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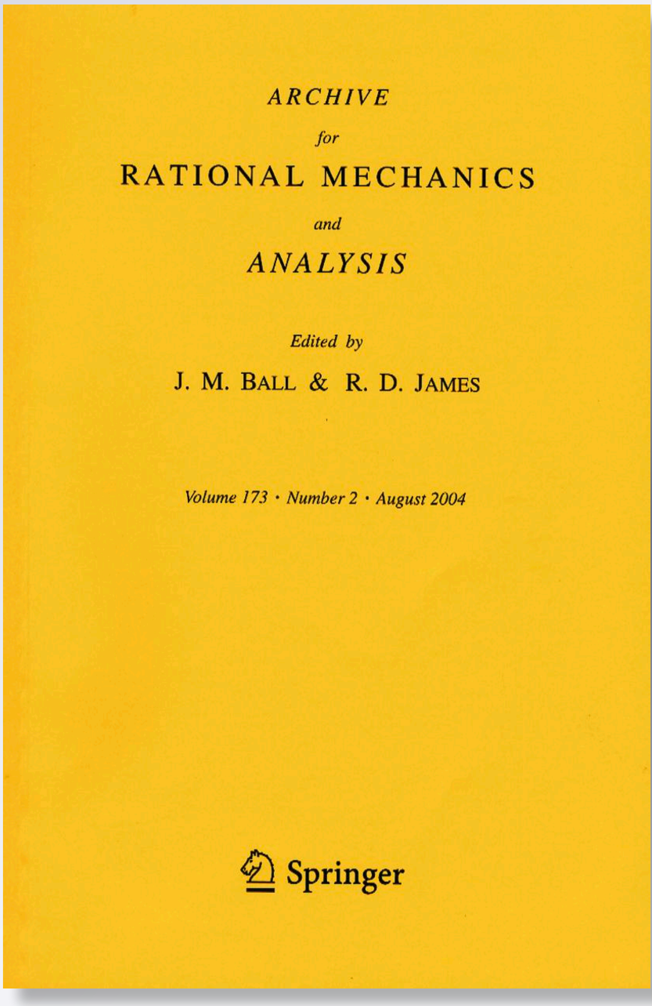

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Asymptotics for Turbulent Flame Speeds of the Viscous G-Equation Enhanced by Cellular and Shear Flows

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Abstract

G-equations are well-known front propagation models in turbulent combustion which describe the front motion law in the form of local normal velocity equal to a constant (laminar speed) plus the normal projection of fluid velocity. In level set formulation, G-equations are Hamilton–Jacobi equations with convex (L^1 type) but non-coercive Hamiltonians. Viscous G-equations arise from either numerical approximations or regularizations by small diffusion. The nonlinear eigenvalue $\bar{H}$ from the cell problem of the viscous G-equation can be viewed as an approximation of the inviscid turbulent flame speed s_T. An important problem in turbulent combustion theory is to study properties of s_T, in particular how s_T depends on the flow amplitude A. In this paper, we study the behavior of $\bar{H} = \bar{H}(A, d)$ as $A \to +\infty$ at any fixed diffusion constant $d > 0$. For cellular flow, we show that

$$\bar{H}(A, d) \leqq C(d) \quad \text{for all } d > 0,$$

where $C(d)$ is a constant depending on d, but independent of A. Compared with $\bar{H}(A, 0) = O(A/\log A)$, $A \gg 1$, of the inviscid G-equation ($d = 0$), presence of diffusion dramatically slows down front propagation. For shear flow, $\lim_{A\to+\infty} \frac{\bar{H}(A,d)}{A} = \lambda(d) > 0$ where $\lambda(d)$ is strictly decreasing in d, and has zero derivative at $d = 0$. The linear growth law is also valid for s_T of the curvature dependent G-equation in shear flows.

1. Introduction

The G-equation has been a very popular field model in combustion and physics literature for studying premixed turbulent flame propagation [1,5,6,9,14,19–21,23–26,28]. The inviscid G-equation on a flame moving in a steady flow has the following form:

$$G_t + V(x) \cdot DG + s_l |DG| = 0, \tag{1.1}$$

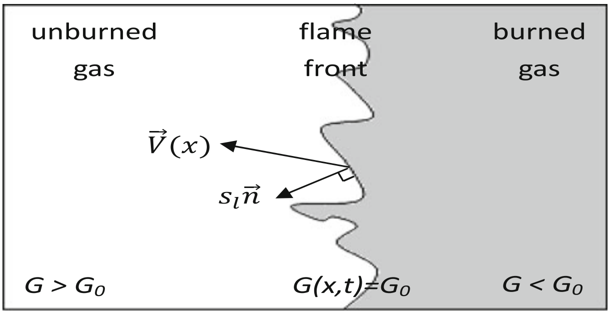

Fig. 1. Illustration of local interface velocities in the G-equation and a flame front

where G is the level set function of the flame, V is the ambient fluid velocity field, and the positive constant s_l is called *laminar flame speed*. The constant s_l describes how quickly the flame propagates when the fluid is at rest. The G-equation can be derived through the level set method based on a simple motion law: the flame propagation speed along the normal direction is equivalent to s_l plus the normal projection of the fluid velocity (see Fig. 1). The level set $\{(x,t) : G(x,t) = G_0\}$ of the solution $G = G(x,t)$ represents the flame front at time t. We assume that the flow field V is periodic and incompressible.

Suppose that the initial flame front is planar and the flame is propagating in the direction P $(G(x,0) = P \cdot x)$ with $|P| = 1$. Due to the movement of the fluid, the flame front will be wrinkled in time. Eventually, the front will evolve into an asymptotic state moving at a constant speed s_T which depends on P; this is called "turbulent flame speed" in combustion literature. It can be computed as $s_T = -\lim_{T\to+\infty} \frac{G(x,T)}{T}$. The s_T is conjectured to exist even when V is stochastic and is used in the combustion community to describe the average speed of a fluctuating front [20]. To predict and analyze properties of s_T is a fundamental problem in turbulent combustion theory. When V is periodic in space, the s_T can be studied in the framework of the periodic homogenization theory of the Hamilton–Jacobi equation [7,12]. It is same as the effective Hamiltonian $\bar{H}(P)$ of a nonlinear eigenvalue problem (so called cell problem):

$$s_l|P + Dw| + V(x) \cdot (P + Dw) = \bar{H}(P). \tag{1.2}$$

Due to the lack of coercivity of the Hamiltonian of the G-equation, the periodic homogenization and the existence of $\bar{H}(P)$ have been rigorously established only very recently by two of the authors [27] and CARDALIAGUET ET AL. [4], independently. When $n = 2$, NOLEN and NOVIKOV [15] proved the existence of $\bar{H}(P)$ for stationary ergodic flows.

In computation of the hyperbolic equation (1.1), a certain amount of numerical diffusion is often present, as in Lax–Friedrichs type schemes [19]. On the other hand, it is known that as t gets large, the level set $\{G(x,t) = 0\}$ might become quite irregular and cause numerical difficulties. Among the various regularizations to fix this problem, one way is to add a diffusion term [9,10] to (1.1) which leads to the following viscous G-equation

$$-d\Delta G + G_t + V(x)\cdot DG + s_l|DG| = 0, \tag{1.3}$$

$\frac{d}{s_l} > 0$ is called the Markstein diffusivity. If we consider $-\lim_{T\to+\infty}\frac{G(x,T)}{T}$, the limit $\bar{H}(P,d)$ is given by the cell problem

$$-d\Delta w + s_l|P+Dw| + V(x)\cdot(P+Dw) = \bar{H}(P,d). \tag{1.4}$$

The existence of $\bar{H}(P,d)$ and classical solutions to (1.4) (unique up to a constant) can be easily deduced from the standard elliptic regularity theory. $\bar{H}(P,d)$ can be viewed as an approximation of the turbulent flame speed s_{T}.

A central issue we address here is the comparison of qualitative behavior of $\bar{H}(P,d)$ and $\bar{H}(P)$ as we vary d and the amplitude of the flow field. To this end, let us scale V to AV for some positive constant A (flow intensity), so:

$$-d\Delta w + s_l|P+Dw| + AV(x)\cdot(P+Dw) = \bar{H}(P,A,d). \tag{1.5}$$

An interesting question is to figure out how $\bar{H}$ behaves as a function of A. There are a few results in the combustion literature in this direction when A is small, see the references of [22]. In this paper, we are interested in the asymptotic behavior of $\bar{H}(P,A,d)$ as $A\to+\infty$. The usual inf-max formula

$$\bar{H}(P,A,d) = \inf_{h\in C^2(\mathbb{T}^n)}\max_{\mathbb{T}^n}(-d\Delta h + s_l|P+Dh| + AV(x)\cdot(P+Dh)) \tag{1.6}$$

provides only that $\bar{H}(P,A,d) \leqq C(d)\,(A+1)$ which is, in general, too rough. Experimental studies show that the turbulent flame speed may grow at a slower than linear rate in some situations. This is the so called "bending effect", see [22,24] among others.

The paper is organized as follows. In Section 2, we look at the case when V is a two dimensional cellular flow, $V = \nabla^{\perp}\mathcal{H} \equiv \nabla^{\perp}\sin 2\pi x_1 \sin 2\pi x_2$. It is known that $\bar{H}(P,A,0) = O(\frac{A}{\log A})$ for the inviscid case. The "bending effect" occurs marginally. When the diffusion is large ($d \gg 1$), it is proved in [16] that $\bar{H}(P,A,d)$ drops dramatically and has an upper bound as $\sqrt{\log A}$. In the small diffusion regime ($d \ll 1$), the analysis becomes much more subtle since the nonlinear L^1 term begins to compete with the linear diffusion term. In this section, we establish the finite bound for any positive diffusivity $d>0$. Precisely speaking, the main result of this section is that for a positive d-dependent constant $C(d)$ which is independent of A and P

$$\bar{H}(P,A,d) \leqq C(d), \quad \text{for all } d>0,\ |P|=1,\ A\geqq 2. \tag{1.7}$$

The uniform bound (1.7) is very different from the $O(A^{1/4})$ speed growth asymptotics for reaction-diffusion fronts in cellular flows [3,18,29,30].

