v = 0.$$

1.2.1. Uniqueness case. If $|\mu| \geq \int_0^1 \sqrt{(1-v)^2 - 1}\, dy$, $\overline{H}(\mu) \geq 1$ is the unique number such that

$$|\mu| = \int_0^1 \sqrt{(\overline{H}(\mu) - v(y))^2 - 1}\, dy.$$

Also, the inviscid Eq. (1.8) has a unique solution up to a constant, i.e.,

$$w_0(x) = (sign(\mu)) \int_0^x \sqrt{(\overline{H}(\mu) - v(y))^2 - 1}\, dy - \mu x + c$$

for some $c \in \mathbb{R}$ since $w_0' + \mu$ cannot change signs. Accordingly, by $w(0) = 0$,

$$\lim_{d \to 0^+} w = (sign(\mu)) \int_0^x \sqrt{(\overline{H}(\mu) - v(y))^2 - 1}\, dy - \mu x.$$

1.2.2. Non-uniqueness case. When $|\mu| < \int_0^1 \sqrt{(1-v)^2 - 1}\, dy$, $\overline{H}_d(\mu) = 1$. The limiting problem is more interesting since solutions to the inviscid Eq. (1.8) are not unique if the set

$$\mathcal{M}_0 = \{x \in [0, 1) \mid v(x) = \max_{\mathbb{R}} v = 0\}$$

has multiple points. For example, assume that $x_i \in \mathcal{M}_0$ for $i = 1, 2$. Choose $x_{\mu,i} \in (x_i, x_i + 1)$ such that

$$\int_{x_i}^{x_{\mu,i}} \sqrt{(1-v)^2 - 1}\, dy - \int_{x_{\mu,i}}^{x_i+1} \sqrt{(1-v)^2 - 1}\, dy = \mu.$$

Then

$$w_i(x) = \begin{cases} \int_{x_i}^x \sqrt{(1 - v(y))^2 - 1}\, dy - \mu x, \ \forall x \in [x_i, x_{\mu,i}] \\ \int_{x_i}^{x_{\mu,i}} \sqrt{(1 - v(y))^2 - 1}\, dy - \int_{x_{\mu,i}}^x \sqrt{(1 - v(y))^2 - 1}\, dy - \mu x, \\ \forall x \in [x_{\mu,i}, x_i + 1] \end{cases}$$

(extended periodically) are both viscosity solutions to (1.8) and $w_1 - w_2$ is not a constant. So a very interesting problem is to identify the solution selected by the limiting process, i.e., the physical fluctuation associated with the inviscid G-equation model. Hereafter, we assume that

$$\mathcal{M}_0 \text{ is finite and } v''(x) \text{ is distinct for } x \in \mathcal{M}_0. \tag{1.9}$$

Choose the unique $\bar{x} \in \mathcal{M}_0$ such that

$$-v''(\bar{x}) = \min_{x \in \mathcal{M}_0} \{-v''(x)\}.$$

Choose $x_\mu \in (\bar{x}, \bar{x}+1)$ such that

$$\int_{\bar{x}}^{x_\mu} \sqrt{(1-v)^2 - 1}\, dy - \int_{x_\mu}^{\bar{x}+1} \sqrt{(1-v)^2 - 1}\, dy = \mu.$$

Clearly, such x_μ is unique. The following is our selection result.

Theorem 1.2.

$$\lim_{d \to 0^+} w = w_0(x) - w_0(0) \quad \textit{uniformly in } \mathbb{R}.$$

Here

$$w_0(x) = \begin{cases} \int_{\bar{x}}^{x} \sqrt{(1-v)^2 - 1}\, dy - \mu x, \ \forall x \in [\bar{x}, x_\mu] \\ \int_{\bar{x}}^{x_\mu} \sqrt{(1-v)^2 - 1}\, dy - \int_{x_\mu}^{x} \sqrt{(1-v)^2 - 1}\, dy - \mu x, \\ \forall x \in [x_\mu, \bar{x}+1]. \end{cases} \tag{1.10}$$

We would like to point out that selection problems of similar spirit have been studied for the vanishing viscosity limit ([1,2,7], etc), after which the viscosity solution was originally named. In these references, the authors aim to identify $\lim_{\epsilon \to 0^+} v_\epsilon$. Here v_ϵ is the unique smooth solution to

$$-\epsilon \Delta v_\epsilon + H(p + Dv_\epsilon, x) = \overline{H}(p, \epsilon) \quad \text{in } \mathbb{R}^n.$$

The most important case is the mechanical Hamiltonian $H(p, x) = |p|^2 + G(x)$ with a potential function G. The limiting process resembles the passage from quantum mechanics to classical mechanics ([1,6]). The works [1,2] deal with some special cases in high dimensions by employing advanced tools from dynamical systems and random perturbations. Assumptions therein are very hard to check, however. The method in [7] is purely 1-d. Based on simple comparison principles of PDEs/ODEs, our arguments are simpler and more robust. In particular, they can be easily extended to handle certain cases in high dimensions. The rest of the paper contains the proofs of the main theorems.

