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PERIODIC ORBITS OF THE ABC FLOW WITH $A = B = C = 1$[*]

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Abstract. In this paper, we prove that the celebrated Arnold–Beltrami–Childress (ABC) flow with parameters $A = B = C = 1$, given by $\dot{x} = \sin z + \cos y$, $\dot{y} = \sin x + \cos z$, $\dot{z} = \sin y + \cos x$, has periodic orbits on $(2\pi\mathbb{T})^3$ with rotation vectors parallel to $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Despite ABC flows being studied since the 1960s, this seems to be the first time that existence of nonperturbative periodic orbits has been established for them. The main difficulty here is the lack of a variational structure for these flows, and our proof instead relies on their symmetry properties. As an application of our result, we show that the well-known G-equation model of turbulent combustion with this ABC flow on $\mathbb{R}^3$ has a linear (i.e., maximal possible) flame speed enhancement rate as the flow amplitude grows to infinity. To the best of our knowledge, this is the first time an asymptotic flame speed growth law has been established for a natural three-dimensional incompressible flow with such a complex structure.

Key words. ABC flow, periodic orbits, flame speed enhancement by flows

AMS subject classifications. 34C25, 34C14, 34C11

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1. Introduction.

Arnold–Beltrami–Childress (ABC) flows are important examples of three-dimensional periodic incompressible flows [1, 7, 8]. The general form is

$$u_{A,B,C}(x, y, z) = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x),$$

with $A, B, C \in \mathbb{R}$ some given parameters. These flows are steady solutions of the Euler equation, and the corresponding system of ODEs,

$$\dot{X}(t) = u_{A,B,C}(X(t)),$$

becomes integrable when one of the parameters is zero. We refer the reader to [7] and to section 2.6 in [5] for more background information.

The most interesting case is when $A = B = C = 1$ (the 1-1-1 ABC flow), introduced by Childress [3, 4] in connection with dynamo studies. The 1-1-1 ABC flow system is known to be chaotic in the sense that a web of chaos occupies a (fairly complex) region of the phase space, as seen from the Poincaré section plots of Figure 10 in [7]. Though chaotic orbits for other ABC parameters have been identified analytically (see, e.g., [1, 16]), it is not known how to construct them rigorously for the 1-1-1 ABC flow. In fact, despite the study of ABC flows dating back to the 1960s, we are not aware of any previous nonperturbative results involving even nonchaotic orbits (e.g., periodic ones). This is likely due to the lack of a variational structure of ABC flows, which means that classical variational approaches, such as critical point theory [12], do not apply to them.

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In this paper we address the question of existence of periodic orbits for the 1-1-1 ABC flow system. We prove here that for each standard basis vector e this flow does, in fact, have unbounded orbits that are periodic up to shifts by $2\pi\mathbb{Z}$-multiples of e.

Besides this question being of independent interest, our work is also motivated by recent progress and open questions in the study of flame speed enhancement by turbulent fluid motion in combustion models (see, e.g., [15, 17]), especially in three spatial dimensions. Indeed, linear (and hence maximal possible) growth of flame front speeds as the amplitude of the flow grows has been proved to be related to the existence of unbounded (roughly) "periodic" orbits of the turbulent flow (see below), rather than to existence of chaotic orbits (which one might expect). The study of the former orbits thus becomes natural and is a distinct departure from the long history of the study of chaos in ABC flows.

Note that since the 1-1-1 ABC flow is not a small perturbation of an integrable case, classical dynamical systems tools (such as KAM, Melnikov analysis, or Smale horseshoe) are difficult if not entirely impossible to apply. Nevertheless, using a careful analysis of the orbits of the flow inside a triangular prism region (see Figure 1) and certain symmetries of the flow, we are able to establish the following main result of this paper.