From the inf-max formula (1.6), it is easy to see that $\bar{H}$ is positive homogeneous of degree 1. The proof of (1.7) is divided into two parts. In the first part, by a novel $\mathcal{H}$-weighted gradient estimate of solutions of the cell problem, we reveal the retention of positive mass of the gradient of solution in the boundary layers as $A\to+\infty$,

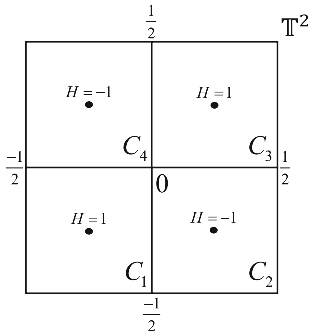

Fig. 2. Decomposition of a unit cell into quarter cells C_i, $i = 1, 2, 3, 4$

or the loss of the gradient mass in the interior of each of the quarter cells (Fig. 2). This allows us to show that $\bar{H}(P, A, d)$ is comparable to $||DT||_{L^1(\mathbb{T}^2)}$, where T is the solution of a linear advection-diffusion equation arising in advection enhanced diffusion problem [17]. In essence, we have overcome the obstacle of nonlinearity. In the second part, we derive the uniform bound of $||DT||_{L^1(\mathbb{T}^2)}$. Combining some energy decay estimates in [17] and the Cauchy inequality, it is easy to show that $||DT||_{L^1(\mathbb{T}^2)}$ has an upper bound of $\sqrt{\log(A)}$. To prove that it is actually uniformly bounded, we need careful analysis around cell corners which is done in Section 3.

In Section 4, we study the case when V is a shear flow. We prove that the limit $\lim_{A\to+\infty} \frac{\bar{H}(P,A,d)}{A}$ is a positive constant (no "bending effect") and is strictly decreasing with respect to the diffusivity d. The converging rate as $d \to 0$ is also discussed. Our approach can be used to recover an earlier result in [11]. We also investigate the limit for the curvature dependent G-equation, i.e, replace the diffusion term by the mean curvature of the flame front. We showed that the limit is the same as for the inviscid G-equation. Our results in this section are consistent with the natural intuition on the front propagation speed:

$$\text{Viscous speed} \leqq \text{Curvature dependent speed} \leqq \text{Inviscid speed}. \tag{1.8}$$

We remark that if the flow field is compressible, the situation is very different. Firstly, positive diffusivity may increase the propagation speed. Secondly, there may be flame trapping (in the inviscid case) or exponential decay of front speed (in the viscous case) due to the high turbulence intensity ($A \gg 1$). Explicit analytical results of this sort for the one space dimensional G-equations are reported in [13].

In Section 5, we show numerical results of $\bar{H}$ in viscous G-equations and cellular flows. It is increasing in A and decreasing in d. The paper ends with concluding remarks in Section 6.

2. Uniform speed bound in cellular flows

Without loss of generality, we assume that $s_l = 1$. Consider the cell problem

$$d\Delta w + |P + Dw| + AV(x) \cdot (P + Dw) = \bar{H}(P, A, d). \tag{2.1}$$

Here we switch $-d\Delta w$ to $d\Delta w$ through a simple change of variables. In this section, let us look at front speeds in cellular flows. A typical example is $V = \nabla^{\perp}\mathcal{H}$, where the stream function $\mathcal{H}(x) = \sin(2\pi x_1)\sin(2\pi x_2)$. For simplicity, we will work with this example and write $\mathcal{H}$ as H hereafter. The following is the main result of this section which says that the (viscous) turbulent flame speed is uniformly bounded for fixed d.

Theorem 2.1. *Assume that* $|P| = 1$. *Then*

$$\bar{H}(P, A, d) \leqq C(d) \quad \textit{for all } d > 0,$$

where $C(d)$ *is a constant depending on d, but is independent of A and P.*

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Proofs for $P = e_1$ and $P = e_2$ are similar. Also, the inf-max formula (1.6) implies that $\bar{H}$ is a convex and positive homogeneous of degree one as function of P. Hence it suffices to prove the above theorem for $P = (1, 0) = e_1$. Let us denote $\bar{H}(e_1, A, d) = \lambda_A$, and omit d dependence for the moment. Clearly $1 \leqq \lambda_A \leqq C(A+1)$. Hereafter, C denotes a constant independent of the flow intensity A. Note that C might depend on the diffusivity constant d. We also assume that $A \geqq 2$. In addition, we split $\mathbb{T}^2 = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ into four cells C_1, C_2, C_3, C_4 shown as in Fig. 2.

Let $G = e_1 \cdot x + w(x)$ and $\int_{\mathbb{T}^2} G\,\mathrm{d}x = 0$. Then

$$d\Delta G + |DG| + AV(x) \cdot DG = \lambda_A = \bar{H}(e_1, A, d). \tag{2.2}$$

Note that $G = G(A, x)$ depends on A. To simplify notation, we drop the dependence on A and write $G(A, x) = G(x)$. Integrating both sides of (2.2) and using the incompressibility of V and $V \cdot DH = 0$, we have that

$$\int_{\mathbb{T}^2} |DG|\,\mathrm{d}x = \lambda_A.$$

Let us denote $v_A = \frac{G}{\lambda_A}$. Then

$$d\Delta v_A + |Dv_A| + AV(x) \cdot Dv_A = 1 \tag{2.3}$$

and

$$\int_{\mathbb{T}^2} |Dv_A|\,\mathrm{d}x = 1. \tag{2.4}$$

Since $\int_{\mathbb{T}^2} v_A\,\mathrm{d}x = 0$, we have that

$$||v_A||_{W^{1,1}(\mathbb{T}^2)} \leqq C. \tag{2.5}$$

Owing to the Sobolev inequality,

$$\int_{\mathbb{T}^2} v_A^2\,\mathrm{d}x \leqq C.$$

Upon a subsequence, we may assume that

$$v_A \rightharpoonup v \quad \text{in } L^2(\mathbb{T}^2)$$

We now prove several lemmas. The first one says that the H^1 norm of v_A is locally bounded.

Lemma 2.1.

$$\int_{\mathbb{T}^2} |Dv_A|^2 H^2 \,\mathrm{d}x \leqq C.$$

Proof. Multiply (2.3) by $H^2 v_A$. The incompressibility of V and $V \cdot DH = 0$ imply that

$$\int_{\mathbb{T}^2} V \cdot Dv_A v_A H^2 \,\mathrm{d}x = 0.$$

Then using integration by parts and Cauchy's inequality, we get that

$$\int_{\mathbb{T}^2} |Dv_A|^2 H^2 \,\mathrm{d}x \leqq C(1 + \int_{\mathbb{T}^2} v_A^2 \,\mathrm{d}x) \leqq C.$$

So the above lemma holds. □

Remark 2.1. From the above lemma, it is clear that for all $\epsilon > 0$, $v \in H^1(\{|H| > \epsilon\})$ and

$$\int_{\{|H|>\epsilon\}} |Dv|^2 H^2 \,\mathrm{d}x \leqq C.$$

Then Sobolev embedding implies that

$$\lim_{A\to+\infty} ||v_A - v||_{L^2(\{|H|>\epsilon\})} = 0.$$

Next we show that the oscillation of v_A along nonzero level curves of H will tend to zero.

Lemma 2.2.

$$\int_{\mathbb{T}^2} H^4 |V(x) \cdot Dv_A|^2 \,\mathrm{d}x \leqq \frac{C}{A}.$$

In particular,

$$V(x) \cdot Dv = 0 \quad \textit{for a.e. } x \in \mathbb{T}^2. \tag{2.6}$$

Proof. To ease notation, we write $v_A = U$ in this proof. Then

$$d\Delta U + |DU| + AV(x) \cdot DU = 1.$$

Multiplying $H^4 V(x) \cdot DU$ on both sides of the above equation and integrating over $\mathbb{T}^2$ show that

$$d \int \Delta U (V(x) \cdot DU) H^4 \mathrm{d}x + \int |DU| (V(x) \cdot DU) H^4 \mathrm{d}x$$
$$+ A \int (V(x) \cdot DU)^2 H^4 \mathrm{d}x = \lambda_A \int V(x) \cdot DU H^4 \,\mathrm{d}x = 0$$

The last equality is due to the incompressibility of V and $V \cdot DH = 0$. Note that

$$\int \Delta U (V(x) \cdot DU) H^4 \mathrm{d}x = \int \left(\sum_{i=1}^{2} U_{x_i x_i} \right) \left(\sum_{k=1}^{2} V_k U_{x_k} \right) H^4 \mathrm{d}x$$

where $V = (V_1, V_2)$ and integration by parts,

$$= - \underbrace{\int \sum_{i=1}^{2} \sum_{k=1}^{2} H^4 U_{x_i} V_{k x_i} U_{x_k} \, \mathrm{d}x}_{\mathbf{I}} - \underbrace{\int \sum_{i=1}^{2} \sum_{k=1}^{2} H^4 U_{x_i} V_k U_{x_k x_i} \, \mathrm{d}x}_{\mathbf{II}}$$
$$\underbrace{-4 \int_{\mathbb{T}^2} H^3 (DU \cdot DH)(V(x) \cdot DU) \, \mathrm{d}x}_{\mathbf{III}}$$

Note that

$$\mathbf{I} + \mathbf{III} \leqq C \int_{\mathbb{T}^2} H^2 |DU|^2 \, \mathrm{d}x.$$

Moreover,

$$\mathbf{II} = \frac{1}{2} \int_{\mathbb{T}^2} \sum_{k=1}^{2} V_k \left(|DU|^2 \right)_{x_k} H^4 \, \mathrm{d}x \; = 0$$

The last inequality is due to the incompressibility of V and $V \cdot DH = 0$. Furthermore, Cauchy inequality implies that

$$\left| \int |DU| (V(x) \cdot DU) H^4 \mathrm{d}x \right| \leqq \frac{1}{2} \left(\int_{\mathbb{T}^2} |V(x) \cdot DU|^2 H^4 \, \mathrm{d}x + \int_{\mathbb{T}^2} |DU|^2 H^4 \, \mathrm{d}x \right).$$

Hence Lemma 2.2 follows from Lemma 2.1. □

The next lemma says that v_A converges to v locally in H^1 norm.

Lemma 2.3. *Let* $W = g(H^2)$ *for some* $g \in C_c^\infty((0, 1])$. *Then*

$$\lim_{A \to +\infty} \int_{\mathbb{T}^2} |Dv_A - Dv|^2 W \, \mathrm{d}x = 0$$

Proof. In fact

$$\int_{\mathbb{T}^2} |Dv_A - Dv|^2 W \, \mathrm{d}x = \int_{\mathbb{T}^2} (Dv_A - Dv) \cdot (Dv_A - Dv) W \, \mathrm{d}x$$
$$= \underbrace{\int_{\mathbb{T}^2} Dv_A \cdot (Dv_A - Dv) W \, \mathrm{d}x}_{\mathbf{I}} - \underbrace{\int_{\mathbb{T}^2} Dv \cdot (Dv_A - Dv) W \, \mathrm{d}x}_{\mathbf{II}}$$

Due to Lemma 2.1, $Dv \in L^2_{loc}(\mathbb{T}^2\backslash\{H=0\})$ and $Dv_A \rightharpoonup Dv$ in $L^2_{loc}(\mathbb{T}^2\backslash\{H=0\})$. Hence $\mathrm{II} \to 0$ as $A \to +\infty$. Also,

$$I = -\int_{\mathbb{T}^2} \Delta v_A (v_A - v) W \,\mathrm{d}x - \underbrace{\int_{\mathbb{T}^2} Dv_A (v_A - v) \cdot DW \,\mathrm{d}x}_{\mathbf{III}}$$

$$= \underbrace{\frac{1}{d}\int_{\mathbb{T}^2} |Dv_A|(v_A - v) W \,\mathrm{d}x}_{\mathbf{IV}} + \underbrace{\frac{A}{d}\int_{\mathbb{T}^2} V(x)\cdot Dv_A (v_A - v) W \,\mathrm{d}x}_{\mathbf{V}} - \mathbf{III} + \mathbf{VI},$$

where

$$\mathbf{VI} = -\frac{1}{d}\int_{\mathbb{T}^2} (v_A - v) W \,\mathrm{d}x.$$

By Remark 2.1 and Lemma 2.1, III, IV, VI $\to 0$ as $A \to +\infty$.

