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Sharp asymptotic growth laws of turbulent flame speeds in cellular flows by inviscid Hamilton–Jacobi models ☆

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Abstract

We study the large time asymptotic speeds (turbulent flame speeds s_T) of the simplified Hamilton–Jacobi (HJ) models arising in turbulent combustion. One HJ model is G-equation describing the front motion law in the form of local normal velocity equal to a constant (laminar speed) plus the normal projection of fluid velocity. In level set formulation, G-equations are HJ equations with convex (L^1 type) but non-coercive Hamiltonians. The other is the quadratically nonlinear (L^2 type) inviscid HJ model of Majda–Souganidis derived from the Kolmogorov–Petrovsky–Piskunov reactive fronts. Motivated by a question posed by Embid, Majda and Souganidis (1995) [10], we compare the turbulent flame speeds s_T's from these inviscid HJ models in two-dimensional cellular flows or a periodic array of steady vortices via sharp asymptotic estimates in the regime of large amplitude. The estimates are obtained by analyzing the action minimizing trajectories in the Lagrangian representation of solutions (Lax formula and its extension) in combination with delicate gradient bound of viscosity solutions to the associated cell problem of homogenization. Though the inviscid turbulent flame speeds share the same leading order asymptotics, their difference due to nonlinearities is identified as a subtle double logarithm in the large flow amplitude from the sharp growth laws. The turbulent flame speeds differ much more significantly in the corresponding viscous HJ models.

MSC: 70H20; 76M50; 76M45; 76N20

Keywords: Inviscid Hamilton–Jacobi equations; Cellular flows; Sharp front speed asymptotics

1. Introduction

Turbulent combustion is a complex nonlinear and multiscale dynamic phenomenon [25,29,24,23,13,14,21,26]. The first principle approach requires a system of reaction–diffusion–advection equations coupled with the Navier–Stokes equations. Progress in theoretical understanding and efficient modeling of the turbulent flame propagation however often relies on simplified models such as the advective Hamilton–Jacobi equations (HJ) and passive scalar reaction–diffusion–advection equations (RDA), as documented in books [25,21,27] and research papers [1,2,7,8,10,14,16,20,22–24,26,29].

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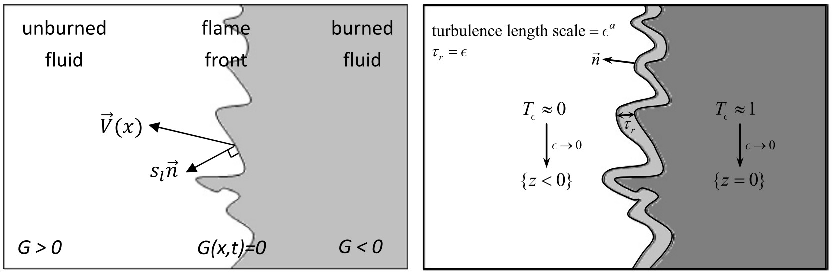

Fig. 1. Left: G-equation model. Right: Majda–Souganidis model.

• **G-equation model:** A popular phenomenological approach in turbulent combustion is the level set formulation [20] of flame front motion laws with the front width ignored [21]. The motion law is in the hands of a modeler based on theory and experiments. The simplest motion law is that the normal velocity of the front (V_n) is equal to a constant s_l (the laminar speed) plus the projection of fluid velocity $V(x,t)$ along the normal $\vec{n}$. See Fig. 1 (left picture). The laminar speed is the flame speed due to chemistry (reaction–diffusion) when fluid is at rest. As the fluid becomes active, the flame front will be wrinkled by the fluid velocity. However it is observed under suitable conditions that the front location eventually moves to leading order at a well-defined steady speed s_T in each specified direction, which is the so-called "turbulent burning velocity". The study of the existence and properties of the turbulent flame speed s_T is a fundamental problem in turbulent combustion theory and experiments [25,22,21]. Let the flame front be the zero level set of a function $G(x,t)$, then the normal direction is $DG/|DG|$, the normal velocity is $-G_t/|DG|$. The motion law becomes the so-called G-equation in turbulent combustion [25,21]:

$$G_t + V(x,t)\cdot DG + s_l|DG| = 0. \tag{1.1}$$

Chemical kinetics and diffusion rates are all included in the laminar speed s_l which is provided by a modeler. Formally under the G-equation model, for a specified unit direction p,

$$s_T(p) = -\lim_{t\to+\infty} \frac{G(x,t)}{t}.$$

Here $G(x,t)$ is the solution of Eq. (1.1) with initial data $G(x,0) = p\cdot x$. The existence of s_T has been rigorously established in [28] and [4] independently for incompressible periodic flows (the assumptions in [4] are more general), and [15] for two-dimensional incompressible random flows. Very recently, Cardaliaguet and Souganidis [5] proved the homogenization of the G-equation for stationary ergodic flows in any dimension. For simplicity, we assume that $V = V(x)$ is spatially periodic and has no time dependence. Then $s_T = s_T(p)$ is equal to the effective Hamiltonian of the following cell problem

$$s_l|p + DG| + V(x)\cdot(p + DG) = s_T(p). \tag{1.2}$$

Here $s_T(p)$ is the unique number such that the above equation admits periodic approximate solutions. Change V to AV for some positive constant A (turbulence intensity). A very important problem in turbulent combustion is to study the dependence of the turbulent flame speed on A (i.e., $s_T = s_T(A)$). Interestingly, for the cellular flow (a prototypical flow in dynamo and convection-enhanced diffusion problems [6,9]):

$$V(x) = V(x_1,x_2) = (-H_{x_2}, H_{x_1}), \quad H = \sin x_1 \sin x_2, \tag{1.3}$$

it is known [1,17,19] that $s_T = O(A/\log A)$ in the limit of large A, which also shows the weak bending phenomenon.

• **RDA model:** The passive scalar reaction–diffusion–advection (RDA) model is:

$$u_t + V(x)\cdot Du = \kappa\Delta u + \frac{1}{\tau_r} f(u), \quad x \in \mathbb{R}^n, \tag{1.4}$$

where u represents the reactant temperature or concentration, D is the spatial gradient operator, $V(x)$ is a prescribed fluid velocity, f is a nonlinear reaction function; κ is the molecular diffusion constant, $\tau_r > 0$ is reaction time scale. In this paper, we will choose the Kolmogorov–Petrovsky–Piskunov–Fisher (KPP–Fisher) type reaction function f.

A typical example is $f(u) = Cu(1-u)$ for some $C > 0$. Under this model, the turbulent flame speed $c_T^*(p)$ along a given direction p is defined to be the large time spreading rate of solution from compactly supported non-negative initial data [16]. In case of periodic flow, it also agrees with the minimal traveling wave speed [3,27]. It is known even for more general time dependent (stationary and ergodic) flows [26,3,16,27] that $c_T^*(p)$ obeys a variational principle in terms of the large time growth rate of a viscous quadratically nonlinear Hamilton–Jacobi equation (QHJ). In spatially periodic flows,

$$c_T^*(p) = \inf_{\lambda>0} \frac{\tau_r^{-1} f'(0) + H^*(p\lambda)}{\lambda}, \tag{1.5}$$

where H^* is the effective Hamiltonian of the following cell problem

$$-\kappa \Delta W + V(x)\cdot(p+DW) + |p+DW|^2 = H^*(p).$$

• **Majda–Souganidis model:** Turbulent combustion always involves small scales. Following the model proposed by Majda and Souganidis [13], we write $V = V(\frac{x}{\epsilon^\alpha})$ for some $\alpha \in (0,1]$. In [13], the authors actually considered more general case, i.e., V is in a scale-separation form $V = V(x,t,\frac{x}{\epsilon^\alpha},\frac{t}{\epsilon^\alpha})$. The flame thickness is in general much smaller than the turbulence scale. As in [13], we set $\kappa = d\epsilon$ and $\tau_r = \epsilon$. See Fig. 1 (right picture). Suppose that $T = T(x,t)$ is the solution of (1.4) with compactly supported initial data $T(x,0)$. The limiting behavior of $T = T^\epsilon$ is [13]: $\lim_{\epsilon\to 0} T^\epsilon = 0$ locally uniformly in $\{(x,t)\colon Z<0\}$ and $T^\epsilon \to 1$ locally uniformly in the interior of $\{(x,t)\colon Z=0\}$, where $Z \in C(\mathbb{R}^n \times [0,+\infty))$ is the unique viscosity solution of the variational inequality

$$\max\bigl(Z_t - \hat{H}(D_x Z) - f'(0), Z\bigr) = 0, \quad (x,t)\times\mathbb{R}^n\times(0,+\infty),$$

with initial data $Z(x,0) = 0$ in the support of $T(x,0)$, and $Z(x,0) = -\infty$ otherwise. The effective Hamiltonian $\hat{H} = \hat{H}(p)$ is defined as a solution of the following cell problem: for each $p \in \mathbb{R}^n$, there are a unique number $\hat{H}(p)$ and a function $\hat{F}(x) \in C^{0,1}(\mathbb{T}^n)$ such that

$$a(\alpha) d\Delta\hat{F} + d|p + D\hat{F}|^2 - V(x)\cdot(p + D\hat{F}) = \hat{H}(p), \tag{1.6}$$