2. Proof of Theorem 1.1

Proof. (1) is trivial. Let us prove (2) which is the most difficult and interesting part. Fix (γ, μ). Denote $\phi = \frac{\mu + w'}{\gamma}$. Then ϕ is the unique periodic solution to

$$-\frac{d\phi'}{1+\phi^2} + \sqrt{1+\phi^2} + v(y) = E(d) = \frac{\overline{H}_d(p)}{\gamma} \quad \text{in } \mathbb{R}$$

subject to $\int_0^1 \phi(x)\,dx = \frac{\mu}{\gamma}$. To prove (2) is equivalent to showing that

$$E'(d) < 0.$$

Due to the uniqueness of ϕ and $E(d)$, their dependence on d is smooth. Taking derivative on both sides of the above equation with respect to d, we obtain that

$$-dF' + b(x)F = E'(d)(1+\phi^2) + \phi',$$

where $b(x) = \frac{2d\phi'\phi}{1+\phi^2} + \phi\sqrt{1+\phi^2}$ and $F(x) = \phi_d(x)$, i.e., the derivative of ϕ with respect to d. Clearly, F is periodic and has zero mean, i.e., $\int_{[0,1]} F = 0$. Note that v is not constant is equivalent to saying the ϕ is not constant. Then (2) follows immediately from Lemma 2.1.

(3) Integrating both sides of (1.6), we obtain:

$$\overline{H}_d(p) = \int_0^1 \sqrt{\gamma^2 + (\mu + w')^2}\,dy + \gamma \int_0^1 v(y)\,dy. \tag{2.11}$$

So due to the convexity of $s(t) = \sqrt{\gamma^2 + t^2}$,

$$\overline{H}_d(p) \geq |p| + \gamma \int_0^1 v(y)\,dy.$$

Also, by maximum principle, we have that

$$\overline{H}_d(p) \leq |p| + \max_{\mathbb{R}} \gamma v$$

and

$$\max_{\mathbb{R}} |\mu + w'| \leq \overline{H}_d(p) - \min_{\mathbb{R}} \gamma v \leq |p| + 2\max_{\mathbb{R}} |\gamma v|.$$

Hence, up to a sequence, we may assume that

$$\lim_{d\to 0} \overline{H}_d = \overline{H}_0 \quad \text{and} \lim_{d\to 0^+} w = w_0 \quad \text{uniformly in } \mathbb{R}.$$

Then the stability of viscosity solution immediately implies that w_0 is a continuous periodic viscosity solution to

$$\sqrt{\gamma^2 + \left(\mu + w_0'\right)^2} + \gamma v(y) = \overline{H}_0(p) \quad \text{in } \mathbb{R}.$$

Note that $\overline{H}_0(p)$ is unique number such that the above equation has a periodic viscosity solutions w_0 although w_0 might not be unique. See [8] for general cases.

(4). If $\gamma = 0$, this is trivial. So we assume that $\gamma \neq 0$. Note that estimates of $\overline{H}_d$ and $\mu + w'$ in (3) are independent of d. Since

$$w'' = \frac{1}{d\gamma^2}(\gamma^2 + (\mu + w')^2)\left(\sqrt{\gamma^2 + (\mu + w')^2} + v - \overline{H}_d(\mu)\right),$$

we have that

$$\max_{\mathbb{R}} |w''| \leq \frac{C}{d}$$

for a constant C independent of d. Due to the periodicity of w and $w(0) = 0$, it is obvious that

$$\lim_{d \to +\infty} w = \lim_{d \to +\infty} w' = 0 \quad \text{uniformly in } \mathbb{R}.$$

Combining with (2.11), (4) holds. □

Lemma 2.1. *Let $d > 0$ and ϕ be a non-constant C^1 periodic function. If the following equation has a mean-zero, periodic solution F*

$$-dF' + b(x)F = \phi' + \alpha(1 + \phi^2) \quad \textit{in } \mathbb{R}$$

for some $\alpha \in \mathbb{R}$ and

$$b(x) = \frac{2d\phi'\phi}{1 + \phi^2} + \phi\sqrt{1 + \phi^2},$$

then

$$\alpha < 0.$$

Proof. It suffices to prove this for $d = 1$. The proof for other d is similar. We can solve F in terms of ϕ and α. Using F is periodic and mean zero (i.e., $F(0) = F(1)$ and $\int_0^1 F(s)\,ds = 0$), it is easy to obtain that

$$\alpha = -\frac{e^{g(1)} \int_0^1 \phi' e^{-g(x)}\,dx \int_0^1 e^{g(x)}\,dx - (e^{g(1)} - 1) \int_0^1 e^{g(x)} \int_0^x \phi' e^{-g(y)}\,dydx}{e^{g(1)} \int_0^1 (1 + \phi^2) e^{-g(x)}\,dx \int_0^1 e^{g(x)}\,dx - (e^{g(1)} - 1) \int_0^1 e^{g(x)} \int_0^x (1 + \phi^2) e^{-g(y)}\,dydx}.$$

Here

$$g(x) = \int_0^x b(y)\,dy = \log(1 + \phi^2(x)) - \log(1 + \phi^2(0)) + \int_0^x \phi\sqrt{1 + \phi^2}\,dx.$$