Since $V(x)\cdot Dv = 0$ and $V \cdot DH = 0$, integration by parts and the incompressibility of V imply that the fifth term $\mathbf{V} = 0$. □

Remark 2.2. It follows immediately from Lemma 2.3 that for all $\epsilon > 0$

$$\lim_{A\to+\infty} \int_{\{|H| \geqq \epsilon\}} |Dv_A - Dv| \,\mathrm{d}x = 0.$$

Lemma 2.4. (i) *For* $i = 1, 3$, $v = f_i(H)$ *in* C_i, *where* $f_i \in C^2((0,1))$ *and* $f_i' < 0$. (ii) *For* $i = 2, 4$, $v = f_i(H)$ *in* C_i, *where* $f_i \in C^2((-1,0))$ *and* $f_i' > 0$.

Proof. Since $v \in H^1_{loc}(\mathbb{T}^2\backslash\{H=0\})$ and $V(x)\cdot Dv = 0$ almost everywhere, it is not hard to show that $v = f_i(H)$ in C_i, where $f_i \in H^1_{loc}((0,1))$ for $i = 1, 3$ and $f_i \in H^1_{loc}((-1,0))$ for $i = 2, 4$.

It suffices to prove Lemma 2.4 for $i = 1$. The other cases are similar.

Step 1. We first show that $f_1 \in C^2((0,1))$. In fact, let $\varphi(s)$ in $C_c^\infty(0,1)$. Since

$$d\Delta v_A + |Dv_A| + AV(x)\cdot Dv_A = 1$$

Multiplying $\varphi(H)$ on both sides and integrating by parts over the cell C_1 give

$$\Rightarrow -d\int_{C_1} Dv_A \varphi'(H) DH \mathrm{d}x + \int_{C_1} |Dv_A| \varphi(H) \mathrm{d}x = \int_{C_1} \varphi(H) \mathrm{d}x$$

Owing to Remark 2.2, sending $A \to +\infty$,

$$-d\int_{C_1} Dv \varphi'(H) DH \mathrm{d}x + \int_{C_1} |Dv| \varphi(H) \mathrm{d}x = \int_{C_1} \varphi(H) \mathrm{d}x$$

Since $Dv = f_1'(H)DH$, by Coarea formula, it is easy to see that $f_1 = f_1(t) \in H^1_{loc}((0,1))$ is a weak solution of

$$d(f_1'(t)a(t))' + |f_1'(t)|b(t) = c(t) \tag{2.7}$$

where $a(t), b(t), c(t) > 0$ and $\in C^\infty((0,1)) \Rightarrow f_1 \in C^2((0,1))$.

Step 2. We then prove that $f_1' < 0$. In fact, since

$$\int_{C_1} |v_A - v|^2 \mathrm{d}x \to 0 \ \text{ and } \ \int_{C_1} |V(x) \cdot Dv_A|^2 H^4 \mathrm{d}x \to 0,$$

for any $0 < a < b < 1$, $\epsilon > 0$, when A is sufficiently large, $\exists 0 < a_\epsilon < b_\epsilon < 1$ such that $|a_\epsilon - a| < \epsilon$, $|b_\epsilon - b| < \epsilon$ and

$$\oint_{\{H=a_\epsilon\}} |v_A - v| \mathrm{d}s \leqq \epsilon, \quad \oint_{\{H=b_\epsilon\}} |v_A - v| \mathrm{d}s \leqq \epsilon \tag{2.8}$$

where $\oint$ denotes the integral average over a closed streamline, and

$$\max_{x,y \in \{H=a_\epsilon\}} |v_A(x) - v_A(y)| \leqq \epsilon, \quad \max_{x,y \in \{H=b_\epsilon\}} |v_A(x) - v_A(y)| \leqq \epsilon. \tag{2.9}$$

Since $d\Delta v_A + |Dv_A| + AV(x) \cdot Dv_A \geqq 0$, v_A satisfies the maximum principle,

$$\max_{H=a_\epsilon} v_A \geqq \max_{H=b_\epsilon} v_A$$

According to (2.8), (2.9),

$$f_1(a_\epsilon) \geqq f_1(b_\epsilon) - 4\epsilon$$

Sending $\epsilon \to 0$,

$$f_1(a) \geqq f_1(b).$$

Hence $f_1' \leqq 0$. Owing to (2.7), f_1' cannot attain 0. □

The following lemma says that there is a mass loss of $|Dv|$ as $A \to +\infty$. This implies the presence of boundary layers (see Remark 2.3 and Fig. 3) where a positive amount of mass of $|Dv|$ is collected. More precisely,

Lemma 2.5.

$$\lim_{\epsilon \to 0} \int_{\{|H| \geqq \epsilon\}} |Dv| \, \mathrm{d}x = \tau < 1.$$

Proof. Owing to (2.4), it is obvious that $\tau \leqq 1$. Our goal is to exclude the case $\tau = 1$. We argue by contradiction. Let us assume that

$$\tau = 1. \tag{2.10}$$

We first prove the following lemma.

Lemma 2.6. *For* $i = 1, 2, 3, 4$,

$$\lim_{t \to 0} f_i'(t) = 0.$$

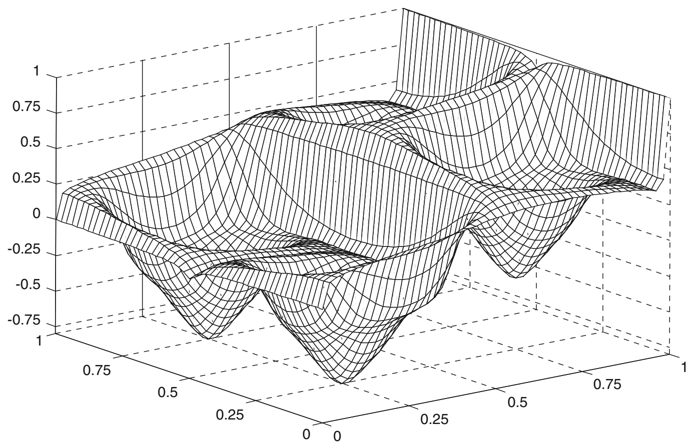

Fig. 3. Graph of solution G to equation (2.2) in cellular flow at $A = 16$, $d = 0.01$: presence of both internal wells and layers at quarter cell boundaries

Proof. Since $d\Delta v_A + |Dv_A| + AV(x) \cdot Dv_A = 1$, for $a \in (0, 1)$,

$$\int_{\{|H| \geqq a\}} d\Delta v_A \mathrm{d}x + \int_{\{|H| \geqq a\}} |Dv_A| \mathrm{d}x = \left|\left\{|H| \geqq a\right\}\right|.$$

Here $|\mathcal{K}|$ represents the measure of the set $\mathcal{K}$. So

$$d \int_{\{|H|=a\}} \frac{\partial v_A}{\partial n} \mathrm{d}s + \int_{\{|H| \geqq a\}} |Dv_A| \mathrm{d}x = \left|\left\{|H| \geqq a\right\}\right|.$$

Hence for $l \in (0, 1)$ and small $\epsilon > 0$,

$$d \int_{\ell-\epsilon}^{\ell+\epsilon} \mathrm{d}a \int_{\{|H|=a\}} \frac{\partial v_A}{\partial n} \mathrm{d}s + \int_{\ell-\epsilon}^{\ell+\epsilon} \mathrm{d}a \int_{\{|H \geqq a\}} |Dv_A| \mathrm{d}x$$
$$= \int_{\ell-\epsilon}^{\ell+\epsilon} \left|\left\{|H| \geqq a\right\}\right| \mathrm{d}a$$

Sending $A \to +\infty$, and by Remark 2.2, we deduce that

$$\Rightarrow d \int_{\ell-\epsilon}^{\ell+\epsilon} \mathrm{d}a \int_{\{|H|=a\}} \frac{\partial v}{\partial n} \mathrm{d}s + \int_{\ell-\epsilon}^{\ell+\epsilon} \mathrm{d}a \int_{\{|H| \geqq a\}} |Dv| \mathrm{d}x$$
$$= \int_{\ell-\epsilon}^{\ell+\epsilon} \left|\left\{|H| \geqq a\right\}\right| \mathrm{d}a$$

Dividing ϵ on both sides, we derive that

$$d \int_{\{|H|=a\}} \frac{\partial v}{\partial n} \mathrm{d}s + \int_{\{|H| \geqq a\}} |Dv| \mathrm{d}x = \left|\left\{|H| \geqq a\right\}\right| \tag{2.11}$$

Since

$$\lim_{a\to 0}\int_{\{|H|\geqq a\}}|Dv|\mathrm{d}x=\tau=1,\quad \text{by (2.10)}$$
$$\Rightarrow \lim_{a\to 0}\int_{\{|H|=a\}}\frac{\partial v}{\partial n}\mathrm{d}s=0. \tag{2.12}$$

Note that

$$\int_{\{|H|=a\}}\frac{\partial v}{\partial n}\mathrm{d}s=|f_1'(a)|\int_{\{|H|=a\}\cap C_1}|\nabla H|\mathrm{d}s+|f_2'(-a)|\int_{\{|H|=-a\}\cap C_2}|\nabla H|\mathrm{d}s$$
$$+|f_3'(a)|\int_{\{|H|=a\}\cap C_3}|\nabla H|\mathrm{d}s+|f_4'(-a)|\int_{\{|H|=-a\}\cap C_4}|\nabla H|\mathrm{d}s. \tag{2.13}$$

Clearly for $i=1,2,3,4$

$$\lim_{a\to 0}\int_{\{|H|=a\}\cap C_i}|\nabla H|\mathrm{d}s>0$$

So Lemma 2.6 holds. □

Now let us finish the proof of Lemma 2.5. By (2.11), for any $0<a<b<1$,

$$\Rightarrow d\int_{\{|H|=a\}}\frac{\partial v}{\partial n}\mathrm{d}s-d\int_{\{|H|=b\}}\frac{\partial v}{\partial n}\mathrm{d}s+\int_{\{a\leqq|H|\leqq b\}}|Dv|\mathrm{d}x$$
$$=\left|\left\{a\leqq|H|\leqq b\right\}\right|$$

According to (2.12) and (2.13),

$$\lim_{a\to 0}\int_{\{|H|=a\}}\frac{\partial v}{\partial n}\mathrm{d}s=0,\quad \int_{\{|H|=b\}}\frac{\partial v}{\partial n}\mathrm{d}s\geqq 0$$
$$\Rightarrow \iint_{\{0<|H|\leqq b\}}|Dv|\mathrm{d}x\geqq\left|\left\{0<|H|\leqq b\right\}\right|$$
$$\Rightarrow 1\leqq\frac{\iint_{\{0<|H|\leqq b\}}|Dv|\mathrm{d}x}{|\{0<|H|\leqq b\}|}=\sum_{i=1}^{4}\frac{\iint_{\{0<|H|\leqq b\}\cap C_i}\left|\nabla H||f_i'(H)\right|\mathrm{d}x}{\left|\left\{0<|H|\leqq b\right\}\right|}\longrightarrow 0$$

as $b\to 0$ according to Lemma 2.6. This is a contradiction.