THEOREM 1.1. *There exist $t_0 > 0$ and a solution $X(t) = (x(t), y(t), z(t))$ to the* 1-1-1 *ABC flow system such that for each $t \in \mathbb{R}$ we have*

$$X(t + t_0) = X(t) + (2\pi, 0, 0).$$

Then $Y(t) = (z(t), x(t), y(t))$ and $Z(t) = (y(t), z(t), x(t))$ are clearly also solutions and satisfy

$$Y(t + t_0) = Y(t) + (0, 2\pi, 0) \qquad \text{and} \qquad Z(t + t_0) = Z(t) + (0, 0, 2\pi).$$

Remarks. 1. Then $\tilde{X}(t) = X(-t) - (\pi, \pi, \pi)$, $\tilde{Y}(t) = Y(-t) - (\pi, \pi, \pi)$, and $\tilde{Z}(t) = Z(-t) - (\pi, \pi, \pi)$ are also solutions, and they satisfy $\tilde{X}(t+t_0) = \tilde{X}(t) - (2\pi, 0, 0)$, $\tilde{Y}(t + t_0) = \tilde{Y}(t) - (0, 2\pi, 0)$, and $\tilde{Z}(t + t_0) = \tilde{Z}(t) - (0, 0, 2\pi)$.

2. Our method can be adjusted to obtain unbounded "periodic" orbits along the x-, y-, and z directions for some other values of A, B, C. For instance, if A, B and C are all close to 1, then this follows from the proof of our result via a simple perturbation argument. Another example is the case $0 < A \ll 1$ and $B = C = 1$, which is a perturbation of the integrable case $A = 0$ and $B = C = 1$. We note that the corresponding system now possesses a large KAM region as well as a chaotic thin layer near the separatrix walls of the integrable flow (i.e., near $\{\sin y + \cos x = 0\}$), and we refer to [10] for both theoretical and numerical analysis of this case.

Applications to combustion models. Finding the turbulent flame speed (or effective burning velocity) is one of the most important unsolved problems in turbulent combustion. Roughly speaking, turbulent flame speed is the flame propagation speed in the presence of (and enhanced by) a strong flow of the ambient fluid medium. Two typical examples are the spread of wildfires fanned by winds and combustion of rotating air-gasoline mixtures inside internal combustion engines.

For simplicity, let us assume that the flow velocity profile $V : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, periodic, and incompressible (i.e., $\nabla \cdot V = 0$). A well-known approach to the study of flame propagation and turbulent flame speed is the G-equation model (see, e.g., [11, 13]): the level set Hamilton–Jacobi equation

$$G_t + KV(x) \cdot \nabla G + s_l |\nabla G| = 0,$$

with $K \geq 0$ the amplitude of the turbulent flow and $s_l > 0$ the laminar flame speed. A basic question is to understand how the turbulent flame speed depends on the flow amplitude as $K \to \infty$ (i.e., for strong flows). Let $s_T(p, K)$ be the turbulent flame speed given by the G-equation model along a fixed unit direction $p \in \mathbb{R}^n$ (see [15] for the precise definition and further references). Then it is proved in [15] that

$$\lim_{K\to\infty} \frac{s_T(p,K)}{K} = \max_{\{\xi |\ \dot{\xi}=V(\xi)\}} \limsup_{t\to\infty} \frac{p\cdot\xi(t)}{t}.$$

In particular, $s_T(p, K)$ grows linearly (which is the maximal possible growth rate) as $K \to \infty$ precisely when there exists an orbit of $\dot{\xi} = V(\xi)$ which travels roughly linearly in the direction p.

This yields the following corollary of Theorem 1.1 (and of Remark 1), which is, to the best of our knowledge, the first time an asymptotic turbulent flame speed growth law has been established for a natural three-dimensional incompressible flow with such a complex structure.

Corollary 1.1. *If V is the* 1-1-1 *ABC flow, then for any $p \in \mathbb{R}^3$ we have*

$$\lim_{K\to\infty} \frac{s_T(p,K)}{K} > 0.$$

We note that another well-known model used in the study of turbulent flame speeds involves traveling front solutions of the reaction-advection-diffusion equation

$$T_t + KV(x)\cdot\nabla T = d\Delta T + f(T).$$

Here T represents the temperature of the reactant, $d > 0$ is the molecular diffusivity, and f is a nonlinear reaction function (see, e.g., [2, 14]). Consider the case of a KPP (Kolmogorov–Petrovsky–Piskunov) reaction f (e.g., $f(T) = T(1-T)$), and let $c^*(p, K)$ be the turbulent flame speed in the direction p given by this model (i.e., the minimal speed of a traveling front in direction p; see [2, 14] for details). It is established in [17] that

$$\lim_{K\to\infty} \frac{c^*(p,K)}{K} = \sup_{w\in\Gamma} \int_{\mathbb{T}^n} (V\cdot p) w^2 dx,$$

where

$$\Gamma = \left\{ w \in H^1(\mathbb{T}^n) \,\middle|\, V\cdot\nabla w = 0,\ ||w||_{L^2(\mathbb{T}^n)} = 1,\ \|\nabla w\|^2_{L^2(\mathbb{T}^n)} \leq f'(0) \right\}.$$