In particular, $g(1) = \int_0^1 \phi\sqrt{1 + \phi^2}\,dx$. The denominator is obviously positive. Hence $\alpha < 0$ is equivalent to proving the inequality

$$e^{g(1)} \int_0^1 \phi' e^{-g(x)}\,dx \int_0^1 e^{g(x)}\,dx > (e^{g(1)} - 1) \int_0^1 e^{g(x)} \int_0^x \phi' e^{-g(y)}\,dydx$$

for every non-constant C^1 periodic function ϕ. Denote that

$$h(x) = \int_0^x \phi\sqrt{1 + \phi^2}\,dy.$$

Then it is equivalent to showing that

$$e^{h(1)} \int_0^1 \frac{\phi'}{1+\phi^2} e^{-h(x)}\, dx \int_0^1 (1+\phi^2) e^{h(x)}\, dx$$
$$> (e^{h(1)} - 1) \int_0^1 (1+\phi^2) e^{h(x)} \int_0^x \frac{\phi'}{1+\phi^2} e^{-h(y)}\, dy$$

Write $\lambda(\phi) = \arctan \phi$. Using integration by parts and $\phi(0) = \phi(1)$, we have that

$$LHS = e^{h(1)} \left(\lambda(\phi(1)) e^{-h(1)} - \lambda(\phi(1)) + \int_0^1 \lambda(\phi) e^{-h(x)} \phi \sqrt{1+\phi^2}\, dx \right) \int_0^1 (1+\phi^2) e^{h(x)}\, dx.$$

and the RHS is

$$RHS = (e^{h(1)} - 1) \left(\int_0^1 \lambda(\phi)(1+\phi^2)\, dx - \lambda(\phi(1)) \int_0^1 (1+\phi^2) e^{h(x)}\, dx \right)$$
$$+ (e^{h(1)} - 1) \left(\int_0^1 (1+\phi^2) e^{h(x)} \int_0^x \lambda(\phi) e^{-h(y)} \phi \sqrt{1+\phi^2}\, dy dx \right).$$

By Fubini Theorem,

$$\int_0^1 (1+\phi^2) e^{h(x)} \int_0^x \lambda(\phi) e^{-h(y)} \phi \sqrt{1+\phi^2}\, dy dx$$
$$= \int_0^1 \lambda(\phi) e^{-h(x)} \phi \sqrt{1+\phi^2} \int_x^1 (1+\phi^2) e^{h(y)}\, dy dx.$$

Then $LHS{-}RHS$ is $A + B - C$ for

$$A(\phi) = e^{h(1)} \int_0^1 \lambda(\phi) e^{-h(x)} \phi \sqrt{1+\phi^2} \int_0^x (1+\phi^2) e^{h(y)}\, dy dx,$$
$$B(\phi) = \int_0^1 \lambda(\phi) e^{-h(x)} \phi \sqrt{1+\phi^2} \int_x^1 (1+\phi^2) e^{h(y)}\, dy dx.$$

and

$$C(\phi) = (e^{h(1)} - 1) \int_0^1 \lambda(\phi)(1+\phi^2)\, dx.$$

If $h(1) = 0$, then $A + B - C = A + B \geq 0$ since $s\lambda(s) \geq 0$. Cleary, "$= 0$" if and only if $\phi \equiv 0$. So we assume that

$$h(1) \neq 0.$$

Also, note that for $\tilde{\phi}(x) = -\phi(-x)$, the correspsonding

$$\tilde{b}(x) = \frac{2\tilde{\phi}'\tilde{\phi}}{1+\tilde{\phi}^2} + \tilde{\phi}\sqrt{1+\tilde{\phi}^2} = -b(-x)$$

and $\tilde{F}(x) = -F(-x)$ satisfies that

$$-\tilde{F}' + \tilde{b}(x)\tilde{F} = \tilde{\phi}' + \alpha(1+\tilde{\phi}^2).$$

Hence, without lost of generality, we may further assume that

$$h(1) > 0.$$

Denote $\phi_+ = \max\{\phi, 0\}$ and $\phi_- = \min\{\phi, 0\}$. Aso write

$$h^{\pm}(x) = \int_0^x \phi_{\pm}\sqrt{1+\phi_{\pm}^2}\, dy.$$

Note that $h(x) = h^+ + h^-$. Now let us prove the following lemma. □

Lemma 2.2. *We have that*

$$A(\phi) + B(\phi) - C(\phi) \geq e^{h^-(1)}\left(A(\phi_+) + B(\phi_+) - C(\phi_+)\right).$$

The equality holds if only if $\phi \geq 0$, i.e., $\phi_- = 0$.