Remark 2.3. (*Presence of boundary layer*) Combining Remark 2.2 and Lemma 2.5, we have for any fixed $\epsilon>0$, that

$$\lim_{A\to+\infty}\int_{\{|H|\leqq\epsilon\}}|Dv_A|\,\mathrm{d}x=1-\tau>0.$$

Let T_A be the smooth solution of the following steady linear advection-diffusion problem

$$d\Delta T_A+AV(x)\cdot DT_A=0, \tag{2.14}$$

subject to $T_A-e_1\cdot x$ being periodic and $\int_{\mathbb{T}^2}T_A\,\mathrm{d}x=0$. To simplify notation, we shall drop the dependence on A and write $T_A=T$. The following lemma says that the analysis of λ_A boils down to that of the L^1 norm of DT.

Lemma 2.7. *There exists a constant C such that*

$$\frac{1}{C}||DT||_{L^1(\mathbb{T}^2)} \leqq \lambda_A \leqq C||DT||_{L^1(\mathbb{T}^2)}.$$

In particular,

$$\limsup_{A\to+\infty} \frac{||DT||_{L^1(\mathbb{T}^2)}}{\lambda_A} \leqq 1. \tag{2.15}$$

Proof. We may assume that $v_A \rightharpoonup v$ in $L^2(\mathbb{T}^2)$ as $A \to +\infty$. Otherwise, we can argue by contradiction and use a subsequence. Let $S = G - T$ and $\beta_A \equiv \|DT\|_{L^1(\mathbb{T}^2)}$. Jensen's inequality and $T - e_1 \cdot x$ being periodic imply that $\beta_A \geqq \|e_1\|_{L^1(\mathbb{T}^2)} = 1$. The S function satisfies the equation:

$$d\Delta S + |DG| + AV(x) \cdot DS = \lambda_A. \tag{2.16}$$

Multiplying S on both sides of (2.16), using integration by parts and the incompressibility of V, we derive that

$$d\int_{\mathbb{T}^2} |DS|^2 \,\mathrm{d}x = \int_{\mathbb{T}^2} S|DG| \,\mathrm{d}x.$$

A modification of Proposition 4 in [18] says that for a constant C

$$G \leqq 1 + ||DG||_{L^1(\mathbb{T}^2)} = 1 + \lambda_A$$

and

$$|T| \leqq 1 + ||DT||_{L^1(\mathbb{T}^2)} = 1 + \beta_A.$$

See Lemma 2.8 for the proof. Hence

$$\int_{\mathbb{T}^2} |DS|^2 \,\mathrm{d}x \leqq C(\lambda_A^2 + \beta_A^2).$$

For $\epsilon > 0$ which will be chosen later,

$$\int_{\mathbb{T}^2} |DS|^2 \,\mathrm{d}x \geqq \int_{\{|H|\leqq\epsilon\}} |DS|^2 \,\mathrm{d}x \geqq \frac{1}{|\{|H| \leqq \epsilon\}|}\left(\int_{\{|H|\leqq\epsilon\}} |DS| \,\mathrm{d}x\right)^2. \tag{2.17}$$

Note that

$$\int_{\{|H|\leqq\epsilon\}} |DG| \,\mathrm{d}x \leqq \int_{\{|H|\leqq\epsilon\}} |DS| \,\mathrm{d}x + \int_{\{|H|\leqq\epsilon\}} |DT| \,\mathrm{d}x. \tag{2.18}$$

According to Remark 2.3, when A is large enough,

$$\int_{\{|H|\leqq\epsilon\}} |DG| \,\mathrm{d}x \geqq \frac{(1-\tau)}{2}\lambda_A. \tag{2.19}$$

It follows from (2.17)–(2.19) that:

$$\frac{(1-\tau)}{2}\lambda_A \leqq C\left(\lambda_A^2+\beta_A^2\right)^{1/2}|\{|H|\leqq\epsilon\}|^{1/2}+\beta_A. \tag{2.20}$$

Since $\lim_{\epsilon\to 0}|\{|H|\leqq\epsilon\}|=0$, we may choose ϵ small enough such that the first term on the right hand side of (2.20) is bounded from above by

$$\frac{(1-\tau)}{4}(\lambda_A+\beta_A),$$

implying:

$$\lambda_A \leqq C\beta_A. \tag{2.21}$$

To finish the proof, it suffices to verify (2.15). In fact,

$$\begin{aligned}
\lambda_A &\geqq \int_{\{|H|\leqq\epsilon\}}|DG|\,\mathrm{d}x\\
&\geqq \int_{\{|H|\leqq\epsilon\}}|DT|\,\mathrm{d}x-\left|\int_{\{|H|\leqq\epsilon\}}(|DG|-|DT|)\,\mathrm{d}x\right|\\
&\geqq -\int_{\{|H|\leqq\epsilon\}}|DS|\,\mathrm{d}x+\int_{\{|H|\leqq\epsilon\}}|DT|\,\mathrm{d}x\\
\text{by (2.17)}\ &\geqq -C(\lambda_A^2+\beta_A^2)^{1/2}|\{|H|\leqq\epsilon\}|^{1/2}+\int_{\{|H|\leqq\epsilon\}}|DT|\,\mathrm{d}x.
\end{aligned}$$

For fixed $\delta>0$, we may choose ϵ sufficiently small such that

$$C(\lambda_A^2+\beta_A^2)^{1/2}|\{|H|\leqq\epsilon\}|^{1/2}\leqq\delta(\lambda_A+\beta_A).$$

Then

$$\lambda_A(1+\delta)\geqq-\delta\beta_A+\int_{\{|H|\leqq\epsilon\}}|DT|\,\mathrm{d}x. \tag{2.22}$$

According to (3.25), for fixed $(\epsilon,\delta)>0$, we can choose A large enough such that

$$\int_{\{|H|\geqq\epsilon\}}|DT|\,\mathrm{d}x\leqq\delta\leqq\delta\beta_A,$$

where the last inequality is due to $\beta_A\geqq 1$. It follows that

$$\int_{\{|H|\leqq\epsilon\}}|DT|\,\mathrm{d}x\geqq(1-\delta)\beta_A,$$

which implies that

$$\frac{\beta_A}{\lambda_A}\leqq\frac{1+\delta}{1-2\delta}.$$

Therefore

$$\limsup_{A\to+\infty}\frac{\beta_A}{\lambda_A}\leqq\frac{1+\delta}{1-2\delta}.$$

Then (2.15) follows by sending $\delta\to 0$. $\square$

Proof of Theorem 2.1. Combining Lemma 2.7 and Theorem 3.1, we obtain λ_A is uniformly bounded. □

Remark 2.4. (*Difference between T and G*) As A increases, it is known that T from equation (2.14) becomes more and more like a constant inside each quarter cell [17]. This is not true for G from (2.2) due to nonlinearity. In fact, according to Lemma 2.4, $v_A = \frac{G}{\lambda_A}$ converges to v, which is a strictly decreasing function of $|H|$ in each cell. Hence internal wells will emerge as $\frac{A}{d} \gg 1$. See Fig. 3 for numerical computations.

The following lemma is a modification of Proposition 4 in [18].

Lemma 2.8. *Let* $\mathbb{T}^2 = \left[\frac{-1}{2}, \frac{1}{2}\right] \times \left[\frac{-1}{2}, \frac{1}{2}\right]$. *Suppose that* $W = W(x_1, x_2) \in C^1(\mathbb{R}^2)$ *satisfies that*

(1) $W - x_1$ *is periodic;*

(2) $\int_{\mathbb{T}^2} W \,\mathrm{d}x = 0$*;*

(3) *For all* $t \in \mathbb{R}$, W *satisfies the following maximal principle*

$$\max_{[t,t+1]\times\mathbb{R}} W = \max_{x_2\in\mathbb{R}} W(t+1, x_2).$$

Then

$$\max_{\mathbb{T}^2} W \leqq ||DW||_{L^1(\mathbb{T}^2)} + 1.$$

Proof. For $s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, let

$$h(s) = \int_{\frac{-1}{2}}^{\frac{1}{2}} W(s, x_2)\,\mathrm{d}x_2$$

and

$$M(s) = \max_{x_2\in\mathbb{R}} W(s, x_2).$$

Clearly,

$$M(s) - h(s) \leqq \int_{\frac{-1}{2}}^{\frac{1}{2}} |DW(s, x_2)|\,\mathrm{d}x_2.$$

Also, owing to assumption (3), $M(s) \geqq \max_{\mathbb{T}^2} W - 1$. Integrating with respect to s on both sides of the above inequality, we obtain

$$\max_{\mathbb{T}^2} W \leqq ||DW||_{L^1(\mathbb{T}^2)} + 1.$$

□

3. Uniform bound of $||DT||_{L^1(\mathbb{T}^2)}$

The following is the main result of this section.

Theorem 3.1.

$$||DT||_{L^1(\mathbb{T}^2)} \leqq C$$

where C is a constant independent of A.

Owing to Lemma 2.8,

$$||T||_{L^\infty(\mathbb{T}^2)} \leqq 1 + ||DT||_{L^1(\mathbb{T}^2)}. \tag{3.23}$$

Hence we have the following Corollary.

Corollary 3.1.

$$||T||_{L^\infty(\mathbb{T}^2)} \leqq C$$

where C is a constant independent of A.