Hence, in contrast to the G-equation model, one now needs a positive measure of orbits of $\dot{\xi} = V(\xi)$ which travel roughly linearly in the direction p to obtain linear-in-K turbulent flame speed enhancement. Such *percolating* flows were first shown to linearly enhance turbulent flame speeds in [6, 9].

When $n = 2$, stability of periodic orbits was used in [15] to establish that $\lim_{K\to\infty} \frac{c^*(p,K)}{K} = 0$ if and only if $\lim_{K\to\infty} \frac{s_T(p,K)}{K} = 0$. However, this is not true, in general, in three dimensions. An example from [15] is the so-called Robert cell flow, for which $\lim_{K\to\infty} \frac{s_T(p,K)}{K} > 0$ and $\lim_{K\to\infty} \frac{c^*(p,K)}{K} = 0$ when $p = (0, 0, 1)$.

The analysis becomes much more difficult for the much more interesting 1-1-1 ABC flow due to the presence of chaotic structures. Nevertheless, numerical simulations suggest that there are “vortex tubes” composed of orbits which travel roughly linearly in the $\pm x$-, $\pm y$-, and $\pm z$ directions [7], suggesting that $c^*(p,K)$ should also grow linearly in K for each p. One could expect to find such vortex tubes, if they indeed exist, in the vicinity of the "periodic" orbits constructed in this paper, but a rigorous proof of their existence is currently not known, and is expected to be an order of magnitude harder than KAM.

2. Proof of Theorem 1.1. Let R be the open triangle in the xy-plane with vertices $(0,-\frac{\pi}{2})$, $(0,\frac{3\pi}{2})$, $(-\pi,\frac{\pi}{2})$, and let $D = R\times(0,\frac{\pi}{2})$. Our proof is based on showing that there exists a solution $X_{\bar a}$ to the 1-1-1 ABC flow system which starts from $(-\frac{\pi}{2},0,\bar a)$ for some $\bar a\in[0,\frac{\pi}{2})$ and passes through the segment $\{(0,y,\frac{\pi}{2})|\ y\in[-\frac{\pi}{2},\frac{3\pi}{2}]\}$ (see Figure 1). We do this in Step 1 below, and then use the symmetries of the flow to construct the desired solution X in Step 2.

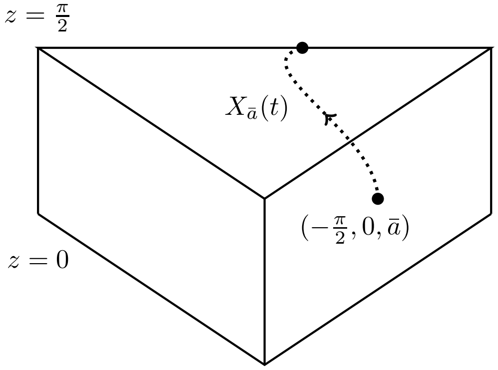

FIG. 1. *The region D, rotated counterclockwise by 90 degrees, and $X_{\bar a}$.*

Step 1. For any $a\in[0,\frac{\pi}{2})$, let $X_a(t)=(x(t),y(t),z(t))$ satisfy $X_a(0)=(-\frac{\pi}{2},0,a)$ and

$$\begin{aligned}\dot x &= \sin z+\cos y,\\ \dot y &= \sin x+\cos z,\\ \dot z &= \sin y+\cos x.\end{aligned}$$

Obviously, $(x(t),y(t))\in R$ for all small $t>0$ (and any $a\in[0,\frac{\pi}{2})$). Since $\cos x+\sin y>0$ for $(x,y)\in R$, we have that $X_a(t)\in D$ for all small $t>0$.