Proof. Clearly

$$\begin{aligned} A(\phi) &\geq e^{h(1)} \textstyle\int_0^1 \lambda(\phi_+) e^{-h(x)} \phi_+ \sqrt{1+\phi_+^2} \int_0^x (1+\phi_+^2) e^{h(y)}\, dydx \\ &= e^{h(1)} \textstyle\int_0^1 \lambda(\phi_+) e^{-h^+(x)} \phi_+ \sqrt{1+\phi_+^2} \int_0^x (1+\phi_+^2) e^{h^+(y)} e^{h^-(y)-h^-(x)}\, dydx \\ &\geq e^{h^-(1)} A(\phi_+), \quad \text{since } h^-(x) \leq h^-(y) \text{ for } x \geq y. \end{aligned}$$

Also,

$$\begin{aligned} B(\phi) &\geq \textstyle\int_0^1 \lambda(\phi_+) e^{-h(x)} \phi_+ \sqrt{1+\phi_+^2} \int_x^1 (1+\phi_+^2) e^{h(y)}\, dydx \\ &= e^{h^-(1)} \textstyle\int_0^1 \lambda(\phi_+) e^{-h^+(x)} \phi_+ \sqrt{1+\phi_+^2} \int_x^1 (1+\phi_+^2) e^{h^+(y)} e^{h^-(y)-h^-(1)} e^{-h^-(x)}\, dydx \\ &\geq e^{h^-(1)} B(\phi_+) \quad \text{since } 0 \geq h^-(y) \geq h^-(1) \text{ for all } y \in [0,1] \end{aligned}$$

and

$$\begin{aligned} C(\phi) &\leq (e^{h(1)} - 1) \textstyle\int_0^1 \lambda(\phi_+)(1+\phi_+^2)\, dx \\ &= \frac{(e^{h(1)}-1)}{(e^{h^+(1)}-1)} C(\phi_+) \\ &\leq e^{h^-(1)} C(\phi_+). \end{aligned}$$

Obviously, for all inequalities to hold, we must have $h^- \equiv 0$ and $\phi_- \equiv 0$. □

Now let us continue the proof of Lemma 2.1. Since $h(1) > 0$, that ϕ is not constant implies ϕ_+ is not constant either. By a small perturbation like $\phi_+ + \epsilon$, we may assume that $\phi_+ > 0$ in computations below. Then h^+ is strictly increasing. After changing of variables $h^+(x) \to x$ and writing $\psi(h^+(x)) = \phi_+(x)$ and $T = h^+(1)$, we obtain that

$$A(\phi_+) = A_{T,\psi} = e^T \int_0^T \lambda(\psi) e^{-x} \int_0^x \frac{\sqrt{1+\psi^2}}{\psi} e^y\, dydx,$$

$$B(\phi_+) = B_{T,\psi} = \int_0^T \lambda(\psi) e^{-x} \int_x^T \frac{\sqrt{1+\psi^2}}{\psi} e^y\, dydx$$

and

$$C(\phi_+) = C_{T,\psi} = (e^T - 1)\int_0^T \lambda(\psi)\frac{\sqrt{1+\psi^2}}{\psi}\,dx.$$

So

$$\begin{aligned} A_{T,\psi} + B_{T,\psi} - C_{T,\psi} = e^T &\int_0^T \lambda(\psi)e^{-x}\int_0^x \frac{\sqrt{1+\psi^2}}{\psi}e^y\,dydx \\ &+ \int_0^T \lambda(\psi)e^{-x}\int_x^T \frac{\sqrt{1+\psi^2}}{\psi}e^y\,dydx \\ &- (e^T-1)\int_0^T \lambda(\psi)\frac{\sqrt{1+\psi^2}}{\psi}\,dx. \end{aligned}$$

Let $M = \max_{[0,T]}\psi = \max_{[0,1]}\phi_+ > 0$. According to Theorem 2.1 by taking $f(x) = \lambda(\psi) = \arctan(\psi)$, $g(y) = \frac{1}{\sin y}$, $L = \arctan(M)$ and $\theta = \frac{1}{\sqrt{1+M^2}}$, we have that $\frac{\sqrt{1+\psi^2}}{\psi} = g(f)$ and

$$\begin{aligned} A_{T,\psi} + B_{T,\psi} - C_{T,\psi} &\geq \tfrac{1}{2\sqrt{1+M^2}}\textstyle\int_{[0,T]^2}|\lambda(\psi(x)) - \lambda(\psi(y))|^2\,dxdy \\ &= \tfrac{1}{2\sqrt{1+M^2}}\textstyle\int_{[0,1]^2}|\lambda(\phi_+(x)) - \lambda(\phi_+(y))|^2 J(x)J(y)\,dxdy \\ &> 0 \quad \text{since } \phi_+ \text{ is not constant.} \end{aligned}$$

Here $J(x) = \phi_+(x)\sqrt{1+\phi_+^2}$. Combining with Lemma 2.2, $A(\phi) + B(\phi) - C(\phi) > 0$. □

2.1. The key inequalities. Given $n \in \mathbb{N}$. Let $\{b_{ik}\}_{1\leq i,k\leq n}$ and $\{\tilde{b}_{ik}\}_{1\leq i,k\leq n}$ be two given sequences of positive numbers satisfying that for all i, k

$$\sum_{l=1}^{i} b_{il} + \sum_{l=i}^{n}\tilde{b}_{il} = \sum_{l=k}^{n} b_{lk} + \sum_{l=1}^{k}\tilde{b}_{lk} = c.$$

Here c is a constant independent of i and k. Also,

$$\min\{\min_{1\leq k\leq i\leq n} b_{ik}, \min_{1\leq i\leq k\leq n}\tilde{b}_{ik}\} \geq \tau > 0. \tag{2.12}$$