Without loss of generality, we assume that $d = 1$. For the sake of readability, we will again work with $H = \sin(x_1)\ \sin(x_2)$ and let $\mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$. We omit the factor of 2π on the stream function for convenience. We will not use any special properties of this stream function. Otherwise, the L^∞ bound will simply follow from the symmetry and the uniqueness of the solution of equation (2.14) (see [8]). Hereafter we denote

$$\epsilon = \frac{1}{A}.$$

Throughout this section, C always denotes a constant independent of ϵ. It is known that (see [17] for instance)

$$||DT||^2_{L^2(\mathbb{T}^2)} = O\left(\frac{1}{\sqrt{\epsilon}}\right). \tag{3.24}$$

Also, according to Theorem 4.2 in [17], $|DT|$ decays very rapidly away from those stream lines $\{H = 0\}$. Precisely speaking, for $N \geqq 1$:

$$||DT||^2_{L^2(\{x\in\mathbb{T}^2:|H|\geqq N\sqrt{\epsilon}\})} \leqq \frac{C}{\sqrt{\epsilon}N^4}. \tag{3.25}$$

Although [17] is associated with Dirichlet boundary conditions rather than the periodic setting in this paper, the above interior estimate still holds. For the reader's convenience, we will explain below how to deduce this dissipation rate by modifying the proof of Theorem 3.2 in [17].

Proof of (3.25). Note that the the proof of Theorem 3.2 in [17] relies on an interior estimate, Proposition 3.4 in [17], whose proof does not involve boundary conditions. For $t \geqq 0$, denote

$$F(t) = \int_{\mathcal{D}(t)} |DT|^2 \,\mathrm{d}x,$$

where $\mathcal{D} = \{|H| \geqq t\}$. The following inequality was derived in the proof of Theorem 3.2 in [17] (page 880, line 14),

$$tF(2t) \leqq C \left(\frac{\epsilon}{t^2}\right)^{\frac{3}{4}} (tF(t))^{\frac{1}{2}}.$$

Denote $\tilde{F}(t) = \frac{tF(t)}{4C^2}$. Then

$$\tilde{F}(2t) \leqq \left(\frac{\epsilon}{t^2}\right)^{\frac{3}{4}} (\tilde{F}(t))^{\frac{1}{2}}.$$

For $k = 0, 1, 2, \ldots$, we write

$$a_k = \tilde{F}(2^k\sqrt{\epsilon}).$$

The above nonlinear recurrence relation implies that

$$a_{k+1} \leqq 2^{\frac{-3k}{2}} \sqrt{a_k}.$$

Since $F(0) = O\left(\frac{1}{\sqrt{\epsilon}}\right)$, we have that

$$a_0 \leqq C.$$

Let $C_0 = \max\{C, 64\}$. Using induction, it is easy to see that

$$a_k \leqq 2^{-3k} C_0.$$

For $N \in \mathbb{N}$, choose $k \in \mathbb{N}$ such that $2^k \leqq N < 2^{k+1}$. Then

$$\tilde{F}(N\sqrt{\epsilon}) \leqq 2\tilde{F}(2^k\sqrt{\epsilon}) \leqq 2^{-3k+1} C_0 \leqq \frac{16C_0}{N^3}.$$

So (3.25) holds. □

Then, by Cauchy inequality, it is easy to show that

$$||DT||_{L^1(\mathbb{T}^2)} \leqq C\sqrt{\log(A)}. \tag{3.26}$$

In fact, let us denote $A_N = \{x \in \mathbb{T}^2 : (N-1)\sqrt{\epsilon} \leqq |H| \leqq N\sqrt{\epsilon}\}$ and use the Cauchy inequality to control the L^1 norm of $|DT|$.

$$\begin{aligned}
\int_{\mathbb{T}^2} |DT| \,\mathrm{d}x &= \sum_{N=1}^{\frac{1}{\sqrt{\epsilon}}} \int_{A_N} |DT| \,\mathrm{d}x \\
&\leqq \sum_{N=1}^{\frac{1}{\sqrt{\epsilon}}} \left(\int_{A_N} |DT|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} |A_N|^{\frac{1}{2}} \\
&\leqq C\epsilon^{\frac{-1}{4}} |A_N|^{\frac{1}{2}} \left(\sum_{N=1}^{\infty} \left(1 + \frac{1}{N^2}\right)\right),
\end{aligned}$$

An easy computation shows that

$$|A_N| \leqq C\sqrt{\epsilon}|\log(\epsilon)|.$$

The extra factor $|\log(\epsilon)|$ is due to the hyperbolicity of H near corners. Hence (3.26) holds. □

To prove that $||DT||_{L^1(\mathbb{T}^2)}$ is actually uniformly bounded, the argument is much more involved. We need delicate analysis around cell corners.

Let us first make some preparations. It suffices to show that the L^1 norm of $|DT|$ is uniformly bounded on the cell $C_\pi = [0, \pi] \times [0, \pi]$. The other cells are similar. For $\mu > 0$, denote

$$\Pi_\mu = [0, \mu] \times [0, \mu] \cup [0, \mu] \times [\pi - \mu, \pi] \cup [\pi - \mu, \pi] \times [\pi - \mu, \pi] \\ \cap [\pi - \mu, \pi] \times [0, \mu].$$

This is the union of four corners. Let $G_\mu = C_\pi \backslash \Pi_\mu$ and $\Omega_N = \{x \in \mathbb{T}^2 : (N-1)\sqrt{\epsilon} \leqq |H| \leqq N\sqrt{\epsilon}\} \cap G_\mu$. The following calculation is the same as the derivation of (3.26). By Cauchy inequality,

$$\begin{aligned} \int_{G_\mu} |DT| \, \mathrm{d}x &= \sum_{N=1}^{\frac{1}{\sqrt{\epsilon}}} \int_{\Omega_N} |DT| \, \mathrm{d}x \\ &\leqq \sum_{N=1}^{\frac{1}{\sqrt{\epsilon}}} \left(\int_{\Omega_N} |DT|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} |\Omega_N|^{\frac{1}{2}} \\ &\leqq C\epsilon^{\frac{-1}{4}} |\Omega_N|^{\frac{1}{2}} \left(\sum_{N=1}^{\infty} \left(1 + \frac{1}{N^2} \right) \right), \end{aligned}$$

It is easy to see that

$$|\Omega_N| \leqq C\sqrt{\epsilon},$$

where C is a constant depending on μ. So for fixed μ,

$$||DT||_{L^1(G_\mu)} \leqq C. \tag{3.27}$$

Hence, to prove that $||DT||_{L^1}$ is uniformly bounded, we need to show only that the L^1 norm is uniformly bounded around four corners. It suffices to look at the first corner $\tilde{Q}_\mu = [0, \mu] \times [0, \mu]$. The other cases are similar. Choose μ small such that

$$x_1 x_2 \leqq 2 \, \sin(x_1) \sin(x_2) \quad \text{in } \tilde{Q}_{4\mu}. \tag{3.28}$$

We introduce an orthogonal coordinate system $h = \frac{x_1 x_2}{\sqrt{\epsilon}}$ and $\theta = \frac{1}{2}(x_1^2 - x_2^2)$ and denote $T(x_1, x_2) = f(h, \theta)$. Then the equation (2.14) becomes

$$f_{hh} + \epsilon f_{\theta\theta} - f_\theta a(h, \theta) + h f_h b(h, \theta) = 0, \tag{3.29}$$

where

$$a(h, \theta) = \frac{x_1 \sin(x_1) \cos(x_2) + x_2 \sin(x_2) \cos(x_1)}{|x|^2}$$

and

$$b(h,\theta) = \frac{x_1 \sin(x_2)\cos(x_1) - x_2 \sin(x_1)\cos(x_2)}{(|x|^2)x_1 x_2}.$$

By Taylor expansion, it is easy to see that when μ is small,

$$\frac{1}{2} \leqq a \leqq 1$$

and

$$|b| \leqq 1.$$

Since $DT = f_\theta D\theta + f_h Dh$, we have that

$$||DT||_{L^1(\tilde{Q}_\mu)} \leqq \underbrace{||f_\theta D\theta||_{L^1(\tilde{Q}_\mu)}}_{\mathbf{I}} + \underbrace{||f_h Dh||_{L^1(\tilde{Q}_\mu)}}_{\mathbf{II}}. \tag{3.30}$$

Note that

$$|f_\theta D\theta| = \frac{|DT \cdot D\theta|}{|D\theta|}.$$

When μ is small, $|D\theta| \sim |DH|$ and $|V(x) + D\theta| \sim |DH|^3$. So

$$\frac{|DT \cdot D\theta|}{|D\theta|} \leqq C\left(\frac{|DT \cdot V(x)|}{|DH|} + |DT||DH|\right).$$

Lemma 3.1.

$$\int_{C_\pi} \frac{|DT \cdot V(x)|}{|DH|} + |DT||DH|\,\mathrm{d}x \leqq C.$$

Proof. Denote $A_N = \{(N-1)\sqrt{\epsilon} \leqq H \leqq N\sqrt{\epsilon}\}$. By Coarea formula,

$$\int_{A_N} |DH|\,\mathrm{d}x \leqq C\sqrt{\epsilon}.$$

Also Cauchy inequality implies that

$$\begin{aligned}\int_{A_N} |DT||DH|\,\mathrm{d}x &\leqq ||DT||_{L^2(A_N)}||DH||_{L^2(A_N)}\\ &\leqq C||DT||_{L^2(A_N)}\sqrt{||DH||_{L^1(A_N)}}\\ &\leqq \frac{C}{N^2} \quad \text{owing to (3.25).}\end{aligned}$$

As the derivation of (3.27), we have that

$$\int_{C_\pi} |DT||DH|\,\mathrm{d}x \leqq \sum_{N \geqq 1} \frac{C}{N^2} \leqq C.$$

As for the term $\frac{|DT\cdot V|}{|DH|}$, since $\frac{|DT\cdot V|}{|DH|} \leqq |DT|$, by (3.27), we need only to control integration around cell corners. Let us look at the corner $\tilde{Q}_\mu$.

$$\int_{\tilde{Q}_\mu} \frac{|DT\cdot V(x)|}{|DH|}\,\mathrm{d}x = \int_{\{|x|\leqq\epsilon\}\cap\tilde{Q}_\mu} \frac{|DT\cdot V(x)|}{|DH|}\,\mathrm{d}x + \int_{\{|x|\geqq\epsilon\}\cap\tilde{Q}_\mu} \frac{|DT\cdot V(x)|}{|DH|}\,\mathrm{d}x$$
$$\leqq C\left(\epsilon||DT||_{L^2(\mathbb{T}^2)} + ||DT\cdot V||_{L^2(\mathbb{T}^2)}\sqrt{\int_{\{|x|\geqq\epsilon\}\cap\tilde{Q}_\mu} \frac{1}{|DH|^2}\,\mathrm{d}x}\right).$$

Multiplying $DT\cdot V$ on both sides of (2.14) and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{T}^2} |DT\cdot V|^2\,\mathrm{d}x &= -\epsilon\int_{\mathbb{T}^2} \Delta T(DT\cdot V)\,\mathrm{d}x\\ &= \epsilon\int_{\mathbb{T}^2}\frac{1}{2}D(|DT|^2)\cdot V\,\mathrm{d}x + \epsilon\sum_{1\leqq i,k\leqq 2}\int_{\mathbb{T}^2} T_{x_i}T_{x_k}V_{k,x_i}\,\mathrm{d}x\\ &= \epsilon\sum_{1\leqq i,k\leqq 2}\int_{\mathbb{T}^2} T_{x_i}T_{x_k}V_{k,x_i}\,\mathrm{d}x\\ &\leqq C\epsilon||DT||^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