The question now is whether and where X_a will (first) exit D. Clearly $\dot z(t)>0$ when $X_a(t)\in D$, and we also have

$$\dot X_a(t)\cdot(1,1,0)=\sin z(t)+\cos z(t)>0$$

when $X_a(t)\in\{x+y=-\frac{\pi}{2}\}\cap\partial D$. Hence X_a cannot exit D through either the plane $\{z=0\}$ or the plane $\{x+y=-\frac{\pi}{2}\}$.

Let us now consider any $T>0$ such that $X_a(t)\in\bar D=\bar R\times[0,\frac{\pi}{2}]$ for all $t\in[0,T]$. The following result shows, in particular, that the first exit cannot happen through the plane $\{y-x=\frac{3\pi}{2}\}$ either.

LEMMA 2.1. *We have $y(t)<\frac{\pi}{4}$ for $t\in[0,T]$.*

Proof. We argue by contradiction. If not, then by $y(0)=0$, there are $0\le t_1<t_2\le T$ such that $y(t)\in[0,\frac{\pi}{4}]$ for $t\in[t_1,t_2]$, $y(t_1)=0$, and $y(t_2)=\frac{\pi}{4}$. Let us choose

the smallest such t_1, t_2. Then $y(t) \in [-\frac{\pi}{2}, \frac{\pi}{4}]$ for all $t \in [0, t_2]$, and therefore $\dot{x}(t) \geq 0$ for $t \in [0, t_2]$. Therefore

$$w(t) = x(t) + \frac{\pi}{2}$$

satisfies $w(0) = 0$ and $w(t) \in [0, \frac{\pi}{2}]$ for $t \in [0, t_2]$.

In the following consider only $t \in [t_1, t_2]$. We have

$$\dot{z} - \dot{y} = \sin y + \cos x - \sin x - \cos z \geq \sin y + 1 - \cos z \geq \sin y \geq 0;$$

hence $z(t) \geq y(t)$ for $t \in [t_1, t_2]$ due to $z(t_1) \geq 0 = y(t_1)$. Then

$$\begin{aligned}
\dot{w} &= \sin z + \cos y \geq \sin y + \cos y \geq 1, \\
\dot{y} &= \cos z - \cos w \leq 1 - \cos w \leq \min\left\{\frac{w^2}{2}, 1\right\},
\end{aligned}$$

since $w \in [0, \frac{\pi}{2}]$. In particular, $\dot{y} \leq 1$ shows that $t_2 - t_1 \geq \frac{\pi}{4}$. Let us now consider three cases.

Case 1: $w(t_1) \geq \frac{\pi}{3}$. Then $\dot{w} \geq 1$ shows $w(t_2) \geq \frac{7\pi}{12} > \frac{\pi}{2}$, a contradiction.

Case 2: $w(t_1) < \frac{\pi}{3}$ and $w(t_2) \leq \frac{\pi}{3}$. We have $\dot{y} \leq \dot{w}\frac{w^2}{2}$, which after integration over $[t_1, t_2]$ yields $y(t_2) \leq \frac{1}{6}w^3(t_2) \leq \frac{\pi^3}{162} < \frac{\pi}{4}$, a contradiction.

Case 3: $w(t_1) < \frac{\pi}{3}$ and $w(t_2) > \frac{\pi}{3}$. Then there exists $t^* \in [t_1, t_2]$ such that $w(t^*) = \frac{\pi}{3}$, and the computation in Case 2 shows $y(t^*) \leq \frac{\pi^3}{162}$. Then

$$y(t_2) - y(t^*) \geq \frac{\pi}{4} - \frac{\pi^3}{162} > \frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{3} \geq w(t_2) - w(t^*),$$

a contradiction with $\dot{w} \geq 1 \geq \dot{y}$ on $[t_1, t_2]$. The proof is finished. ☐

LEMMA 2.2. *If $a = 0$, then $x(t) + \frac{\pi}{2} > z(t)$ for all $t \in (0, T]$.*