Lemma 2.3. *Assume that $L > 0$ and $g \in C((0, L])$ satisfies*

$$g'(a) \leq -\theta \quad \textit{for some } \theta \geq 0.$$

Then

$$\sum_{i=1}^{n} a_i \sum_{k=1}^{i} g(a_k)b_{ik} + \sum_{i=1}^{n} a_i \sum_{k=i}^{n} g(a_k)\tilde{b}_{ik} \geq c\sum_{i=1}^{n} a_i g(a_i) + \frac{\theta\tau}{2}\sum_{1\leq i,k\leq n}(a_i - a_k)^2.$$

for all $(a_1, a_2, \ldots, a_n) \in (0, L]^n$. Here τ is from (2.12). Moreover, if $\theta > 0$, the equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. By approximation, we may assume that $\theta > 0$. For convenience, denote

$$W(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} a_i \sum_{k=1}^{i} g(a_k) b_{ik} + \sum_{i=1}^{n} a_i \sum_{k=i}^{n} g(a_k) \tilde{b}_{ik}$$

and

$$H(a_1, a_2, \ldots, a_n) = c \sum_{i=1}^{n} a_i g(a_i) + \frac{\theta\tau}{2} \sum_{1 \le i,k \le n} (a_i - a_k)^2.$$

It suffices to show that for any fixed $r \in (0, L)$,

$$\min_{[r,L]^n} (W - H) = 0$$

and the minimum is attained when all a_i are the same.

Choose $(\hat{a}_1, \hat{a}_2, \hat{a}_3, ..\hat{a}_n) \in [r, L]^n$ such that

$$W(\hat{a}_1, \hat{a}_2, \hat{a}_3, ..\hat{a}_n) - H(\hat{a}_1, \hat{a}_2, \hat{a}_3, ..\hat{a}_n) = \min_{[r,L]^n} (W - H).$$

Assume that $\hat{a}_j = \max_{1 \le i \le n} \{\hat{a}_i\}$. If $\hat{a}_j = r$, then $\hat{a}_1 = \hat{a}_2 = \cdots = \hat{a}_n = r$ and we are done. So let us assume that

$$\hat{a}_j > r.$$

Then

$$W_{a_j} - H_{a_j} \le 0 \quad \text{at } (\hat{a}_1, \hat{a}_2, \hat{a}_3, ..\hat{a}_n).$$

Here we include < 0 since $\hat{a}_j$ might be equal to L. Accordingly,

$$\begin{aligned} &\textstyle\sum_{k=1}^{j} g(\hat{a}_k) b_{jk} + \sum_{k=j}^{n} g(\hat{a}_k) \tilde{b}_{jk} \\ &\quad + g'(\hat{a}_j) \textstyle\sum_{k=j}^{n} \hat{a}_k b_{kj} + g'(\hat{a}_j) \sum_{k=1}^{j} \hat{a}_k \tilde{b}_{kj} \\ &\le c(g(\hat{a}_j) + \hat{a}_j g'(\hat{a}_j)) + 2 \textstyle\sum_{k \ne j} \theta\tau(\hat{a}_j - \hat{a}_k) \end{aligned}$$

On the other hand, since $g' \le -\theta < 0$, we also have that

$$\begin{aligned} &\textstyle\sum_{k=1}^{j} g(\hat{a}_k) b_{jk} + \sum_{k=j}^{n} g(\hat{a}_k) \tilde{b}_{jk} \\ &\quad + g'(\hat{a}_j) \textstyle\sum_{k=j}^{n} \hat{a}_k b_{kj} + g'(\hat{a}_j) \sum_{k=1}^{j} \hat{a}_k \tilde{b}_{kj} \\ &\ge c(g(\hat{a}_j) + \hat{a}_j g'(\hat{a}_j)) + 2 \textstyle\sum_{k \ne j} \theta\tau(\hat{a}_j - \hat{a}_k). \end{aligned}$$

Hence all equalities should hold and $\hat{a}_1 = \hat{a}_2 \ldots = \hat{a}_n$ follows from that g is strictly decreasing. Then $W(\hat{a}_1, \hat{a}_2, \hat{a}_3, ..\hat{a}_n) - H(\hat{a}_1, \hat{a}_2, \hat{a}_3, ..\hat{a}_n) = 0$. □

Now we are ready to state a specific continuous version for our purpose.