$a(\alpha) = 1$ if $\alpha = 1$ and $a(\alpha) = 0$ if $\alpha \in (0,1)$. The set $\Gamma_t = \partial\{x \in \mathbb{R}^n\colon Z(x,t) < 0\}$ can be viewed as a front which moves with normal velocity which is the turbulent flame speed predicted by Majda–Souganidis model:

$$v_{\vec{n}} = c_T(\vec{n}).$$

In order to be consistent with the G-equation, we choose $\vec{n}$ to be the unit normal vector pointing to the propagation direction, i.e., $\vec{n} = -\frac{DZ}{|DZ|}$. Then

$$c_T(p) = \inf_{\lambda>0} \frac{f'(0) + \overline{H}(p\lambda)}{\lambda}, \tag{1.7}$$

with

$$\overline{H}(p) = \hat{H}(-p).$$

Note when $\alpha = 1$ (viscous case), $\overline{H}$ is the same as H^* in (1.5) and $c_T(p) = c_T^*(p)$. If $\alpha \in (0,1)$, $\overline{H}$ is the effective Hamiltonian of the following inviscid QHJ (F-equation):

$$d|p+DF|^2 + V(x)\cdot(p+DF) = \overline{H}(p). \tag{1.8}$$

In this paper, we show the difference between the inviscid s_T and c_T in cellular flows (1.3) through their sharp asymptotics at large A. Therefore we set $s_l = 2\sqrt{df'(0)}$. In the paper [10] by Embid, Majda and Souganidis, the comparison of s_T and c_T for periodic shear flows on non-zero mean showed $c_T > s_T$ under certain conditions of the mean flow. At the end of [10], the authors raised the following question: "*It is very interesting to develop further comparisons of enhanced flame speeds between the complete nonlinear averaging theory summarized in Section 2 and the averaged G-equation from Section 3 for more realistic periodic flow fields such as arrays of vortices*". The following is our main result which provides an answer to this question at least for two-dimensional cellular flow. For clarity of presentation, we assume $s_l = d = 1$ and $f'(0) = \frac{1}{4}$ as in [10].

Theorem 1.1. *For the V given by (1.3), scale $V(x)$ to $AV(x)$. Consider the inviscid G-equation front speed s_T (1.2) and the inviscid KPP front speed c_T (1.7)–(1.8) in cellular flows (1.3). Let $p=(p_1,p_2)$ be a unit vector. There exist positive constants $0<C_1\leqslant C_2$ independent of A and p such that for $A\geqslant 4$,*

$$\frac{A\pi(|p_1|+|p_2|)}{2\log A+C_2}\leqslant s_T(p,A)\leqslant\frac{A\pi(|p_1|+|p_2|)}{2\log A+C_1}. \tag{1.9}$$

Also, there exist positive constants C_3, C_4 and $A_0\geqslant 4$ independent of A and p such that when $A\geqslant A_0$

$$\frac{A\pi(|p_1|+|p_2|)}{2\log A-\log\log A+C_3}\leqslant c_T(p,A)\leqslant\frac{A\pi(|p_1|+|p_2|)}{2\log(A)-2\log\log A-C_4}. \tag{1.10}$$

Note that (1.9) and (1.10) imply that

$$\lim_{A\to+\infty}\frac{\log(A)s_T(p,A)}{A}=\lim_{A\to+\infty}\frac{\log(A)c_T(p,A)}{A}=\frac{\pi}{2}\big(|p_1|+|p_2|\big),$$

and

$$c_T(p,A)-s_T(p,A)=O\left(\frac{A\log\log A}{\log^2 A}\right),\quad \text{as } A\to+\infty.$$

For general n-dimensional incompressible flow, we have the following relatively rough comparison.

Theorem 1.2. *Assume that $V(x):\mathbb{T}^n\to\mathbb{R}^n$ is periodic and incompressible. Suppose that $s_T(p,A)=O(\frac{A}{\log A})$. Then*

$$c_T(p,A)=O\left(\frac{A}{\log A}\right).$$

The proof of Theorem 1.1 is much more delicate and the symmetric structure of the stream function $H=\sin x_1\sin x_2$ around hyperbolic critical points will play essential role. We would like to mention that for the 2d cellular flow, the turbulent flame speed predicted by the RDA model (1.5) obeys the growth law of $O(A^{\frac{1}{4}})$ [18].

Remark 1.1 *(Difference between cellular flow and shear flow).* According to inf-max formulas,

$$s_T(p,A)=\inf_{\phi\in C^1(\mathbb{T}^n)}\max_{\mathbb{T}^n}\big\{|p+D\phi|+AV(x)\cdot(p+D\phi)\big\} \tag{1.11}$$

and

$$\overline{H}(p,A)=\inf_{\phi\in C^1(\mathbb{T}^n)}\max_{\mathbb{T}^n}\big\{|p+D\phi|^2+AV(x)\cdot(p+D\phi)\big\}. \tag{1.12}$$

Here $\mathbb{T}^n$ is the n-dimensional flat Torus. For the specific shear flow $V(x_1,x_2)=(v(x_2),0)$ and $p=(1,0)$. It is easy to see that $s_T(p,A)=1+A\max_{\mathbb{T}^1}v$ and $\overline{H}(\lambda p,A)=\lambda^2+A\lambda\max_{\mathbb{T}^1}v$. Then

$$c_T(p,A)=1+A\max_{\mathbb{T}^1}v=s_T(p,A).$$

Note c_T will never exceed s_T no matter how large A is. This is different from the cellular flow case.

The paper is organized as follows. In Section 2, we provide some straightforward comparison results based on inf-max formulas. In particular, we show that $s_T\leqslant c_T$ and $\lim_{A\to+\infty}\frac{s_T}{A}=\lim_{A\to+\infty}\frac{c_T}{A}$. In Section 3, we give the proof of Theorem 1.1 by estimating the travel time of the controlled characteristics in the Lagrangian representation of the G-equation, and the Lax formula [12] of the quadratically nonlinear F-equation (1.8). In order to establish the upper bound of $c_T(p,A)$, we prove an almost sharp estimate of $\sup_{\mathbb{R}^2}|DF|$ based on the (Eulerian) corrector equation (1.8) with V replaced by AV through very delicate analysis. In Section 4, we prove Theorem 1.2. Concluding remarks are in Section 5.

Assumption and Notations.

(1) Throughout this section, $A \geqslant 4$ and $|p| = 1$. Also, C, $\hat{C}$, $\tilde{C}$, C_1 and C_2 represent positive constants independent of A and p. Moreover, we set $d = s_l = 1$ and $f'(0) = \frac{1}{4}$. For convenience, we also use notations like $O(\Phi(A))$ which means $C\Phi(A)$. Here $\Phi(A)$ is a function of A.

(2) To simplify notations, we omit the dependence on p and write

$$\begin{cases} s_T(p, A) = \alpha_A, \\ \overline{H}(p, A) = \beta_A, \\ c_T(p, A) = \gamma_A. \end{cases}$$

(3) $\mathbb{T}^n$ is n-dimensional flat Torus. $f \in C^k(\mathbb{T}^n)$ means that $f \in C^k(\mathbb{R}^n)$ and is periodic. Sometimes we also identify $\mathbb{T}^n$ with the cube $[-\frac{1}{2}, \frac{1}{2}]^n$.

2. Some simple comparisons based on inf-max formulas

The following says that the G-equation model always predicts slower turbulent flame speeds than the Majda–Souganidis model.

Lemma 2.1.

$$\alpha_A(p) \leqslant \gamma_A(p) \leqslant \beta_A(p) + \frac{1}{4}.$$

Proof. The right inequality is obvious by choosing $\lambda = 1$ in (1.7). Let us prove the left inequality. Since $t^2 \geqslant t - \frac{1}{4}$, according to the inf-max formulas (1.11)–(1.12),

$$\beta_A(p) \geqslant \alpha_A(p) - \frac{1}{4}.$$

Combining with the degree 1 homogeneity of $\alpha_A(p)$ with respect to the p variable, the above lemma holds. □

The following theorem says that α_A/A, β_A/A and γ_A/A have the same asymptotic limit.

Theorem 2.1. *Given $p \in \mathbb{R}^n$. Denote*

$$c_p = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{\mathbb{T}^n} \{V(x) \cdot (p + D\phi)\}.$$

Then

$$\lim_{A \to +\infty} \frac{\alpha_A}{A} = \lim_{A \to +\infty} \frac{\beta_A}{A} = \lim_{A \to +\infty} \frac{\gamma_A}{A} = c_p.$$

In particular, G-equation and Majda–Souganidis models predict the bending effect simultaneously.

Proof. The proof is simple. Owing to the inf-max formula (1.11),

$$\frac{\alpha_A}{A} \geqslant c_p.$$

Now fix $\epsilon > 0$ and choose $\phi_\epsilon \in C^1(\mathbb{T}^n)$ such that

$$\max_{\mathbb{T}^n} \{V(x) \cdot (p + D\phi_\epsilon)\} \leqslant \epsilon + c_p.$$

Then

$$\frac{\alpha_A}{A} \leqslant \frac{1}{A} \max_{\mathbb{T}^n} |D\phi_\epsilon| + c_p + \epsilon.$$

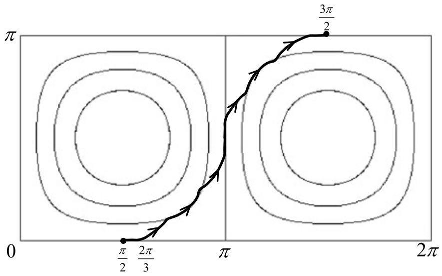

Fig. 2. An illustration of a controlled trajectory ξ traveling between two vortices of the cellular flow in the proof of Lemma 3.1.