The third equality is due to the incompressibility of V. Hence (3.24) leads to

$$||DT\cdot V||_{L^2(\mathbb{T}^2)} \leqq C\epsilon^{\frac{1}{4}}.$$

Also, since $|DH|^2 \sim |x|^2$, using polar coordinates we have that

$$\int_{\{|x|\geqq\epsilon\}\cap\tilde{Q}_\mu} \frac{1}{|DH|^2}\,\mathrm{d}x \sim -\log(\epsilon).$$

Hence $\int_{\mathbb{T}^n} \frac{|DT\cdot V(x)|}{|DH|}\,\mathrm{d}x$ is also uniformly bounded. So term **I** in (3.30) is uniformly bounded. Now let us look at term **II** in (3.30). By a change of variables, we derive that

$$\int_{\tilde{Q}_\mu} |f_h Dh|\,\mathrm{d}x = \int_{\tilde{Q}_\mu} \frac{|f_h|}{|x|}\,\mathrm{d}h\mathrm{d}\theta \leqq \int_{\tilde{Q}_\mu} \frac{|f_h|}{\sqrt{2|\theta|}}\,\mathrm{d}h\mathrm{d}\theta. \tag{3.31}$$

Also (3.24) and a change of variables imply that

$$\begin{aligned}\int_{\tilde{Q}_{4\mu}} f_h^2\,\mathrm{d}h\mathrm{d}\theta &= \sqrt{\epsilon}\int_{\tilde{Q}_{4\mu}} f_h^2|Dh|^2\,\mathrm{d}x\\ &\leqq \sqrt{\epsilon}||DT||^2_{L^2(\mathbb{T}^2)} \leqq C.\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\epsilon\int_{\tilde{Q}_{4\mu}} f_\theta^2\,\mathrm{d}h\mathrm{d}\theta &= \sqrt{\epsilon}\int_{\tilde{Q}_{4\mu}} f_\theta^2|D\theta|^2\,\mathrm{d}x\\ &\leqq \sqrt{\epsilon}||DT||^2_{L^2(\mathbb{T}^2)} \leqq C.\end{aligned} \tag{3.33}$$

Clearly,

$$\tilde{Q}_\mu \subset \left\{(x_1, x_2) | \, 0 \leqq h \leqq \frac{\mu^2}{\sqrt{\epsilon}}, \; -\frac{\mu^2}{2} \leqq \theta \leqq \frac{\mu^2}{2}\right\} \leqq \tilde{Q}_{4\mu}. \tag{3.34}$$

Owing to (3.28),

$$\{xy \geqq N\sqrt{\epsilon}\} \cap \tilde{Q}_{4\mu} \subset \left\{H \geqq \frac{N\sqrt{\epsilon}}{2}\right\} \cap \tilde{Q}_{4\mu}.$$

Therefore

$$\int_{\{h \geqq N\} \cap \tilde{Q}_{4\mu}} |DT|^2 \, \mathrm{d}x \leqq \frac{C}{\sqrt{\epsilon} N^4}.$$

Hereafter we will work with the coordinates (h, θ) and use Q instead of $\tilde{Q}$ as the rectangle.

Theorem 3.2. *For $\delta > 0$ and $M > 4$. Denote $Q_{M,\delta} = \{(h, \theta) | 0 \leqq h \leqq M, \; -\delta \leqq \theta \leqq \delta\}$. Assume that $f \in C^\infty(Q_{M,\delta})$ is a solution of equation (3.29) and satisfies*

(1) $\int_{Q_{M,\delta}} |f_h|^2 \, \mathrm{d}h\mathrm{d}\theta \leqq C$;
(2) $\int_{[N,N+1] \times [-\delta,\delta]} |f_h|^2 \, \mathrm{d}h\mathrm{d}\theta \leqq \frac{C}{N^4}$ *for* $N \geqq 1$;
(3) $\epsilon \int_{Q_{M,\delta}} f_\theta^2 \, \mathrm{d}h\mathrm{d}\theta \leqq C$.

Then

$$\int_{Q_{\frac{M}{4}, \frac{\delta}{4}}} \frac{|f_h|}{\sqrt{|\theta|}} \mathrm{d}h\mathrm{d}\theta \leqq C_\delta(\sqrt{||f||_{L^\infty}} + 1),$$

the constant C_δ depends on C and δ and is independent of M and ϵ.

Note that (1) and (2) above imply that

$$\int_{Q_{M,\delta}} h^2 |f_h|^2 \, \mathrm{d}h\mathrm{d}\theta \leqq C. \tag{3.35}$$

Denote $\lambda = ||f||_{L^\infty(Q_{M,\delta})}$. We first prove several lemmas.

Lemma 3.2. *Let $Q = [a, b] \times [c, d]$. Suppose that $f \in C^\infty(Q)$ is a solution of the following equation*

$$f_{hh} + \epsilon f_{\theta\theta} - f_\theta a(h, \theta) = g \quad \text{in } Q \tag{3.36}$$

and

$$f, \, f_\theta|_{\partial Q} = 0.$$

Assume that $\frac{1}{2} \leqq a(h, \theta) \leqq 1$. Then

$$||f_{hh}||_{L^2(Q)} \leqq 3||g||_{L^2(Q)}.$$

Proof. Multiply f_θ on both sides of (3.36). Using integration by parts and the Cauchy inequality, we derive that

$$||f_\theta||_{L^2} \leqq 2||g||_{L^2}. \tag{3.37}$$

Multiply f_{hh} on both sides of (3.36). Notice that

$$\int_Q f_{hh} f_{\theta\theta}\, \mathrm{d}h\mathrm{d}\theta = ||f_{h\theta}||^2_{L^2(Q)} \geqq 0.$$

Then

$$\begin{aligned}\int_Q f_{hh}^2\, \mathrm{d}h\mathrm{d}\theta &\leqq \int_Q g f_{hh}\, \mathrm{d}h\mathrm{d}\theta + \int_Q |f_\theta f_{hh}|\, \mathrm{d}h\mathrm{d}\theta \\ &\leqq ||g||_{L^2(Q)}||f_{hh}||_{L^2(Q)} + ||f_\theta||_{L^2(Q)}||f_{hh}||_{L^2(Q)}.\end{aligned}$$

By (3.37), we get that

$$||f_{hh}||_{L^2} \leqq 3||g||_{L^2}.$$

□

The following is a boundary energy estimate.

Lemma 3.3. *Let* $Q_{r,\tau} = [0, r] \times [-\tau, \tau] \subset Q_{M,\delta}$ *and* f *satisfy assumptions in Theorem* 3.2. *Then*

$$||hf_h||_{L^4\left(Q_{\frac{r}{4},\frac{\tau}{4}}\right)} \leqq C(r, \tau)(\lambda + 1),$$

where $\lambda = ||f||_{L^\infty(Q_{M,\delta})}$.

Proof. To simplify notation, we denote $Q_s = Q_{sr,s\tau}$ and drop the dependence of C on r and τ. Choose a smooth function $\phi(h, \theta)$ satisfying that $\phi = 0$ near $\partial Q_1 \backslash \{(0, \theta)| -\tau \leqq \theta \leqq \tau\}$ and $\phi = 1$ in $Q_{\frac{1}{2}}$. Let $F = h\phi f$. Clearly, F is a smooth solution of

$$F_{hh} + \epsilon F_{\theta\theta} - F_\theta a(h, \theta) = \tilde{g} \quad \text{in } Q_1,$$

where

$$\begin{aligned}\tilde{g} = &-h\phi g + 2f_h \frac{\partial(h\phi)}{\partial h} + f\frac{\partial^2(h\phi)}{\partial^2 h} + \epsilon\left(2f_\theta \frac{\partial(h\phi)}{\partial\theta} + f\frac{\partial^2(h\phi)}{\partial^2\theta}\right) \\ &-a(h, \theta) f \frac{\partial(h\phi)}{\partial\theta}\end{aligned}$$

and

$$g = hf_h b(h, \theta).$$

According to assumptions (1) and (3) in Theorem 3.2

$$||\tilde{g}||_{L^2(Q_1)} \leqq C(\lambda + 1).$$

Hence, due to Lemma 3.2,

$$||F_{hh}||_{L^2(Q_1)} \leqq C(\lambda+1).$$

Therefore,

$$||hf_{hh}||_{L^2(Q_{\frac{1}{2}})} \leqq C(\lambda+1). \tag{3.38}$$

Integration by parts leads to

$$\begin{aligned}&\int_{Q_{\frac{1}{2}}} h^4\left(\frac{r}{2}-h\right)^2 f_h^4\,\mathrm{d}h\mathrm{d}\theta\\ &\quad= -3\int_{Q_{\frac{r}{2}}} h^4\left(\frac{r}{2}-h\right)^2 ff_h^2 f_{hh}\,\mathrm{d}h\mathrm{d}\theta - \int_{Q_{\frac{r}{2}}} w(h)ff_h^3\,\mathrm{d}h\mathrm{d}\theta,\end{aligned}$$

where $w(h) = 3h^3\left(\frac{r}{2}-h\right)^2 + h^4\left(h-\frac{r}{2}\right)$. Integration by parts again,

$$\int_{Q_{\frac{1}{2}}} w(h)ff_h^3\,\mathrm{d}h\mathrm{d}\theta = -\int_{Q_{\frac{1}{2}}} w(h)f^2 f_{hh}f_h\,\mathrm{d}h\mathrm{d}\theta - \frac{1}{2}\int_{Q_{\frac{1}{2}}} w'(h)f_h^2 f^2\,\mathrm{d}h\mathrm{d}\theta.$$

Let $M = \int_{Q_{\frac{1}{2}}} h^4\left(\frac{r}{2}-h\right)^2 f_h^4\,\mathrm{d}h\mathrm{d}\theta$. Then by (3.38), assumption (1) and Cauchy inequality,

$$M \leqq C(\lambda(\lambda+1)\sqrt{M} + \lambda^3 + 1).$$

Hence

$$M \leqq C(\lambda^4+1).$$

So Lemma 3.3 holds. □

We also have the following interior energy estimate.

Lemma 3.4. *Let* $Q_{h_0,\theta_0,r,\tau} = [h_0-r, h_0+r]\times[\theta_0-\tau,\theta_0+\tau] \subset Q_{M,\delta}$. *Then*

$$||f_h||_{L^4\left(Q_{h_0,\theta_0,\frac{r}{2},\frac{\tau}{2}}\right)} \leqq C_{r,\tau}(\lambda+1).$$

Here the constant $C_{r,\tau}$ *is independent of* h_0 *and* θ_0.