Theorem 2.1. *Let $T > 0$ and $f \in C([0,T])$ be a continuous positive function. Suppose that $g \in C^1((0,L])$ for $L = \max_{[0,T]} f$.*
(1) If $g' \le -\theta$ for some $\theta \ge 0$, then

$$e^T \int_0^T f(x)e^{-x} \int_0^x g(f(y))e^y\,dydx + \int_0^T f(x)e^{-x} \int_x^T g(f(y))e^y\,dydx$$

$$\ge (e^T - 1) \int_0^T f(x)g(f(x))\,dx + \frac{\theta}{2} \int_{[0,T]^2} |f(x) - f(y)|^2\,dxdy.$$

(2) If If $g' \ge \theta$ for some $\theta \ge 0$, then

$$e^T \int_0^T f(x)e^{-x} \int_0^x g(f(y))e^y\,dydx + \int_0^T f(x)e^{-x} \int_x^T g(f(y))e^y\,dydx$$

$$\le (e^T - 1) \int_0^T f(x)g(f(x))\,dx - \frac{\theta}{2} \int_{[0,T]^2} |f(x) - f(y)|^2\,dxdy.$$

Proof. (1) For $n \in N$, let $x_i = \frac{iT}{n}$ for $i = 1, 2, .., n$. Note that for $i, k = 1, 2, 3, \ldots n$,

$$\sum_{l=1}^{i} e^{T - x_i + x_l} + \sum_{l=i}^{n} e^{x_l - x_i} = \frac{e^{T + \frac{T}{n}} - 1}{e^{\frac{T}{n}} - 1} = \sum_{l=k}^{n} e^{T - x_l + x_k} + \sum_{l=1}^{k} e^{x_k - x_l}.$$

Then desired inequality in (1) follows from Lemma 2.3 and Riemann sum approximation by taking $a_i = f(x_i)$, $c = \frac{e^{T+\frac{T}{n}} - 1}{e^{\frac{T}{n}} - 1}$, $\tau = 1$,

$$b_{ik} = e^{T - x_i + x_k} \quad \text{and} \quad \tilde{b}_{ik} = e^{x_k - x_i} \quad \text{for } 1 \le i, k \le n.$$

(2) follows immediately from (1) by considering $-g$. □

Remark 2.1. Similar to the proof of Theorem 1.1, (1) in the above Theorem 2.1 also implies that the one dimensional viscous effective Hamiltonian $\overline{H}(p,d)$ given by the cell problem

$$-d\,w'' + H(p + w') + G(x) = \overline{H}(p,d) \quad \text{in } \mathbb{R}$$

is strictly decreasing with respect to the diffusivity $d > 0$ for a non-constant function G, and a strictly convex function $H : \mathbb{R} \to \mathbb{R}$. Here we choose $f = p + w'$ and $g = \frac{1}{H'}$ after suitable translations. It remains an interesting problem whether this is also true in high dimensions. For the special case $H(p) = \frac{1}{2}|p|^2$, using integration by parts, it is easy to derive that

$$\frac{\partial \overline{H}(p,d)}{\partial d} = -\frac{\int_{\mathbb{T}^n} |Dw_d|^2 e^{-w_d}\,dx}{\int_{\mathbb{T}^n} e^{-w_d}\,dx} \le 0 \tag{2.13}$$

and "=" holds if and only if G is a constant. Here w_d represents the derivative of w with respect to d. On the other hand, if H is non-convex, then (2) in the above Theorem 2.1 implies that for some p, $\overline{H}(p,d)$ could be strictly increasing with respect to d.

2.2. Calculations in high dimensions in perturbative cases. Consider the case of weak flow or δV for $0 \le \delta \ll 1$. Let $p \in \mathbb{R}^n$ be a unit vector satisfying the Diophantine condition, i.e., there exist $\beta, C > 0$ such that

$$|p \cdot \vec{k}| \ge \frac{C}{|\vec{k}|^\beta} \quad \text{for all } \vec{k} \in \mathbb{Z}^n \backslash \{0\}.$$

Owing to [9], when δ is small enough, the cell problem (1.4) has a viscosity solution. Formally, we can write the solution as

$$w = \delta w_1 + \delta^2 w_2 + O(\delta^3)$$

and the constant (turbulent flame speed)

$$\overline{H}_d(p) = |p| + \delta\alpha_1(p) + \delta^2\alpha_2(p) + O(\delta^3). \tag{2.14}$$

By comparing coefficients of δ and δ^2, w_1 and w_2 are determined by inhomogeneous linear equations. They can be solved in terms of Fourier series. For example, w_1 satisfies

$$-d(\Delta w_1 - p \cdot D^2 w_1 \cdot p) + p \cdot Dw_1 + p \cdot V = \alpha_1(p).$$

The equation for w_2 is more messy. Applying Fredholm alternatives to both equations, we have that

$$\alpha_1(p) = p \cdot \int_{\mathbb{T}^n} V\, dx = p \cdot \lambda_0$$

and

$$\alpha_2(p) = \frac{1}{2}\int_{\mathbb{T}^n} |Dw_1|^2\, dx = \frac{1}{2}\sum_{\vec{k}\in\mathbb{Z}^n\backslash\{0\}} \frac{|p \cdot \lambda_{\vec{k}}|^2|\vec{k}|^2}{d^2(|\vec{k}|^2 - |p \cdot \vec{k}|^2)^2 4\pi^2 + |p \cdot \vec{k}|^2},$$

where $\lambda_{\vec{k}} \in \mathbb{C}^n$ are Fourier coefficients of V, i.e., $V = \sum_{\vec{k}\in\mathbb{Z}^n} \lambda_{\vec{k}} e^{i2\pi\vec{k}\cdot x}$. Clearly, $\overline{H}_d(p)$ is strictly decreasing with respect to d. The approximation of $\overline{H}_d(p)$ (2.14) can actually be proved easily through maximum principles of viscosity solutions, i.e., evaluating at where $w - \delta w_1 - \delta^2 w_2$ attains maximum/minimum values.