Hence

$$\lim_{A\to+\infty}\frac{\alpha_A}{A}=c_p.$$

The proof for β_A is similar. The proof for γ_A follows immediately from Lemma 2.1. □

3. Proof of Theorem 1.1

Throughout this section, we assume that $V(x)=V(x_1,x_2)=(-H_{x_2},H_{x_1})$ for $H=\sin x_1\sin x_2$. Moreover, in the section, a function f is periodic if $f(x+2\pi\vec{k})=f(x)$ for any $\vec{k}\in\mathbb{Z}^2$. The cellular flow is written in the scaled form $A(x)$. The solution of Eq. (1.1) with initial data $u_0(x)$ is given by a control representation formula:

$$G(x,t)=\inf_{\xi\in\mathcal{W}_x}\bigl\{u_0\bigl(\xi(t)\bigr)\bigr\},$$

where $\mathcal{W}_x$ is the collection of all $\xi\in W^{1,\infty}([0,t];\mathbb{R}^n)$ such that $\xi(0)=x$ and:

$$\xi'(\tau)+A\,V\bigl(\xi(\tau)\bigr)=y(\tau),\ \forall y(\tau)\in L^\infty\bigl([0,t]\bigr),\quad |y|\leqslant 1,$$

where the function y is the dynamic control. The formula (3.13) is an extension of the well-known Lax formula [12,11,27] for strictly convex Lagrangian (L^q, $q>1$) to the L^∞ type Lagrangian due to L^1 type Hamiltonian in the G-equation [16,28,4,15]. We are going to use this formula to bound α_A. Due to the symmetry of the stream function and the inf-max formula (1.11),

$$\alpha_A(p)=\alpha_A(-p)=\lim_{t\to\infty}\frac{1}{t}\sup_{\xi\in\mathcal{W}_x}\bigl\{p\cdot\xi(t)\bigr\}. \tag{3.13}$$

Note that the limit is independent of the choice of x (i.e., the initial position $\xi(0)$). Moreover, the symmetry is just used for convenience and is not essentially necessary. We shall first estimate the travel time of a controlled trajectory ξ passing through the two vortices in cellular flow as shown in Fig. 2. The control $y(\tau)$ makes possible the passage of the ξ-trajectory around the saddle points $(\pi,0)$ and (π,π).

Lemma 3.1. *Given $A\geqslant 4$, there exist a positive constant T' and a Lipschitz continuous curve $\xi:[0,+\infty)\to\mathbb{R}^2$ such that*

(i)

$$\bigl|\dot{\xi}(t)+AV\bigl(\xi(t)\bigr)\bigr|\leqslant 1\quad a.e.;$$

(ii) $T'\leqslant\frac{\log A+C}{A}$ *and for* $k\in\mathbb{N}\cup\{0\}$

$$\xi\bigl(kT'\bigr)=\biggl(\frac{\pi}{2},0\biggr)+k\biggl(\frac{\pi}{2},\frac{\pi}{2}\biggr).$$

Here $C>0$ is a constant which is independent of A.

Proof. Throughout the proof, C represents a positive constant independent of A. Due to symmetry, it suffices to construct a path $\xi : [0, T'] \to \mathbb{R}^2$ such that $\xi(0) = (\frac{\pi}{2}, 0)$, $\xi(T') = (\pi, \frac{\pi}{2})$ (see Fig. 2), and

$$T' \leqslant \frac{\log A + C}{A}.$$

In order to save notations, $x_1(t)$ and $x_2(t)$ in the following different steps represent different curves.

Step 1. Let $\eta_1 : [0, \infty) \to \mathbb{R}$ satisfy $\eta(0) = \frac{\pi}{2}$ and $\dot\eta(t) = A \sin \eta + 1$. Suppose that $\eta_1(t_1) = \frac{2\pi}{3}$. Clearly

$$t_1 = \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{dx}{A \sin x + 1} \leqslant \frac{C}{A}.$$

Step 2. For $t \geqslant t_1$, let $\eta_2 : [t_1, \infty) \to \mathbb{R}^2$ satisfying $\eta_2(t_1) = (\frac{2\pi}{3}, 0)$ and

$$\dot\eta_2(t) = -AV\big(\eta_2(t)\big) + \frac{DH}{|DH|}.$$

Denote $L_1 = \min_{|x-(\frac{2\pi}{3},0)|\leqslant\frac{1}{2}} |DH(x)| > 0$, $L_2 = \min_{|x_2-\frac{\pi}{3}|\leqslant\frac{1}{2}|} |DH(x)| > 0$ and $L = \min\{L_1, L_2, 1\}$. Let t_2 be the moment when $H(\eta_2(t_2)) = \frac{L}{4A}$. Our claim is that $t_2 - t_1 \leqslant \frac{1}{4A}$. In fact, for $t \in [t_1, t_1 + \frac{1}{4A}]$,

$$\left|\eta_2(t) - \left(\frac{2\pi}{3}, 0\right)\right| \leqslant \frac{1}{2}$$

and $H(\eta_2(\frac{1}{4A})) \geqslant \frac{L}{4A}$. Hence our claim holds. Denote $\eta_2(t) = (x_1(t), x_2(t))$. Note that for $t \in [t_1, t_2]$

$$\dot x_1(t) \geqslant A \cos x_2(t) \sin x_1(t) - 1 \geqslant 4 \times \cos\frac{\pi}{6} \sin\frac{\pi}{6} - 1 > 0.$$

Hence

$$x_1(t_2) \in \left(\frac{2\pi}{3}, \pi\right). \tag{3.14}$$

Also it is clear that $x_2(t_2) \in (0, \frac{1}{2})$.

Step 3. For $t \geqslant t_2$, let $\eta_3 : [t_2, +\infty) \to \mathbb{R}^2$ satisfy $\eta_3(t_2) = \eta_2(t_2)$ and

$$\dot\eta_3(t) = -AV\big(\eta_3(t)\big).$$

Assume that $\eta_3(t) = (x_1(t), x_2(t))$. Choose t_3 to be the first time such that $\eta_3(t_3) \cdot e_2 = \frac{\pi}{3}$. Our claim is

$$t_3 \leqslant \frac{\log A + C}{A}.$$

In fact for $t \in [t_2, t_3]$,

$$\sin x_1(t) \sin x_2(t) = \lambda$$

for $\lambda = \frac{L}{4A}$, where L is the same as in Step 2. Since x_2 changes from $\frac{C}{A}$ to $\frac{\pi}{3}$ and $|\cos x_1(t)| \geqslant \frac{1}{2}$ for $t \in [t_2, t_3]$,

$$\begin{aligned} t_3 - t_2 &= \int_{C\lambda}^{\frac{\pi}{3}} \frac{1}{A \sin u \sqrt{1 - \frac{\lambda^2}{\sin^2 u}}}\, du \\ &\leqslant \int_{C\lambda}^{\pi} 31A \sin u\, du + \frac{C}{A^3} \int_{C\lambda}^{\frac{\pi}{3}} \frac{1}{u^3}\, du \\ &\leqslant \frac{C + \log A}{A}. \end{aligned}$$

Hence our claim holds.

Step 4. For $t \geqslant t_3$, define $\eta_4 : [t_3, +\infty) \to \mathbb{R}^2$ such that $\eta_4(t_3) = \eta_3(t_3)$ and

$$\dot{\eta}_4(t) = -AV\big(\eta_4(t)\big) - \frac{DH}{|DH|}.$$

Choose t_4 such that $H(\eta_4(t_4)) = 0$. We claim that $t_4 - t_3 \leqslant \frac{1}{4A}$. Again we denote $\eta_4(t) = (x_1(t), x_2(t))$. Note that for $t \in [t_3, t_3 + \frac{1}{4A}]$, $|x_2(t) - \frac{\pi}{3}| \leqslant \frac{1}{2}$ and $H(\eta_4(t_3 + \frac{1}{4A})) \leqslant H(\eta_4(t_3)) - \frac{L}{4A} = 0$. L is the same as in Step 2. Hence our claim holds. Also, it is clear that $x_1(t_4) = \pi$. Moreover, for $t \in [t_3, t_4]$, $\dot{x}_1(t) \geqslant 0$. Owing to (3.14) and Step 3, $\pi \geqslant x_1(t) > \frac{2\pi}{3}$ for $t \in [t_3, t_4]$. Therefore

$$\dot{x}_2(t) > A\frac{1}{2} \times \frac{1}{2} - 1 > 0.$$

Hence $x_2(t_4) \in (\frac{\pi}{3}, \frac{\pi}{3} + \frac{1}{2}) \subset (\frac{\pi}{3}, \frac{\pi}{2})$.