Proof. Consider $F = \phi f$ where $\phi \in C_c^\infty(Q_{h_0,\theta_0,r,\tau})$ and $\phi = 1$ in $Q_{h_0,\theta_0,\frac{r}{2},\frac{\tau}{2}}$. Then the above lemma follows from the similar calculations of the proof of Lemma 3.3 and (3.35). □

Proof of Theorem 3.2. For $1 \leqq N \leqq \frac{M}{2}$, we denote

$$Q_N = [N-1, N] \times \left[-\frac{\delta}{4}, \frac{\delta}{4}\right].$$

According to Lemma 3.3 and Lemma 3.4,

$$||hf_h||_{L^4(Q_1)} \leqq C(\lambda + 1).$$

and

$$||f_h||_{L^4(Q_N)} \leqq C(\lambda + 1) \quad \text{for } N \geqq 2.$$

Then

$$\begin{aligned}\int_{Q_1} \frac{|f_h|}{\sqrt{|\theta|}} \,\mathrm{d}h\mathrm{d}\theta &= \int_{Q_1} \frac{\sqrt{h|f_h|}\sqrt{|f_h|}}{\sqrt{h\theta}} \,\mathrm{d}h\mathrm{d}\theta \\ &\leqq \sqrt{||hf_h||_{L^4(Q_1)}}\sqrt{||f_h||_{L^2(Q_1)}}||\frac{1}{\sqrt{h|\theta|}}||_{L^{\frac{8}{5}}(Q_1)} \\ &\leqq C(\sqrt{\lambda} + 1)\end{aligned}$$

Also for $N \geqq 2$,

$$\begin{aligned}\int_{Q_N} \frac{|f_h|}{\sqrt{\theta}} \,\mathrm{d}h\mathrm{d}\theta &= \int_{Q_N} \frac{\sqrt{|f_h|}\sqrt{|f_h|}}{\sqrt{\theta}} \,\mathrm{d}h\mathrm{d}\theta \\ &\leqq \sqrt{||f_h||_{L^3(Q_N)}}\sqrt{||f_h||_{L^2(Q_N)}}||\frac{1}{\sqrt{|\theta|}}||_{L^{\frac{12}{7}}(Q_N)} \\ &\leqq C\sqrt{||f_h||_{L^3(Q_N)}}\frac{1}{N} \\ &\leqq \frac{C(1+\lambda^{\frac{1}{3}})}{N^{1+\frac{1}{3}}}.\end{aligned}$$

The last inequality comes from the Cauchy inequality

$$||f_h||^3_{L^3(Q_N)} \leqq ||f_h||^2_{L^4(Q_N)}||f_h||_{L^2(Q_N)}.$$

Hence

$$\int_{Q_{\frac{M}{4},\frac{\delta}{4}}} \frac{|f_h|}{\sqrt{\theta}} \,\mathrm{d}h\mathrm{d}\theta \leqq C(\sqrt{\lambda} + 1).$$

□

Proof of Theorem 3.1. According to (3.34), we choose $M = \frac{\mu^2}{\sqrt{\epsilon}}$ and $\delta = \frac{\mu^2}{2}$. Owing to (3.32), (3.33) and (3.34), $f(h, \theta) = T(x, y)$ satisfies all the assumptions of Theorem 3.2. So by (3.31) and (3.34),

$$||f_h Dh||_{L^1\left(\tilde{Q}_{\frac{\mu}{2}}\right)} \leqq C(\sqrt{\lambda} + 1) \tag{3.39}$$

for $\lambda = ||T||_{L^\infty(\mathbb{T}^2)}$ and

$$\tilde{Q}_{\frac{\mu}{2}} = \left[0, \frac{\mu}{2}\right] \times \left[0, \frac{\mu}{2}\right] \quad \text{for } (x_1, x_2) \text{ coordinate.}$$

Then by calculations before Theorem 3.2, we obtain

$$||DT||_{L^1\left(\Pi_{\frac{\mu}{2}}\right)} \leqq C(\sqrt{\lambda} + 1).$$

Here $\Pi_{\frac{\mu}{2}}$ is the union of four corners. Together with (3.27), we have:

$$||DT||_{L^1(\mathbb{T}^2)} \leqq C(\sqrt{\lambda} + 1).$$

In view of (3.23), we establish Theorem 3.1.

4. Linear law in shear flows

In this section, we will investigate the front speed asymptotics for a shear flow, that is, $V(x, y) = (v(y), 0)$ where $v(y)$ is a smooth periodic function with mean zero, but not identically zero. Unlike the cellular flow, the turbulent flame speed from the shear flow grows linearly with respect to A. For the inviscid G-equation, an explicit formula of $\bar{H}$ is given in [6]. Here we focus on the viscous G-equation. We will also discuss the curvature dependent G-equation.

For $P = (m, n)$, the corresponding cell problem is reduced to an ODE

$$-d\psi'' + \sqrt{m^2 + (n + \psi')^2} + A\, m\, v(y) = \lambda(A). \tag{4.1}$$

To simplify the notation, we write $\bar{H}(P, A, d)$ as $\lambda(A)$. If $m = 0$, it is obvious that $\lambda(A) = |n|$. So throughout this section, we assume that $m \neq 0$. We first show that the turbulent flame speed $\lambda(A)$ is enhanced as A increases.

Theorem 4.1. *$\lambda = \lambda(A)$ as a function of $A \geqq 0$ is convex and strictly increasing.*

Proof. The convexity follows immediately from the inf-max formula

$$\lambda(A) = \inf_{\phi \in C^2(\mathbb{T}^1)} \max_{y \in \mathbb{T}^1} \{-d\phi'' + \sqrt{m^2 + (n + \phi')^2} + Amv(y)\}.$$

To prove that it is strictly increasing, it suffices to show that

$$\lambda(0) = |P| < \lambda(A), \quad \text{for all } A > 0.$$

This follows immediately from Jensen's inequality and the strict convexity of the function $f(t) = \sqrt{m^2 + t^2}$. □

Now let us look at the asymptotic behavior of $\frac{\lambda(A)}{A}$ as $A \to +\infty$. Choose a solution ψ (viscosity solution if $d = 0$) of the cell problem with mean zero. Denote $\psi_A = \frac{\psi}{A}$. Then ψ_A satisfies that

$$-d\psi_A'' + \sqrt{\frac{m^2}{A^2} + \left(\frac{n}{A} + \psi_A'\right)^2} + mv(y) = \frac{\lambda(A)}{A}.$$

Since $\frac{\lambda(A)}{A}$ is bounded, maximal principle implies that ψ_A' is uniformly bounded. Hence ψ_A is equally continuous for both the inviscid ($d = 0$) and the viscous case ($d > 0$). Upon a subsequence, if necessary, we may assume that $(\psi_A, \frac{\lambda(A)}{A})$ converges to $(\phi, \bar{\lambda})$. Stability of viscosity solutions implies that $(\phi, \bar{\lambda})$ satisfies the following cell problem:

$$-d\phi'' + |\phi'| + mv(y) = \bar{\lambda} \tag{4.2}$$

which is a special case of (4.1) for $P = (0, 0)$ and $A = 1$ subject to $\int_{\mathbb{T}^1} \phi \, dx = 0$. Here $\mathbb{T}^1 = [0, 1]$. Therefore $\bar{\lambda}$ and ϕ' are uniquely given. In particular, $\bar{\lambda}$ is positive. Hence $\lambda(A)$ grows linearly for the shear flow.

Theorem 4.2.

$$\lim_{A \to +\infty} \frac{\lambda(A)}{A} = \bar{\lambda} = \bar{\lambda}(d) > 0.$$

When $d = 0$, $\bar{\lambda}(0) = \max_{\mathbb{T}^1} mv$.

Proof. We need to show only that $\bar{\lambda} > 0$. Taking integration on both sides of (4.2) leads to

$$\int_0^1 |\phi'| \, dy = \bar{\lambda}.$$

Since v is not a constant, ϕ' cannot vanish everywhere. So $\bar{\lambda}$ must be positive. □

Next we shall see how $\bar{\lambda}$ depends on the diffusivity constant d. The following result says that diffusion will slow down the front propagation.

Theorem 4.3. *For* $d > 0$, $\bar{\lambda} = \bar{\lambda}(d)$ *is strictly decreasing as a function of* d.

Proof. Let $w(y, d) = \phi'(y, d)$ and take the derivative with respect to d on both sides of (4.2). We get that

$$-w' - dw_d' + \text{sign}(w) w_d = \bar{\lambda}_d.$$

Let $h = -(dw_d + w)$. We have that

$$h' - d^{-1}\text{sign}(w) h = \bar{\lambda}_d + d^{-1} |w|. \tag{4.3}$$

If $\bar{\lambda}_d \geqq 0$,

$$h' - d^{-1}\text{sign}(w) h \geqq 0. \tag{4.4}$$

Since $\int_0^1 h \, \mathrm{d}y = 0$, there exists y_0 such that $h(y_0) = 0$. According to (4.4) and the periodicity of h, we must have that

$$h \equiv 0.$$

Due to (4.3), $w \equiv 0$. So (4.2) implies that v is a constant function. This is a contradiction. $\square$

Apparently, $\lim_{d\to 0^+} \bar{\lambda}(d) = \bar{\lambda}(0) = \max_{\mathbb{T}^1} mv$. A subsequent question is the convergence rate. It is not obvious at all whether $\bar{\lambda}(d)$ is differentiable at $d = 0$ since the inviscid equation ($d = 0$) has multiple solutions and those solutions are not C^1. In the following, we show that $\bar{\lambda}_d(0) = 0$.