3. Proof of Theorem 1.2

Let us first prove some lemmas. Recall that

$$\mathcal{M}_0 = \{x \in [0, 1)|\ v(x) = \max_{\mathbb{R}} v = 0\}.$$

See Sect. 1.2.2 (non-uniqueness case) for the range of μ, defintions of $\bar{x}$ and x_μ and other assumptions like (1.9).

Lemma 3.1. *Assume that $\mathcal{M}_0 = \{\bar{x}\}$, i.e., it contains a single element. Then*

$$\lim_{d\to 0^+} \frac{\overline{H}_d(\mu) - 1}{d} = -\sqrt{-v''(\bar{x})}.$$

Proof. Since $\mathcal{M}_0$ has only one element, $1 - v > 1$ in $(\bar{x}, \bar{x}+1)$. Then it is easy to see that periodic viscosity solutions to

$$\sqrt{1 + (\mu + w_0^{'})^2} + v(y) = 1 \quad \text{in } \mathbb{R}$$

are unique up to a constant. Hence, since $w(0) = 0$,

$$\lim_{d \to 0^+} w = w_0(x) - w_0(0) \quad \text{uniformly in } \mathbb{R}. \tag{3.15}$$

Here w_0 is given by (1.10). Fix $\delta > 0$ and denote

$$u_{\delta,\pm}(x) = \begin{cases} \int_{\bar{x}}^{x} \sqrt{(1-(1\pm\delta)v)^2 - 1}\, dy & \text{for } x \geq \bar{x} \\ \int_{x}^{\bar{x}} \sqrt{(1-(1\pm\delta)v)^2 - 1}\, dy & \text{for } x \leq \bar{x}. \end{cases}$$

Apparently,

$$u_{\delta,-}(x) < u_0(x) = w_0(x) + \mu x < u_{\delta,+}(x) \quad \text{for } x \in [x_\mu - 1, x_\mu]\backslash\{\bar{x}\}$$

and $u_{\delta,-}(\bar{x}) = u_0(\bar{x}) = u_{\delta,+}(\bar{x}) = 0$. See the left picture on Fig. 4. Denote

$$e_\delta = \min_{x = x_\mu \text{ or } x_\mu - 1} \{u_0(x) - u_{\delta,-}(x),\ u_{\delta,+}(x) - u_0(x)\} > 0.$$

and

$$u_{d,\delta,\pm}(x) = w(x) - w(\bar{x}) + \mu(x - \bar{x}) \pm \frac{1}{2} e_\delta.$$

Clearly, by (3.15), when d is small enough, there exist $x_{d,\delta,\pm} \in (x_\mu - 1, x_\mu)$ such that

$$u_{d,\delta,+}(x_{d,\delta,+}) - u_{\delta,+}(x_{d,\delta,+}) \geq u_{d,\delta,+}(x) - u_{\delta,+}(x) \quad \text{for all } x \in (x_\mu - 1, x_\mu)$$

and

$$u_{d,\delta,-}(x_{d,\delta,-}) - u_{\delta,-}(x_{d,\delta,-}) \leq u_{d,\delta,-}(x) - u_{\delta,-}(x) \quad \text{for all } x \in (x_\mu - 1, x_\mu).$$

Hence maximum principle implies that

$$-\frac{d u_{\delta,+}^{''}}{1 + (u_{\delta,+}^{'})^2} + \sqrt{1 + (u_{\delta,+}^{'})^2} + v \leq \overline{H}_d(\mu) \quad \text{at } x_{d,\delta,+}.$$

So

$$-\frac{d u_{\delta,+}^{''}}{1 + (u_{\delta,+}^{'})^2} \leq \overline{H}_d(\mu) - 1 + \delta v \leq \overline{H}_d(\mu) - 1 \quad \text{at } x_{d,\delta,+}.$$

Sending $d \to 0$ first and then $\delta \to 0$, we derive that $x_{d,\delta,+} \to \bar{x}$ and

$$\liminf_{d \to 0^+} \frac{\overline{H}_d(\mu) - 1}{d} \geq -\sqrt{-v''(\bar{x})}.$$

By looking at $x_{d,\delta,-}$, similarly, we can obtain that

$$\limsup_{d \to 0^+} \frac{\overline{H}_d(\mu) - 1}{d} \leq -\sqrt{-v''(\bar{x})}.$$