Step 5. For $t \geqslant t_4$, let $\eta_5 : [t_4, +\infty) \to \mathbb{R}^2$ be $\eta_5 = (0, x_2(t))$, $\eta_5(t_4) = \eta_4(t_4)$ and

$$\dot{x}_2(t) = A \sin x_2(t) + 1.$$

Choose t_5 be the first moment that $x_2(t_5) = \frac{\pi}{2}$. Clearly, $t_5 - t_4 \leqslant \frac{C}{A}$. Finally, we define that for $t \in [0, t_5]$

$$\xi = \begin{cases} \eta_1(t) & \text{for } 0 \leqslant t \leqslant t_1, \\ \eta_2(t) & \text{for } t_1 \leqslant t \leqslant t_2, \\ \eta_3(t) & \text{for } t_2 \leqslant t \leqslant t_3, \\ \eta_4(t) & \text{for } t_3 \leqslant t \leqslant t_4, \\ \eta_5(t) & \text{for } t_4 \leqslant t \leqslant t_5. \end{cases}$$

Then $t_5 \leqslant \frac{\log A + C}{A}$. $\square$

Estimate of $\boldsymbol{\beta_A}$. Next we will start to estimate β_A. Applying similar estimates in the proof of Lemma 3.1 to a closed streamline (i.e., closed level curves of H), we have:

Lemma 3.2. *Suppose that $\mathcal{L}$ is a closed streamline. Let ξ be the controlled trajectory $\dot{\xi} = \sqrt{A}V(\xi) + \frac{1}{A}\frac{V(\xi)}{|V|}$ and T the time that ξ travels through the whole $\mathcal{L}$. Then*

$$T \leqslant \frac{C \log A}{\sqrt{A}}, \qquad \int_0^T \big|\dot{\xi} - \sqrt{A}V(\xi)\big|^2(t)\,dt \leqslant \frac{T}{A^2}.$$

Here C is a positive constant which is independent of A.

Proof. We want to emphasize again that throughout the proof C represents a positive constant which is independent of A. We only need to prove that

$$T \leqslant \frac{C \log A}{\sqrt{A}}. \tag{3.15}$$

Without loss of generality, let us assume that $\mathcal{L}$ lies within the cell $[0, \pi] \times [0, \pi]$ and is the level curve $\{H = a\}$.

Case 1. $a \in [0, \frac{1}{2}]$. It suffices to establish (3.15) for $\mathcal{L}_1 = \mathcal{L} \cap [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \cap \{x_1 \geqslant x_2\}$. The other portions are similar. For the controlled trajectory $\xi = (x_1(t), x_2(t))$ in $\mathcal{L}_1$, we have that

$$\dot{x}_1(t) \leqslant -\sqrt{A} \sin x_1(t) \cos x_2(t) - \frac{1}{A\sqrt{2}} \leqslant -\frac{\sqrt{2A}}{2} \sin x_1(t) - \frac{1}{A\sqrt{2}}.$$

Hence the traveling time within $\mathcal{L}_1$ is no more than

$$\int_0^{\frac{\pi}{4}} \frac{1}{\frac{\sqrt{2A}}{2} \sin x_1 + \frac{1}{A\sqrt{2}}}\,dx_1 = \frac{C \log A}{\sqrt{A}}.$$

Case 2. $a \in [\frac{1}{2}, 1]$. Then

$$\left|\dot{\xi}(t)\right| \geqslant C\sqrt{A}\sqrt{1-a}.$$

Note that total length of $\mathcal{L}$ is $C\sqrt{1-a}$. Therefore, the traveling time is at most $\frac{C}{\sqrt{A}}$. □

Lemma 3.3. *Let a be a positive constant. Suppose that $|x-y| \leqslant a$. Then there exist $T > 0$ and $\eta \in W^{1,\infty}([0,T])$ satisfying $|\dot{\eta} - \sqrt{A}V(\eta(t))| \leqslant 1$ a.e., $\eta(0) = x$, $\eta(T) = y$ and $T \leqslant C(\frac{\log A}{\sqrt{A}} + a\log A)$. Also*

$$\int_0^T \left|\dot{\eta}(t) - \sqrt{A}V\big(\eta(t)\big)\right|^2 dt \leqslant Ca + \frac{T}{A^2}. \tag{3.16}$$

Here C is a positive constant depending only on V (i.e., independent of A, a, x and y).

Proof. Throughout the proof, positive constants C, C_1 and C_2 depend only on V (i.e, independent of A, a, x and y). Suppose $H(x) = s_1$ and $H(y) = s_2$. Without loss of generality, we may assume that x and y are both in the cell $[0,\pi] \times [0,\pi]$, $0 \leqslant s_1 \leqslant s_2 \leqslant 1$. Then $s_2 - s_1 \leqslant a$.

Case 1. Assume that $s_2 > s_1 \geqslant \sin\frac{\pi}{4}\sin\frac{\pi}{4} = \frac{1}{2}$. Define $\eta_1(t) : [0,\infty) \to \mathbb{R}^2$ as $\eta_1(0) = y$ and

$$\dot{\eta}_1(t) = AV\big(\eta_1(t)\big) - \frac{DH(\eta_1)}{|DH|}.$$

Denote t_1 as the first moment when $H(\eta_1(t_1)) = s_1$. I claim that

$$t_1 \leqslant C|y-x|.$$

In fact when $H \geqslant \frac{1}{2}$, there exists $0 < C_1 < C_2$ such that

$$C_1 \leqslant \frac{|DH|}{\sqrt{1-H}} \leqslant C_2.$$

Hence

$$\frac{d\sqrt{1-H(\eta_1(t))}}{dt} \leqslant -C_1 < 0$$

and

$$\begin{aligned}
|x-y| &\geqslant C_2\big(\sqrt{1-H(x)} - \sqrt{1-H(y)}\big) \\
&= C_2\big(\sqrt{1-s_1} - \sqrt{1-s_2}\big) \\
&\geqslant C_1C_2t_1.
\end{aligned}$$

Then we define $\eta_2(t) : [t_1, +\infty) \to +\mathbb{R}^2$ as $\eta_2(t_1) = \eta_1(t_1)$ and

$$\dot{\eta}_2(t) = \sqrt{A}V\big(\eta_2(t)\big).$$

Let t_2 be the first moment such that $\eta_2(t_2) = x$. By Case 2 in the proof of Lemma 3.2, $t_2 \leqslant \frac{C}{\sqrt{A}}$.

Case 2. $s_2 \leqslant \frac{1}{2}$. According to Lemma 3.2, it suffices to show that two streamlines $\{H = s_1\}$ and $\{H = s_2\}$ can be connected by a controlled curve within time $C(\frac{\log A}{\sqrt{A}} + a\log A)$. Let us define a controlled trajectory as $\eta(0) = y$ and

$$\dot{\eta}(t) = \sqrt{A}V\big(\eta(t)\big) + \alpha(\eta).$$

Here $a(\eta)$ satisfies:

$$a(\eta) = \begin{cases} -\frac{DH(\eta)}{|DH|} & \text{if } \eta \notin W, \\ \frac{1}{A}\frac{V(\eta)}{|V|} & \text{if } \eta \in W \end{cases}$$

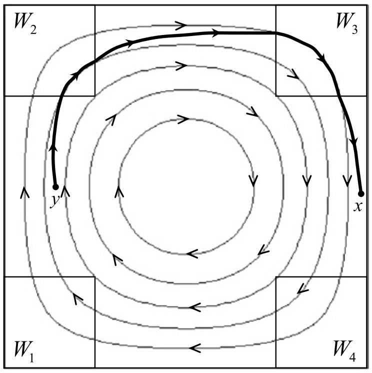

Fig. 3. An illustration of a controlled trajectory η connecting two points in the quarter cell (enclosing a single vortex) of the cellular flow in the proof of Lemma 3.3. The quarter cell (large square) containing the closed streamlines is $[0,\pi]^2$, the smaller squares in the four corners are of size $\pi/4$ by $\pi/4$.

where W is the union of four corners (see Fig. 3), i.e.,

$$W=\bigcup_{i=1}^{4} W_i.$$

Here $W_1=[0,\frac{\pi}{4}]\times[0,\frac{\pi}{4}]$, $W_2=[0,\frac{\pi}{4}]\times[\frac{3\pi}{4},\pi]$, $W_3=[\frac{3\pi}{4},\pi]\times[\frac{3\pi}{4},\pi]$ and $W_4=[\frac{3\pi}{4},\pi]\times[0,\frac{\pi}{4}]$. $A\geqslant 4$ will guarantee that either $\dot{x}_1(t)\neq 0$ or $\dot{x}_2(t)\neq 0$ depending regions between different W_i. Suppose that there are N times that the controlled trajectory η travels between corners before it reaches the streamline $\{H=s_1\}$. Denote t_i as the traveling time for $1\leqslant i\leqslant N$. Then

$$C\sum_{i=1}^{N} t_i\leqslant s_2-s_1.$$

For $2\leqslant i\leqslant N-1$, by considering the x_1 or x_2 components, we have that

$$t_i=O\left(\frac{1}{\sqrt{A}}\right).$$

Hence

$$N-2\leqslant C\sqrt{A}(s_2-s_1).$$

The traveling time of the control within each corner is at most $O(\frac{\log A}{\sqrt{A}})$. Therefore the total travel time is at most

$$C(s_2-s_1)+CN\frac{\log A}{\sqrt{A}}\leqslant C\left(a\log A+\frac{\log A}{\sqrt{A}}\right).$$