Theorem 4.4.

$$\lim_{d\to 0^+} \frac{\bar{\lambda}(0) - \bar{\lambda}(d)}{d} = 0.$$

Proof. Without loss of generality, we assume that $m = 1$ and $v(0) = \max_{\mathbb{T}^1} v = 0$. By Theorem 3.3, $\bar{\lambda}(d) < \bar{\lambda}(0) = 0$ for all $d > 0$.
Case I: Suppose that 0 is the unique maximum point of v in $[0, 1)$. For $d > 0$, let $\phi = \phi(y, d)$ be the unique solution of

$$-d\phi'' + |\phi'| + v(y) = \bar{\lambda}$$

satisfying $\phi(0, d) = 0$. Then

$$\lim_{d\to 0} \phi = s(y) \quad \text{uniformly in } \mathbb{T}^1,$$

where $s(y)$ is the unique viscosity solution of

$$|s'| + v(y) = \bar{\lambda}(0) = 0$$

satisfying that $s(0) = 0$ which is given by the formula

$$s(y) = \begin{cases} \int_0^y (-v(t)) \, \mathrm{d}t & \text{for } 0 \leqq y \leqq \bar{y} \\ \int_y^1 (-v(t)) \, \mathrm{d}t & \text{for } \bar{y} \leqq y \leqq 1. \end{cases}$$

Here $\bar{y} \in (0, 1)$ is the unique point which satisfies that

$$\int_{\bar{y}}^1 (-v(t)) \, \mathrm{d}t = \int_0^{\bar{y}} (-v(t)) \, \mathrm{d}t.$$

Let

$$\hat{s}(y) = \begin{cases} 2\int_0^y (-v(t)) \, \mathrm{d}t & \text{for } y \geqq 0 \\ 2\int_y^0 (-v(t)) \, \mathrm{d}t & \text{for } y \leqq 0. \end{cases}$$

Choose $y_d \in [0, 1]$ such that

$$\phi(y_d, d) - \hat{s}(y_d) = \max_{\mathbb{T}^1} (\phi - \hat{s}).$$

Since

$$s(0) - \hat{s}(0) = 0 > s(y) - \hat{s}(y) \quad \text{for } y \neq 0,$$

we have that

$$\lim_{d \to 0} y_d = 0. \tag{4.5}$$

The maximal principle implies that

$$\phi''(y_d) \leqq \hat{s}''(y_d) \leqq C|y_d|$$

and

$$\phi'(y_d) = \hat{s}'(y_d).$$

Hence

$$0 \geqq \bar{\lambda}(d) = -d\phi''(y_d) + |\phi'(y_d)| + v(y_d) \geqq -Cd|y_d| + |v(y_d)| \geqq -Cd|y_d|.$$

Accordingly, (4.5) implies Theorem 4.4.

Case II: v has more than one maximum point. Choose a smooth periodic function L such that $L(0) = 0$ and

$$L(y) < 0 \quad \text{for } y \in (0, 1).$$

For $\delta > 0$, write

$$v_\delta = v + \delta L.$$

Then v_δ has a unique maximum point at $y = 0$ in $[0, 1)$. Let $\bar{\lambda}_\delta(d)$ be the corresponding asymptotic limit. Owing to the inf-max formula,

$$\bar{\lambda}(d) \geqq \bar{\lambda}_\delta(d).$$

By case I,

$$\lim_{d \to 0} \frac{\bar{\lambda}_\delta(d)}{d} = 0. \tag{4.6}$$

Therefore Theorem 4.4 holds. □

Remark 4.1. It remains an interesting problem to study whether $\bar{\lambda}(d) = \bar{\lambda}(0) + O(d^r)$ for some power $r > 1$. Computation suggests that $r = 2$. Table 1 lists the raw and d^2-scaled $\bar{\lambda}$ values for $d \sim 0$, and suggests the quadratic behavior of $\bar{\lambda}$ in d. The numerical values in Table 1 are obtained from the cell problem (3.2) with $m = 1$, $v(y) = \cos(2\pi y) - 1$, and so $\lambda(0) = 0$. We consider the time dependent problem

$$\phi_t - d\phi'' + |\phi'| + (\cos(2\pi y) - 1) = 0, \quad \phi(t = 0) = 0.$$

Then $\bar{\lambda}$ is extracted from $\phi_t \to -\bar{\lambda}$ uniformly in $(0, 1)$ as $t \to +\infty$.

Spatial derivatives ϕ' and ϕ'' are discretized by central differencing with small enough grid size to ensure accuracy. By symmetry, we have $\phi'(y) > 0$ in $(0, 1/2)$ and $\phi'(y) < 0$ in $(1/2, 1)$ for any $t \geqq 0$. An implicit Euler scheme can be readily used to relax the time step constraint and speed up convergence to steady state. The overall scheme is implicit in time and second order in space.

Table 1. Values of $\bar{\lambda}$ and d^2-scaled $\bar{\lambda}$ at small d for $v(y) = \cos(2\pi y) - 1$

d	$-\bar{\lambda}$	$-\bar{\lambda}/d^2$
4e−2	5.9414e−2	3.7134e+1
2e−2	1.5540e−2	3.8850e+1
1e−2	3.9280e−3	3.9280e+1
4e−3	6.3428e−4	3.9642e+1
2e−3	1.5514e−4	3.8785e+1
1e−3	3.7061e−5	3.7061e+1

The proof of Theorem 4.4 can be easily modified to recover a known interesting result in [11]. Denote

$$\Gamma = \{y \in \mathbb{T}^1 \mid v(y) = \max v\} = \{y_i\}_{i=1}^m.$$

We assume that $v''(y_i) \neq 0$ for all i and $|v''(y_i)|$ is strictly increasing as i varies from 1 to m. Suppose that $\phi = \phi(y, d)$ is the unique solution of

$$-d\phi'' + \frac{1}{2}|\phi'|^2 + v(y) = I_d$$

subject to

$$\int_0^1 \phi \, \mathrm{d}y = 0.$$

Then

Theorem 4.5. (JAUSLIN–KREISS–MOSER [11]) *Assume* $\max_{\mathbb{T}^1} v = 0$. *Then*

$$\lim_{d \to 0} \frac{I_d}{d} = -\sqrt{|v''(y_1)|}$$

and ϕ *uniformly converges to* ϕ_0, *which is the periodic viscosity solution of*

$$\frac{1}{2}|\phi_0'|^2 + v(y) = 0$$

with a unique transition point y_1.

Remark 4.2. The inviscid equation $\frac{1}{2}|\phi'|^2 + v(y) = 0$ has many solutions, even up to a constant when v has multiple maximum points ($m > 1$). A solution is uniquely determined by its transition points, that is, where it changes from decreasing to increasing. The above theorem says that under some nondegeneracy conditions, the vanishing viscosity method will select a unique "physical" solution. Compared to the method in [11], our approach is more elementary and can be easily extended to higher dimensions and more general Hamiltonians, at least when the Aubry set consists only of finitely many points (see other approaches in [2] using stochastic control and random perturbation theories).

In combustion modeling, the laminar flame speed s_l might also depend on the curvature of the flame front. PETERS [21] proposed the following curvature dependent G-equation:

$$G_t - d|DG|\mathrm{div}\left(\frac{DG}{|DG|}\right) + s_l|DG| + V(x)\cdot DG = 0.$$

Here $\kappa = \mathrm{div}\left(\frac{DG}{|DG|}\right)$ is the mean curvature of the flame front. In general, we do not know how to prove the existence of the turbulent flame speed for the curvature dependent G-equation. However, for the shear flow, the corresponding cell problem is reduced to an ODE

$$\frac{-dm^2\phi''}{m^2+(n+\phi')^2} + \sqrt{m^2+(n+\phi')^2} + mv(y) = \lambda.$$

Here we set $s_l = 1$. It is very easy to verify the existence of classical solution ϕ (unique up to an additive constant) and a constant λ. Intuitively, the λ from the curvature G-equation should be between the inviscid and the viscous case. The following theorem says that its asymptotic limit coincides with the inviscid case.

Theorem 4.6. *If v is scaled to $A\,v$, then $\lambda = \lambda(A)$ satisfies the growth law:*

$$\lim_{A\to+\infty}\frac{\lambda(A)}{A} = \max_{\mathbb{T}^1} mv.$$

Proof. Up to a subsequence if necessary, we may assume that

$$\lim_{A\to+\infty}\frac{\lambda(A)}{A} = \bar{\lambda}.$$

Suppose that $\phi(y_0) = \min_{\mathbb{T}^1}\phi$. Then

$$\lambda(A) \leqq \sqrt{m^2+n^2} + Amv(y_0) \leqq \sqrt{m^2+n^2} + A\max_{\mathbb{T}^1} mv(y).$$

Hence

$$\bar{\lambda} \leqq \max_{\mathbb{T}^1} mv.$$

One the other hand, let $mv(y_1) = \max_{\mathbb{T}^1} mv$. Then for any $\delta > 0$,

$$\begin{aligned}\frac{\delta\lambda(A)}{A} &= \frac{1}{A}\int_{y_1}^{y_1+\delta}\frac{-dm^2\phi''}{m^2+(n+\phi')^2}\,\mathrm{d}y + \frac{1}{A}\int_{y_1}^{y_1+\delta}(\sqrt{m^2+(n+\phi')^2} + Amv(y))\,\mathrm{d}y\\ &\geqq -\frac{dm\pi}{A} + \delta\min_{[y_1,y_1+\delta]} mv.\end{aligned}$$

Therefore $\bar{\lambda} \geqq \max_{\mathbb{T}^1} mv$. So Theorem 4.6 holds. □

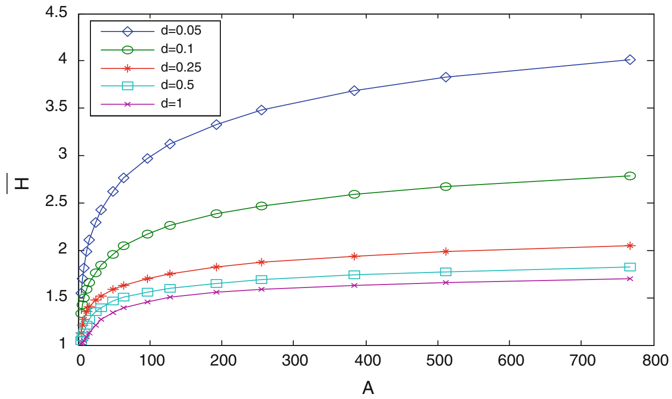

Fig. 4. Plot of $\bar{H}(P, d)$ vs. $A \in [0, 768]$ for $d = 0.05, 0.1, 0.25, 0.5, 1$

5. Numeric results of $\bar{H}$ in cellular flow

Computation is carried out with finite difference discretization and the iteration method on equation (1.4) for $d \geqq 0.1$ and the upwind method on the evolution equation (1.3) for $d < 0.1$ with small enough grid size. We choose $s_l = 1$, $P = e_1$, and $V(x) = (A/2\pi)\nabla^{\perp} \sin 2\pi x_1 \sin 2\pi x_2$. More details can be found in [13]. Numerical values of $\bar{H}(A, d)$ are obtained for $d \in (0, 1]$ and A up to 768. Figure 4 clearly shows that $\bar{H}$ is decreasing with respect to the diffusivity d and increasing with respect to the flow intensity A. This qualitative property of $\bar{H}$ remains to be proved. It is also an interesting problem to identify the limit

$$\lim_{A \to +\infty} \bar{H}(P, d, A).$$

6. Conclusions

We studied the front speed asymptotics in the viscous G-equation by analyzing the related cell problem of homogenization. A new and striking result is that for cellular flows and any positive viscosity in the viscous G-equation, the front speed is uniformly bounded. In contrast, the front speed of the inviscid G-equation grows almost linearly in the large amplitude regime of the cellular flows. In shear flows, the front speed of the G-equation grows linearly in the large flow amplitude. The growth rate is a monotone decreasing function of the viscosity coefficient. The linear growth law in shear flows also persists in the curvature dependent G-equation, with the same growth rate as that of the inviscid G-equation.

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