Proof. Again let $w(t) = x(t) + \frac{\pi}{2}$. From $w(0) = y(0) = z(0) = 0$ and

$$\begin{aligned}
\dot{w} &= \sin z + \cos y, \\
\dot{z} &= \sin w + \sin y,
\end{aligned}$$

we have $\dot{w}(0) > \dot{z}(0)$; hence $w(t) > z(t)$ for all small $t > 0$. Assume that there is $t^* \in (0, T]$ such that $w(t^*) = z(t^*)$ and $w > z$ on $(0, t^*)$. Lemma 2.1 shows that $\cos y > \sin y$ on $[0, T]$, so

$$0 \geq \dot{w}(t^*) - \dot{z}(t^*) > \sin z(t^*) - \sin w(t^*) = 0,$$

a contradiction. ☐

Lemma 2.1 implies $\cos y \geq 0$ on $[0, T]$. Since $z(t) \in (0, \frac{\pi}{2}]$ and $\dot{z}(t) > 0$ for $t \in (0, T]$, it follows that $\dot{x}$ is bounded below by a positive constant on $[\delta, T]$ for each $\delta > 0$. Hence X_a will reach ∂D in finite time, and we denote by $t_a > 0$ the first such positive time. The discussion preceding Lemma 2.1 shows that

$$X_a(t_a) \notin \{z = 0\} \cup \left\{x + y = \frac{\pi}{2}\right\} \cup \left\{y - x = \frac{3\pi}{2}\right\},$$

and hence

$$X_a(t_a) \in \{x = 0\} \cup \left\{z = \frac{\pi}{2}\right\}.$$

Let S_0, S_1 be the sets of all $a \in [0, \frac{\pi}{2})$ such that $X_a(t_a) \in \{x = 0\} \setminus \{z = \frac{\pi}{2}\}$ and $X_a(t_a) \in \{z = \frac{\pi}{2}\} \setminus \{x = 0\}$, respectively. We have $S_0 \neq \emptyset \neq S_1$ since obviously $a \in S_1$ when a is close to $\frac{\pi}{2}$, while Lemma 2.2 implies $0 \in S_0$. Moreover, $\dot{X}_a(t_a)$ is transversal to ∂D for any $a \in S_0 \cup S_1$. (For $a \in S_0$ we have $\dot{x}(t_a) > 0$ by the argument in the last paragraph, while for $a \in S_1$ obviously $\dot{z}(t_a) > 0$.) It follows that S_0, S_1 are both relatively open in $[0, \frac{\pi}{2})$ and that t_a and $X_a(t_a)$ are continuous on them. Hence $[0, \frac{\pi}{2}) \setminus (S_0 \cup S_1) \neq \emptyset$, and then for any $\bar{a}$ from this set we must have

$$X_{\bar{a}}(t_{\bar{a}}) \in \partial D \cap \{x = 0\} \cap \left\{z = \frac{\pi}{2}\right\} = \left\{\left(0, y, \frac{\pi}{2}\right) \,\middle|\, y \in \left[-\frac{\pi}{2}, \frac{\pi}{4}\right]\right\}.$$

Step 2. We now use the symmetry of the ABC flow to show that for any $\bar{a}$ as above, $X_{\bar{a}}(t) = (x(t), y(t), z(t))$ is the desired solution. For $t \in \mathbb{R}$ let

$$\tilde{X}(t) = (-\pi - x(-t),\ -y(-t),\ z(-t))$$

(this is the reflection across the line $(-\frac{\pi}{2}, 0) \times \mathbb{R}$) and

$$\hat{X}(t) = (-x(2t_{\bar{a}} - t),\ y(2t_{\bar{a}} - t),\ \pi - z(2t_{\bar{a}} - t))$$

(this is the reflection across the line $\{x = 0\} \cap \{z = \frac{\pi}{2}\}$). Clearly, both $\tilde{X}$ and $\hat{X}$ are solutions to the 1-1-1 ABC flow system. Since $\tilde{X}(0) = X_{\bar{a}}(0)$ and $\hat{X}(t_{\bar{a}}) = X_{\bar{a}}(t_{\bar{a}})$, we have $X_{\bar{a}} = \tilde{X} = \hat{X}$. Thus

$$(x(-t_{\bar{a}}),\ y(-t_{\bar{a}}),\ z(-t_{\bar{a}})) = X_{\bar{a}}(-t_{\bar{a}}) = \tilde{X}(-t_{\bar{a}}) = \left(-\pi, -y(t_{\bar{a}}), \frac{\pi}{2}\right),$$

and then

$$\hat{X}(3t_{\bar{a}}) = (-x(-t_{\bar{a}}), y(-t_{\bar{a}}), \pi - z(-t_{\bar{a}})) = \left(\pi, -y(t_{\bar{a}}), \frac{\pi}{2}\right).$$