(3.16) follows easily from the construction of η. Combining Case 1 and 2, the lemma holds. □

Lemma 3.4. *Suppose that $F_A\in W^{1,\infty}(\mathbb{R}^2)$ is a periodic viscosity solution of (1.8) with $V(x)$ replaced by $AV(x)$, i.e.,*

$$|p+DF_A|^2+AV(x)\cdot(p+DF_A)=\beta_A. \tag{3.17}$$

Denote $\omega_A=\operatorname{esssup}_{\mathbb{T}^n}|p+DF_A|$ and $\tilde{\alpha}_A$ as the effective Hamiltonian of the following modified G-equation

$$\omega_A|p+D\tilde{G}|+AV(x)\cdot(p+D\tilde{G})=\tilde{\alpha}_A.$$

Then

$$\beta_A\leqslant\tilde{\alpha}_A.$$

Proof. Suppose that $\phi \in C^1(\mathbb{T}^n)$ and

$$\phi(x_0) - F_A(x_0) = \max_{\mathbb{R}^n}(\phi - F_A).$$

Then

$$\left|p + D\phi(x_0)\right|^2 + AV(x_0)\cdot\left(p + D\phi(x_0)\right) \geqslant \beta_A.$$

Also, it is easy to see that

$$\left|p + D\phi(x_0)\right| \leqslant \omega_A.$$

Hence

$$\omega_A\left|p + D\phi(x_0)\right| + AV(x_0)\cdot\left(p + D\phi(x_0)\right) \geqslant \beta_A.$$

So

$$\max_{\mathbb{R}^n}\left\{\omega_A|p + D\phi| + AV\cdot(p + D\phi)\right\} \geqslant \beta_A.$$

By the inf-max formula (1.11),

$$\tilde{\alpha}_A = \inf_{\phi\in C^1(\mathbb{T}^n)} \max_{\mathbb{R}^n}\left\{\omega_A|p + D\phi| + AV\cdot(p + D\phi)\right\} \geqslant \beta_A. \qquad \square$$

Lemma 3.5. *Suppose that $F_A \in W^{1,\infty}(\mathbb{R}^2)$ is a periodic viscosity solution of (3.17) and $\mathcal{M}$ is the set where F_A is differentiable. Then*

$$\sup_{x\in\mathcal{M}} |DF_A| \leqslant O(\sqrt{A}).$$

Proof. To simplify notations, we drop the A dependence and write $F = F_A$. Throughout this proof, C denotes a constant depending only on V. Choose F such that $\int_{(-\pi,\pi)\times(-\pi,\pi)} F\,dx = 0$. Denote $w(x) = \frac{p\cdot x+F}{A}$. Then

$$|Dw|^2 + V(x)\cdot Dw = \frac{\beta_A}{A^2}.$$

Choose $x_0 \in [-\pi,\pi]\times[-\pi,\pi]$ such that w is differentiable at x_0. Our goal is to show that

$$\left|DF(x_0)\right| \leqslant C\sqrt{A}.$$

Let $\xi(t)$, $t \in (-\infty, 0)$, be the backward characteristics with $\xi(0) = x_0$. Then

$$w(x_0) - w\left(\xi(t)\right) = t\frac{\beta_A}{A^2} + \frac{1}{4}\int_t^0 \left|\dot{\xi} - V(\xi)\right|^2 ds. \tag{3.18}$$

Since $Dw(x_0)$ exists, such ξ is unique. Also, ξ satisfies the Euler–Lagrange equation

$$\ddot{\xi} = DV(\xi)\cdot\dot{\xi} - (\dot{\xi} - V)\cdot DV(\xi) \tag{3.19}$$

and the equality

$$Dw(\xi) = \frac{\dot{\xi} - V(\xi)}{2}. \tag{3.20}$$

Note that the initial velocity $\xi(0)$ is determined. Accordingly, $|\dot{\xi}|, |\ddot{\xi}| \leqslant C$. Let us assume that

$$M = \max_{t\in[-1,0]}\left|\dot{\xi} - V(\xi)\right|.$$

Then owing to (3.19)

$$\min_{t\in[-1,0]}\left|\dot{\xi}(t) - V\left(\xi(t)\right)\right| \geqslant CM.$$

Due to the inf-max formula (1.12), it is clear that $\beta_A \leqslant CA$. Choosing $t = -1$ in (3.18), we deduce that

$$\frac{1}{4}\int_{-1}^{0} \left|\dot{\xi} - V(\xi)\right|^2 ds \leqslant \frac{C}{A} + w(x_0) - w\big(\xi(-1)\big).$$

Hence

$$M^2 \leqslant \frac{C}{A} + C\big(w(x_0) - w\big(\xi(-1)\big)\big). \tag{3.21}$$

Now we need to estimate $w(x_0) - w(\xi(-1))$. Let $\gamma(t): \mathbb{R} \to \mathbb{R}^2$ satisfy

$$\begin{cases} \dot{\gamma}(t) = V\big(\gamma(t)\big), \\ \gamma(-1) = \xi(-1). \end{cases}$$

Then

$$\frac{d|\xi(t) - \gamma(t)|}{dt} \leqslant C\left|\xi(t) - \gamma(t)\right| + M.$$

So

$$\left|\gamma(0) - \xi(0)\right| \leqslant CM.$$

Denote $U = \frac{p\cdot x + F}{\sqrt{A}} = \sqrt{A}w$. Then

$$|DU|^2 + \sqrt{A}V(x)\cdot DU = \frac{\beta_A}{A}.$$

Then

$$U\big(\xi(0)\big) - U\big(\xi(-1)\big) = U\big(\xi(0)\big) - U\big(\gamma(0)\big) + U\big(\gamma(0)\big) - U\big(\gamma(-1)\big).$$

Note for any Lipschitz continuous curve $s(t)$ and $t_1 \leqslant t_2$

$$U\big(s(t_2)\big) - U\big(s(t_1)\big) \leqslant \frac{(t_2 - t_1)\beta_A}{A} + \frac{1}{4}\int_{t_1}^{t_2} \left|\dot{s}(t) - \sqrt{A}V\big(s(t)\big)\right|^2 dt.$$

Then owing to Lemmas 3.2 and 3.3,

$$U\big(\xi(0)\big) - U\big(\gamma(0)\big) \leqslant C\left(\frac{\log A}{\sqrt{A}} + M\log A\right)\left(\frac{\beta_A}{A} + \frac{1}{A^2}\right) + CM.$$

Also by choosing $s(t) = \gamma(\sqrt{A}t)$,

$$U\big(\gamma(0)\big) - U\big(\gamma(-1)\big) \leqslant \frac{\beta_A}{A\sqrt{A}}.$$

Therefore

$$U(x_0) - U\big(\xi(-1)\big) \leqslant C\left(\frac{\log A}{\sqrt{A}} + M\log A\right)\left(\frac{\beta_A}{A} + \frac{1}{A^2}\right) + CM + \frac{\beta_A}{A\sqrt{A}}. \tag{3.22}$$

Now we claim

$$\beta_A \leqslant O\left(\frac{A}{\log A}\right).$$

In fact, since $\beta_A \leqslant CA$, by (3.21), (3.22) and $w = \frac{U}{\sqrt{A}}$,

$$M^2 \leqslant C\left(\frac{\log A}{A} + \frac{M\log A}{\sqrt{A}}\right).$$

Then $M \leqslant \frac{\log A}{\sqrt{A}}$ and

$$\sup_{\mathcal{M}} |DF| \leqslant O(\sqrt{A}\log A).$$

Therefore our claim follows from Lemma 3.4 and the known fact that $\alpha_A = O(\frac{A}{\log A})$. Using (3.21), (3.22) and $w = \frac{U}{\sqrt{A}}$ again,

$$M^2 \leqslant C\left(\frac{1}{A} + \frac{M}{\sqrt{A}}\right).$$

Hence $M \leqslant \frac{C}{\sqrt{A}}$ and the lemma holds. □

By looking at points where $V(x)$ vanishes and the inf-max formula, it is easy to see that

$$\sup_{x\in\mathcal{M}} |DF_A| \geqslant \sqrt{\beta_A} = O\left(\sqrt{\frac{A}{\log A}}\right).$$

It remains an open problem whether $\sup_{x\in\mathcal{M}} |DF_A| = O(\sqrt{\frac{A}{\log A}})$.