Hence we finish the proof. □

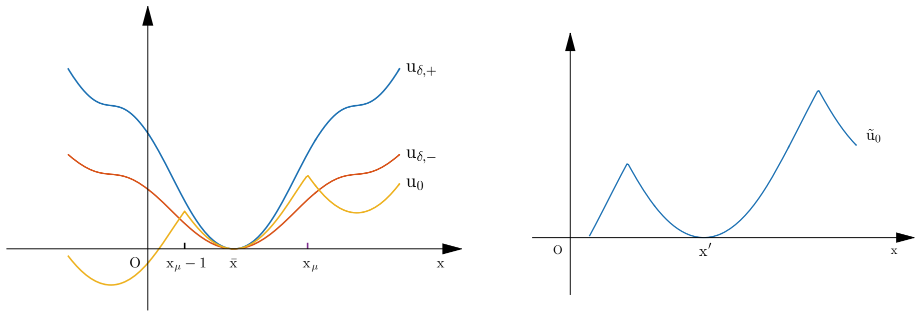

Fig. 4. Left: graphs of $u_{\delta,\pm}$ and u_0. Right: Turning points

Remark 3.1. The above proof based on comparison and maximum principle actually also shows that for any subsequence $\{d_m\} \to 0$, if

$$\lim_{d_m \to 0^+} w(x) = \tilde{w}_0(x)$$

and $\tilde{u}_0 = \mu x + \tilde{w}_0(x)$ has turning point at some $x' \in \mathcal{M}$, i.e. there exists a $\tau > 0$ such that (see the right picture on Fig. 4)

$$\tilde{u}_0(x) - \tilde{u}_0(x') = \begin{cases} \int_{x'}^{x} \sqrt{(1-v)^2 - 1}\, dy & \text{for } x \in [x', x' + \tau] \\ \int_{x}^{x'} \sqrt{(1-v)^2 - 1}\, dy & \text{for } x \in [x' - \tau, x'], \end{cases}$$

then

$$\lim_{m \to +\infty} \frac{\overline{H}_{d_m}(\mu) - 1}{d_m} = -\sqrt{-v''(x')}.$$

Lemma 3.2. *Suppose that $\tilde{w}$ is a periodic viscosity solution to the inviscid equation*

$$\sqrt{1 + (\mu + \tilde{w}')^2} + v = 1 \quad \textit{in } \mathbb{R}.$$

Then $x_0 \in \mathbb{R}$ is a turning point of $\tilde{u}(x) = \mu x + \tilde{w}$ if and only if $\tilde{u}(x)$ attains local minimum at x_0.

Proof. "$\Rightarrow$" is obvious. We only need to show that any local minimum point x_0 must be a turning point. By the definition of viscosity solutions,

$$1 + v(x_0) \geq 1.$$

So $v(x_0) = 0$ and $x_0 \in \mathcal{M}_0$. Choose $\tau > 0$ such that $(x_0, x_0 + \tau) \cap \mathcal{M}_0 = \emptyset$ and $\tilde{u}'(x_0 + \tau) = p + \tilde{w}'(x_0 + \tau) > 0$. Then we must have that

$$\tilde{u}'(y) > 0 \quad \text{for any } y \in (x_0, x_0 + \tau) \text{ where } \tilde{u}' \text{ exists.}$$

Otherwise there will be a local mimimum point in $(x_0, x_0 + \tau)$. Note that any local minimum point belongs to $\mathcal{M}_0$. This will contradict to the choice of τ. Accordingly,

$$\tilde{u}' = \sqrt{(1-v)^2 - 1} \quad \text{in } (x_0, x_0 + \tau).$$

Similarly, we can show that for some $\tau' > 0$,

$$\tilde{u}' = -\sqrt{(1-v)^2 - 1} \quad \text{in } (x_0 - \tau', x_0).$$

□

Proof of Theorem 1.2. **Step 1:** We first show that

$$\liminf_{d \to 0^+} \frac{\overline{H}_d(\mu) - 1}{d} \geq -\sqrt{-v''(\bar{x})}. \tag{3.16}$$

In fact, let $h(x)$ be a smooth periodic function satisfying that $h(\bar{x}) = 0$ and $h(x) > 0$ for $x \notin \bar{x} + \mathbb{Z}$. For $\epsilon > 0$, denote

$$v_\epsilon(x) = v(x) - \epsilon h(x)$$

and $\overline{H}_{d,\epsilon}(p)$ from the cell problem (1.6) with $\gamma = 1$ and v replaced by v_ϵ. It is easy to see that

$$\overline{H}_d(\mu) \geq \overline{H}_{d,\epsilon}(\mu).$$

Choose ϵ small enough such that

$$|\mu| < \int_0^1 \sqrt{(1 - v_\epsilon)^2 - 1}\, dx.$$

Clearly, $\max_{\mathbb{R}} v_\epsilon = 0$ and the maximum is only obtained at $\bar{x} + \mathbb{Z}$. Then (3.16) follows immediately from Lemma 3.1.

Step 2: Suppose $\tilde{u} = \mu x + \tilde{w}$ is the limit of a subsequence of $\mu x + w$ as $d \to 0$. Combining with the above Remark 3.1 and assumption (1.9), (3.16) implies that $\tilde{u}$ can only have a turning point at $\bar{x}$. Owing to Lemma 3.2, $\tilde{u}$ does not have local minimum points in $(\bar{x}, \bar{x}+1)$. Together with $|\mu| < \int_0^1 \sqrt{(1-v)^2 - 1}\, dx$, it is easy to see that there exists a unique $x_\mu \in (\bar{x}, \bar{x}+1)$ such that $\tilde{u}$ is increasing in $(\bar{x}, x_\mu)$ and is decreasing in $(x_\mu, \bar{x}+1)$. Hence $\tilde{w}$ must be uniquely given by the formula (1.10). □

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