So

$$X_{\bar{a}}(3t_{\bar{a}}) = \hat{X}(3t_{\bar{a}}) = X_{\bar{a}}(-t_{\bar{a}}) + (2\pi, 0, 0),$$

and it follows from the 2π-periodicity of the ABC flow that

$$X_{\bar{a}}(t + 4t_{\bar{a}}) = X_{\bar{a}}(t) + (2\pi, 0, 0)$$

for each $t \in \mathbb{R}$. ∎

REFERENCES

[1] V. Arnold, *Sur la topologie des écoulements stationaires des fluides parfaits*, C. R. Acad. Sci. Paris, 261 (1965), pp. 17–20.

[2] H. Berestycki, *The influence of advection on the propagation of fronts in reaction-diffusion equations*, in Nonlinear PDEs in Condensed Matter and Reactive Flows, NATO Science Series C 569, H. Berestycki and Y. Pomeau, eds., Kluwer Academic Publishers, Doordrecht, The Netherlands, 2002, pp. 11–48.

[3] S. Childress, *Construction of Steady-State Hydromagnetic Dynamos.* I. *Spatially Periodic Fields*, Report MF-53, Courant Institute, New York, 1967.

[4] S. Childress, *New solutions of the kinematic dynamo problems*, J. Math. Phys., 11 (1970), pp. 3063–3076.

[5] S. Childress, *Stretch, Twist, Fold: The Fast Dynamo*, Lecture Notes in Phys. Monogr., Springer, New York, 1995.

[6] P. Constantin, A. Kiselev, A. Oberman, and L. Ryzhik, *Bulk burning rate in passive-reactive diffusion*, Arch. Ration. Mech. Anal., 154 (2000), pp. 53–91.

[7] T. Dombre, U. Frisch, J.M. Greene, M. Hénon, A. Mehr, and A.M. Soward, *Chaotic streamlines in the ABC flows*, J. Fluid Mech., 167 (1986), pp. 353–391.

[8] S. Friedlander, A. Gilbert, and M. Vishik, *Hydrodynamic instability for certain ABC flows*, Geophys. Astrophys. Fluid Dynamics, 73 (1993), pp. 97–107.

[9] A. Kiselev and L. Ryzhik, *Enhancement of the traveling front speeds in reaction-diffusion equations with advection*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 18 (2001), pp. 309–358.

[10] T. McMillen, J. Xin, Y. Yu, and A. Zlatoš, *Ballistic orbits and front speed enhancement for ABC flows*, SIAM J. Appl. Dyn. Syst., 15 (2016), pp. 1753–1782, https://doi.org/10.1137/16M1059059.

[11] N. Peters, *Turbulent Combustion*, Cambridge University Press, Cambridge, UK, 2000.

[12] P. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Series 65, AMS, Providence, RI, 1986.

[13] F.A. Williams, *Turbulent combustion*, in The Mathematics of Combustion, J.D. Buckmaster, ed., Frontiers in Appl. Math. 2, SIAM, Philadelphia, 1985, pp. 97–131.

[14] J. Xin, *An Introduction to Fronts in Random Media*, Surveys and Tutorials Appl. Math. Sci. 5, Springer, New York, 2009.

[15] J. Xin and Y. Yu, *Asymptotic growth rates and strong bending of turbulent flame speeds of G-equation in steady two-dimensional incompressible periodic flows*, SIAM J. Math. Anal., 46 (2014), pp. 2444–2467, https://doi.org/10.1137/130929990.

[16] X.-H. Zhao, K.-H. Kwek, J.-B. Li, and K.-L. Huang, *Chaotic and resonant streamlines in the ABC flow*, SIAM J. Appl. Math., 53 (1993), pp. 71–77, https://doi.org/10.1137/0153005.

[17] A. Zlatoš, *Sharp asymptotics for KPP pulsating front speed-up and diffusion enhancement by flows*, Arch. Ration. Mech. Anal., 195 (2010), pp. 441–453.