Lemma 3.6. *There exist positive constants $0 < C_1 \leqslant C_2$ independent of A and p such that for $A \geqslant 4$,*

$$\frac{A\pi(|p_1| + |p_2|)}{2\log A + C_2} \leqslant \alpha_A \leqslant \frac{A\pi(|p_1| + |p_2|)}{2\log A + C_1}. \tag{3.23}$$

There also exist positive constants $K_0 \geqslant 4$, K_1 and K_2 independent of A and p such that when $A \geqslant K_0$

$$\frac{A(|p_1| + |p_2|)\pi}{\log A + \log\log A + K_2} \leqslant \beta_A \leqslant \frac{A(|p_1| + |p_2|)\pi}{\log A - K_1}. \tag{3.24}$$

Proof. By the symmetry of cellular flow and the inf-max formulas (1.11)–(1.12),

$$\alpha_A(p_1, p_2) = \alpha_A(\pm p_1, \pm p_2) \quad \text{and} \quad \beta_A(p_1, p_2) = \beta_A(\pm p_1, \pm p_2).$$

Hence we may assume $p_1, p_2 \geqslant 0$. The symmetry is just used for convenience and is not essentially necessary. We first prove (3.23). The left inequality in (3.23) follows easily from Lemma 3.1 and the control formula (3.13) by choosing $\xi(0) = (\frac{\pi}{2}, 0)$. Now let us prove the right inequality by choosing $\xi(0) = (0, 0)$ in (3.13). Suppose $\xi(t) = (x_1(t), x_2(t)) \in \mathcal{W}$. Then for $i = 1, 2$

$$\dot{x}_i(t) \leqslant A|\sin x_i| + 1.$$

Without loss of generality, we assume that $x_1(T) \geqslant x_2(T)$ and $x_1(T) \in (\frac{k\pi}{2}, \frac{(k+1)\pi}{2}]$. Then we have that

$$T \geqslant k\int_0^{\frac{\pi}{2}} \frac{1}{A\sin x + 1}\,dx \geqslant k\frac{\log A + \log\frac{\pi}{2}}{A}.$$

This immediately leads to the right inequality.

Next we verify (3.24). The right inequality of (3.24) follows easily from Lemmas 3.4 and 3.5. In fact, using same notations as in the statement of Lemma 3.4, we write

$$\omega_A = \operatorname{esssup}_{\mathbb{T}^2} |p + DF_A|.$$

Owing to Lemma 3.5, $\omega_A \leqslant C\sqrt{A}$. Choose A large enough such that $\frac{A}{\omega_A} \geqslant 4$. Then due to (3.23) and Lemma 3.4,

$$\beta_A \leqslant \tilde{\alpha}_A = \omega_A \alpha_{\frac{A}{\omega_A}} \leqslant \frac{A(|p_1| + |p_2|)\pi}{2\log\frac{A}{\omega_A} + C_1} \leqslant \frac{A(|p_1| + |p_2|)\pi}{\log A - K_1}$$

for some positive constant K_1.

Now we prove the left inequality. It is well known that

$$\beta_A(p) = -\lim_{t\to+\infty} \frac{F(x,t)}{t},$$

where $F(x,t)$ is the solution of equation

$$\begin{cases} F_t + |DF|^2 + AV(x)\cdot DF = 0, \\ F(x,0) = p\cdot x. \end{cases}$$

Since $\beta_A(p) = \beta_A(-p)$, according to the Lax formula [12,11,27]:

$$\beta_A(p) = \beta_A(-p) = \lim_{t\to+\infty} \frac{1}{t}\sup_{\xi}\left(p\cdot\xi(t) - \frac{1}{4}\int_0^t \left|\dot{\xi}(s) + AV(\xi)\right|^2 ds\right), \tag{3.25}$$

where ξ runs over all Lipschitz continuous curve $\xi : [0,t] \to \mathbb{R}^2$ satisfying $\xi(0) = x$. The limit does not depend on the choice of x. Again, the symmetry is used here just for convenience and is not essentially necessary. Hence to prove the left part, it suffices to construct a Lipschitz continuous curve $\xi : [0,\infty) \to \mathbb{R}^2$ such that $\xi(0) = (\frac{\pi}{2}, 0)$ and there exists a positive constant T satisfying

(i) for all $k \in \mathbb{N}$

$$\xi(kT) = \left(\frac{\pi}{2}, 0\right) + k\left(\frac{\pi}{2}, \frac{\pi}{2}\right);$$

(ii)

$$T = \frac{\log A + \log\log A + C}{2A};$$

(iii) Moreover, the traveling cost

$$\int_0^{kT} \left|\dot{\xi} + AV\big(\xi(t)\big)\right|^2 dt \leqslant \frac{kC}{\log A}.$$

In fact, by (3.25) and (i)–(iii), we have that

$$\beta_A \geqslant \frac{A(p_1+p_2)\pi}{\log A + \log\log A + C} - \frac{\tilde{C}A}{\log^2 A}.$$

Here both C and $\tilde{C}$ are positive constants independent of A and p. Then when A is large enough,

$$\beta_A \geqslant \frac{A(p_1+p_2)\pi}{\log A + \log\log A + C + \tilde{C}}.$$

Now let us start to construct such a ξ. Owing to the symmetry, it suffices to construct a Lipschitz continuous curve $\xi : [0,\infty) \to \mathbb{R}^2$ such that $\xi(0) = (\frac{\pi}{2}, 0)$, $\xi(T) = (\frac{\pi}{2}, \frac{\pi}{2})$,

$$T = \frac{\log A + \log\log A + C}{2A}$$

with the traveling cost of:

$$\int_0^T \left|\dot{\xi} + A\,V(\xi)\right|^2 dt \leqslant \frac{C}{\log A}.$$

The shape of ξ is similar to that in the proof of Lemma 3.1, shown in Fig. 2. We define ξ as follows:

$$\xi = \begin{cases} \tilde{\eta}_1(t) & \text{for } 0 \leqslant t \leqslant s_1, \\ \tilde{\eta}_2(t) & \text{for } s_1 \leqslant t \leqslant s_2, \\ \tilde{\eta}_3(t) & \text{for } s_2 \leqslant t \leqslant s_3, \\ \tilde{\eta}_4(t) & \text{for } s_3 \leqslant t \leqslant s_4, \\ \tilde{\eta}_5(t) & \text{for } s_4 \leqslant t \leqslant s_5. \end{cases}$$

To save notations, $x_1(t)$ and $x_2(t)$ in the following different steps represent different curves.

Step 6. Definition of $\tilde{\eta}_1 = (x_1(t), 0)$. Let $x_1(t) : [0, +\infty) \to \mathbb{R}$ satisfy $x_1(0) = \frac{\pi}{2}$ and

$$\dot{x}_1(t) = A \sin x_1(t).$$

Let s_1 be the moment when $x_1(s_1) = \frac{2\pi}{3}$. Clearly, $s_1 \leqslant \frac{C}{A}$ and the cost is 0.

Step 7. Definition of $\tilde{\eta}_2 : [s_1, +\infty) \to \mathbb{R}^2$. Let $\tilde{\eta}_2(s_1) = (\frac{2\pi}{3}, 0)$ and

$$\dot{\tilde{\eta}}_2(t) = -AV\big(\xi(t)\big) + \frac{\sqrt{A}}{\sqrt{\log A}} \frac{DH}{|DH|}.$$

Now let $s_2 = s_1 + \frac{1}{8A}$. Then it is clear that for $s \in [s_1, s_2]$

$$\left|\eta_2(s) - \left(\frac{2\pi}{3}, 0\right)\right| < \frac{1}{2} \tag{3.26}$$

and for some $\hat{C} \in [\min_{\{|x-(\frac{2\pi}{3},0)| \leqslant \frac{1}{2}\}} |DH|, \max_{\{|x-(\frac{2\pi}{3},0)| \leqslant \frac{1}{2}\}} |DH|]$,

$$H\big(\eta_2(s_2)\big) = \frac{\hat{C}}{8\sqrt{A}\sqrt{\log A}}. \tag{3.27}$$

Denote $\tilde{\eta}_2 = (x_1(t), x_2(t))$. Owing to (3.26), for $t \in [s_1, s_2]$

$$\dot{x}_1(t) \geqslant A \cos\frac{\pi}{6} \sin\frac{\pi}{6} - \sqrt{A} > 0.$$

Hence

$$x_1(s_2) \in \left(\frac{2\pi}{3}, \frac{5\pi}{6}\right). \tag{3.28}$$

The cost is

$$\int_{s_1}^{s_2} \left|\dot{\tilde{\eta}}_2 + AV(\tilde{\eta}_2)\right|^2(t)\,dt \leqslant \frac{C}{\log A}.$$

Step 8. Definition of $\tilde{\eta}_3 = (x_1(t), x_2(t)) : [s_2, +\infty) \to \mathbb{R}^2$. Let $\tilde{\eta}_3(s_2) = \tilde{\eta}_2(s_2)$ and

$$\dot{\tilde{\eta}}_3(t) = -AV(\tilde{\eta}_3).$$

Let s_3 be the moment when $x_2(s_3) = \frac{\pi}{3}$. Then for $t \in (s_2, s_3)$, we have:

$$\sin x_1(t) \sin x_2(t) = \lambda$$

for $\lambda = \frac{\hat{C}}{8\sqrt{A}\sqrt{\log A}}$, where $\hat{C}$ is the same as in (3.27). Therefore

$$\dot{x}_2(t) = -A \sin x_2(t) \cos x_1(t) = A \sin x_2(t) \sqrt{1 - \frac{\lambda^2}{\sin^2 x_2}}.$$

Due to (3.28), for some $C>0$, $x_2(t)$ changes from $C\lambda$ to $\frac{\pi}{3}$ as t increases from s_2 to s_3. Since $|\cos x_1(t)|\geqslant \frac{1}{2}$,

$$\begin{aligned} s_3-s_2&=\int_{C\lambda}^{\frac{\pi}{3}}\frac{1}{A\sin u\sqrt{1-\frac{\lambda^2}{\sin^2 u}}}\,du\\ &\leqslant \int_{C\lambda}^{\frac{\pi}{3}}\frac{1}{A\sin u}\,du+C\frac{\lambda^2}{A}\int_{C\lambda}^{\frac{\pi}{3}}\frac{1}{u^3}\,du\\ &\leqslant \frac{\frac{1}{2}\log A+\frac{1}{2}\log\log A+C}{A}. \end{aligned}$$

The cost is 0.

Step 9. Definition of $\tilde\eta_4=(x_1(t),x_2(t)):[s_3,+\infty)\to\mathbb{R}^2$. Let $\tilde\eta_4(s_3)=\eta_3(s_3)$ and

$$\dot{\tilde\eta}_4(t)=-AV(\tilde\eta_4)-k\frac{\sqrt{A}}{\sqrt{\log A}}\frac{DH}{|DH|}.$$

Denote $L=\min_{|x_2-\frac{\pi}{3}|\leqslant\frac{1}{2}}|DH|$. Choose k such that

$$k=\frac{\hat C}{L},$$

where the $\hat C$ is same as that in (3.27). Let A be large enough such that

$$\frac{1}{4}+\frac{k}{8\sqrt{A\log A}}\leqslant\frac{1}{2}.$$

Let s_4 be the first moment such that $H(\tilde\eta_4)=0$. Note that

$$\left|x_2\left(\frac{1}{8A}\right)-\frac{\pi}{3}\right|<\frac{1}{2}$$

and

$$H\left(\tilde\eta\left(\frac{1}{8A}\right)\right)\leqslant 0.$$

Hence

$$s_4-s_3\leqslant\frac{1}{8A}$$

and for $t\in[s_3,s_4]$

$$\left|x_2(t)-\frac{\pi}{3}\right|\leqslant\frac{1}{2}.$$

Moreover, $\dot x_1(t)\geqslant 0$. Therefore owing to (3.28) and Step 3, $x_1(t)\in(\frac{2\pi}{3},\pi)$ for $t\in(s_3,s_4)$. Accordingly, when A is large,

$$\dot x_2(t)\geqslant -A\cos\frac{2\pi}{3}\sin\frac{\pi}{6}-k\sqrt{A}>0.$$

So

$$x_2(s_4)\in\left(\frac{\pi}{3},\frac{\pi}{3}+\frac{1}{2}\right)\subset\left(\frac{\pi}{3},\frac{\pi}{2}\right).$$

Moreover, the cost is

$$\int_{s_3}^{s_4} \left|\dot{\tilde{\eta}}_4 + AV(\tilde{\eta}_4)\right|^2 dt \leqslant \frac{C}{\log A}.$$

Step 10. Definition of $\tilde{\eta}_5 = (0, x_2(t)) : [s_4, +\infty) \to \mathbb{R}^2$. Let $\tilde{\eta}_5(s_4) = \tilde{\eta}_4(s_4)$ and

$$\dot{x}_2(t) = A \sin x_2(t).$$

Let s_5 be the moment when $x_2(s_5) = \frac{\pi}{2}$. It is clear that $s_5 - s_4 \leqslant O(\frac{1}{A})$ and the cost is 0.

Conclusion. The total time is

$$s_5 = \frac{\log A + \log\log A + C}{2A}$$

and the total cost is

$$\int_{0}^{s_5} \left|\dot{\xi} + AV(\xi)\right|^2 dt \leqslant \frac{C}{\log A}.$$

The lemma is proved. □

Proof of Theorem 1.1. By the symmetry of cellular flow, we may assume $p_1, p_2 \geqslant 0$. (1.9) has been proved in previous lemma. We only need to establish (1.10). Recall that C denotes a positive constant independent of A and p. According to (1.7), it is very easy to see that

$$\gamma_A = \inf_{\lambda > 0} \left\{ \frac{1}{4\lambda} + \lambda \beta_{\frac{A}{\lambda}} \right\}.$$

Denote $\hat{\lambda} = \frac{\log^2 A}{2A}$. Choose A large enough $\frac{2A^2}{\log^2 A} \geqslant K_0$. Here K_0 is the same constant in the statement of Lemma 3.6. Then $\frac{A}{\hat{\lambda}} \geqslant K_0$,

$$\gamma_A \leqslant \frac{1}{4\hat{\lambda}} + \hat{\lambda} \beta_{\frac{A}{\hat{\lambda}}}$$

and the right inequality of (1.10) follows from Lemma 3.6.

As for the left inequality, suppose that

$$\gamma_A = \frac{1}{4\bar{\lambda}} + \bar{\lambda} \beta_{\frac{A}{\bar{\lambda}}}$$

for some $\bar{\lambda} > 0$.

Case 1. $\bar{\lambda} \geqslant \frac{A}{K_0}$. Since $\beta_A(p) \geqslant |p|^2 = 1$,

$$\gamma_A \geqslant \bar{\lambda} \geqslant \frac{A}{K_0}.$$

Case 2. $\bar{\lambda} < \frac{A}{K_0}$. Denote

$$h(\lambda) = \frac{1}{4\lambda} + \frac{A\pi(|p_1| + |p_2|)}{\log \frac{A}{\lambda} + \log\log \frac{A}{\lambda} + K_2}.$$

Here the constant K_2 is the same as in the statement of Lemma 3.6. Then

$$\gamma_A \geqslant \min_{\lambda \in (0, \frac{A}{K_0}]} h(\lambda),$$

where the minimum is attained, $h(\lambda_0) = \min_{\lambda \in (0, \frac{A}{K_0}]} h(\lambda)$ for some $\lambda_0 \in (0, \frac{A}{K_0}]$.

Case 2.1. If $\lambda_0 = \frac{A}{K_0}$, then

$$\gamma_A \geqslant h\left(\frac{A}{K_0}\right) \geqslant CA.$$

Case 2.2. Now let us assume $\lambda_0 < \frac{A}{K_0}$. Then

$$h'(\lambda_0) = 0.$$

Accordingly,

$$\frac{1}{4\lambda_0} = \frac{A(|p_1| + |p_2|)\pi}{(\log\frac{A}{\lambda_0} + \log\log\frac{A}{\lambda_0} + K_2)^2}\left(1 + \frac{1}{\log A - \log\lambda_0}\right).$$

Let $\mu = \frac{A}{\lambda_0}\log\frac{A}{\lambda_0} > K_0 > 4$. Then

$$\mu \leqslant \frac{CA^2}{\log\mu}. \tag{3.29}$$

This implies that

$$\mu \leqslant C\frac{A^2}{\log A}.$$

Therefore

$$h(\lambda_0) \geqslant \frac{A(|p_1| + |p_2|)\pi}{\log\mu + K_2} \geqslant \frac{A(|p_1| + |p_2|)\pi}{2\log A - \log\log A + C}.$$

Combining all cases, we get that when A is sufficiently large

$$\gamma_A \geqslant \frac{A(|p_1| + |p_2|)\pi}{2\log A - \log\log A + C}. \qquad \square$$

4. Proof of Theorem 1.2

Throughout this section, V is a general periodic n-dimensional incompressible flow. We first prove several lemmas.

Lemma 4.1. *Let $Q_n = [0, 1]^n$ and $f \in W^{1,\infty}(Q_n)$. Assume that*

$$\|f\|_{W^{1,\infty}(Q_n)} \leqslant 1.$$

Then there exist two non-negative constants $\mu_n \in (0, 1]$ and C_n which depend only on n such that

$$\|f\|_{L^\infty(Q_n)} \leqslant C_n\|f\|^{\mu_n}_{H^1(Q_n)}. \tag{4.1}$$

Proof. We prove by induction. When $n = 1$, (4.1) holds obviously by choosing $\mu_1 = 1$ and $C_1 = 2$. Now assume it holds when $n = m - 1$. For $\epsilon \in (0, 1)$ and $x' \in Q_{m-1}$, we define

$$g(x') = \frac{1}{\epsilon}\int_0^\epsilon f(x', s)\,ds.$$

Then $g \in W^{1,\infty}(Q_{m-1})$ and $\|g\|_{W^{1,\infty}(Q_{m-1})} \leqslant 1$. Hence by induction and Cauchy's inequality there exist C_{m-1} and μ_{m-1} such that

$$\|g\|_{L^\infty(Q_{m-1})} \leqslant C_{m-1}\|g\|^{\mu_{m-1}}_{H^1(Q_{m-1})} \leqslant \frac{C_{m-1}}{(\sqrt{\epsilon})^{\mu_{m-1}}}\|f\|^{\mu_{m-1}}_{H^1(Q_m)}.$$

Accordingly,

$$\left|f\left(x',0\right)\right| \leqslant \inf_{\epsilon\in(0,1)}\left\{\left|g\left(x'\right)\right|+\epsilon\right\} \leqslant C_m\|f\|_{H^1(Q_m)}^{\mu_m}$$

for some $C_m > 0$ and $\mu_m = \frac{2\mu_{m-1}}{2+\mu_{m-1}}$. (4.1) follows by translation. □

The following is a rough analogue of Lemma 3.5 for general n-dimensional incompressible flows.

Lemma 4.2. *Suppose that F_A is a periodic viscosity solution of (1.8) with $V(x)$ replaced by $AV(x)$ and $\mathcal{M} \in \mathbb{R}^n$ is the set where F_A is differentiable. Then there exists $\theta \in (0,1)$ which depends only on n and $\max_{\mathbb{R}^n}|V|$ such that*

$$\sup_{x\in\mathcal{M}}|DF_A| \leqslant O\left(A^{1-\theta}\right).$$

Proof. To simplify notations, we drop the A dependence and write $F = F_A$. Throughout this proof, C denotes a constant depending only on V and $|p|$. Choose F such that $\int_{\mathbb{T}^n} F\,dx = 0$ for $\mathbb{T}^n = [-\frac{1}{2},\frac{1}{2}]^n$. Denote $w(x) = \frac{p\cdot x+F}{A}$. Then

$$|Dw|^2 + V(x)\cdot Dw = \frac{\beta_A}{A^2}.$$

Clearly, $||w||_{W^{1,\infty}(\mathbb{T}^n)} \leqslant C$. Also, $||Dw||^2_{L^2(\mathbb{T}^n)} = \frac{\beta_A}{A^2} + \frac{\int_{\mathbb{T}^n} p\cdot V\,dx}{A} \leqslant \frac{C}{A}$. Then by Lemma 4.1 and Poincaré inequality

$$\|w\|_{L^\infty(\mathbb{T}^n)} \leqslant CA^{-\epsilon}$$

for some positive constant $\epsilon \in (0,1]$ independent of A. Choose $x_0 \in \mathbb{T}^n$ such that w is differentiable at x_0. Our goal is to show that

$$\left|DF(x_0)\right| \leqslant CA^{1-\frac{\epsilon}{2}}.$$

Let $\xi(t)$, $t \in (-\infty,0)$, be the backward characteristics with $\xi(0) = x_0$. Then

$$w(x_0) - w\left(\xi(t)\right) = t\frac{\beta_A}{A^2} + \frac{1}{4}\int_t^0 \left|\dot{\xi} - V(\xi)\right|^2 ds. \tag{4.2}$$

Also, ξ satisfies the Euler–Lagrange equation

$$\ddot{\xi} = DV(\xi)\cdot\dot{\xi} - (\dot{\xi} - V)\cdot DV(\xi) \tag{4.3}$$

and the equality

$$Dw(\xi) = \frac{\dot{\xi} - V(\xi)}{2}. \tag{4.4}$$

Accordingly, $|\dot{\xi}|, |\ddot{\xi}| \leqslant C$. Let us assume that

$$M = \max_{t\in[-1,0]}\left|\dot{\xi} - V(\xi)\right|.$$

Then owing to (4.3)

$$\min_{t\in[-1,0]}\left|\dot{\xi}(t) - V\left(\xi(t)\right)\right| \geqslant CM.$$

Due to the inf-max formula (1.12), it is clear that $\beta_A \leqslant CA$. Choosing $t = -1$ in (4.2), we deduce that

$$\frac{1}{4}\int_{-1}^0 \left|\dot{\xi} - V(\xi)\right|^2 ds \leqslant \frac{C}{A} + w(x_0) - w\left(\xi(-1)\right) \leqslant \frac{C}{A^\epsilon}.$$

Hence $M \leqslant \frac{C}{A^{\frac{\epsilon}{2}}}$. Owing to (4.4), $|DF(x_0)| \leqslant CA^{1-\frac{\epsilon}{2}}$. □

Proof of Theorem 1.2. Since $\gamma_A \geqslant \alpha_A$, it suffices to show that $\gamma_A \leqslant O(\frac{A}{\log A})$. Using same notations as in the statement of Lemma 3.4, we write

$$\omega_A = \operatorname{esssup}_{\mathbb{T}^n} |p + DF_A|.$$

Owing to Lemma 4.2, $\omega_A \leqslant CA^{1-\theta}$. Due to Lemma 3.4,

$$\beta_A \leqslant \tilde{\alpha}_A = \omega_A \alpha_{\frac{A}{\omega_A}} \leqslant \frac{CA}{\log A}.$$

The conclusion $\gamma_A \leqslant O(\frac{A}{\log A})$ follows immediately from Lemma 2.1. □

5. Concluding remarks

The sharp asymptotic growth laws have been established for the turbulent flame speeds in inviscid Hamilton–Jacobi models with L^1 and L^2 type nonlinearities and cellular flows. In the regime of large flow amplitude, the growth laws differ by a double logarithm correction while showing weak bending (slightly sublinear growth). Our future work shall consider the growth laws of turbulent flame speeds in more complex flows such as time-dependent two-dimensional incompressible flows and three-dimensional steady flows.

References

[1] M. Abel, M. Cencini, D. Vergni, A. Vulpiani, Front speed enhancement in cellular flows, Chaos 12 (2002) 481–488.
[2] B. Audoly, H. Berestycki, Y. Pomeau, Réaction diffusion en écoulement stationnaire rapide, C. R. Acad. Sci. Paris 328 (IIb) (2000) 255–262.
[3] H. Berestycki, F. Hamel, Front propagation in periodic excitable media, Comm. Pure Appl. Math. 60 (2002) 949–1032.
[4] P. Cardaliaguet, J. Nolen, P.E. Souganidis, Homogenization and enhancement for the G-equation, Arch. Ration. Mech. Anal. 199 (2) (2011) 527–561.
[5] P. Cardaliaguet, P.E. Souganidis, Homogenization and enhancement the G-equation in random environments, Comm. Pure Appl. Math., in press.
[6] S. Childress, A.M. Soward, Scalar transport and alpha-effect for a family of cat's-eye flows, J. Fluid Mech. 205 (1989) 99–133.
[7] P. Clavin, F. Williams, Theory of premixed-flame propagation in large-scale turbulence, J. Fluid Mech. 90 (1979) 598–604.
[8] P. Constantin, A. Kiselev, A. Oberman, L. Ryzhik, Bulk burning rate in passive–reactive diffusion, Arch. Ration. Mech. Anal. 154 (2000) 53–91.
[9] A. Fannjiang, G. Papanicolaou, Convection enhanced diffusion for periodic flows, SIAM J. Appl. Math. 54 (1992) 333–408.
[10] P. Embid, A. Majda, P. Souganidis, Comparison of turbulent flame speeds from complete averaging and the G-equation, Phys. Fluids 7 (8) (1995) 2052–2060.
[11] L.C. Evans, Partial Differential Equations, Graduate Studies in Mathematics, AMS, Providence, RI, 1998.
[12] P.D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 11, SIAM, Philadelphia, 1973.
[13] A. Majda, P. Souganidis, Large scale front dynamics for turbulent reaction–diffusion equations with separated velocity scales, Nonlinearity 7 (1994) 1–30.
[14] A. Majda, P. Souganidis, Flame fronts in a turbulent combustion model with fractal velocity fields, Comm. Pure Appl. Math. 51 (1998) 1337–1348.
[15] J. Nolen, A. Novikov, Homogenization of the G-equation with incompressible random drift in two dimensions, Commun. Math. Sci. 9 (2) (2011) 561–582.
[16] J. Nolen, J. Xin, Asymptotic spreading of KPP reactive fronts in incompressible space–time random flows, Ann. Inst. Henri Poincaré, Anal. Non Lineaire 26 (3) (May–June 2009) 815–839.
[17] J. Nolen, J. Xin, Y. Yu, Bounds on front speeds for inviscid and viscous G-equations, Methods Appl. Anal. 16 (4) (2009) 507–520.
[18] A. Novikov, L. Ryzhik, Boundary layers and KPP fronts in a cellular flow, Arch. Ration. Mech. Anal. 184 (1) (2007) 23–48.
[19] A. Oberman, Ph.D thesis, University of Chicago, 2001.
[20] S. Osher, R. Fedkiw, Level Set Methods and Dynamic Implicit Surfaces, Applied Mathematical Sciences, vol. 153, Springer, New York, 2003.
[21] N. Peters, Turbulent Combustion, Cambridge University Press, Cambridge, 2000.
[22] P. Ronney, Some open issues in premixed turbulent combustion, in: J.D. Buckmaster, T. Takeno (Eds.), Modeling in Combustion Science, in: Lecture Notes in Physics, vol. 449, Springer-Verlag, Berlin, 1995, pp. 3–22.
[23] G. Sivashinsky, Cascade-renormalization theory of turbulent flame speed, Combust. Sci. Techol. 62 (1988) 77–96.
[24] G. Sivashinsky, Renormalization concept of turbulent flame speed, in: Lecture Notes in Physics, vol. 351, 1989.
[25] F. Williams, Turbulent combustion, in: J. Buckmaster (Ed.), The Mathematics of Combustion, SIAM, Philadelphia, 1985, pp. 97–131.
[26] J. Xin, Front propagation in heterogeneous media, SIAM Rev. 42 (2) (June 2000) 161–230.
[27] J. Xin, An Introduction to Fronts in Random Media, Surveys and Tutorials in the Applied Mathematical Sciences, vol. 5, Springer, 2009.
[28] J. Xin, Y. Yu, Periodic homogenization of inviscid G-equation for incompressible flows, Commun. Math. Sci. 8 (4) (2010) 1067–1078.
[29] V. Yakhot, Propagation velocity of premixed turbulent flames, Combust. Sci. Techol. 60 (1988) 191–241.