SIAM J. MATH. ANAL.
Vol. 46, No. 4, pp. 2444–2467

ASYMPTOTIC GROWTH RATES AND STRONG BENDING OF TURBULENT FLAME SPEEDS OF G-EQUATION IN STEADY TWO-DIMENSIONAL INCOMPRESSIBLE PERIODIC FLOWS*

JACK XIN† AND YIFENG YU†

Abstract. The study of turbulent flame speeds (the large time front speeds) is a fundamental problem in turbulent combustion theory. A significant project is to understand how the turbulent flame speed (s_T) depends on the flow intensity (A). The G-equation is a very popular level set flame propagation model in the turbulent combustion community. The main purpose of this paper is to study properties of $\lim_{A\to+\infty} \frac{s_T}{A}$ and $\lim_{A\to+\infty} s_T$ (if finite, or strong bending) in the G-equation model for two-dimensional (2D) divergence-free periodic flows. Our analysis is based on the invariant measures and rotation vectors of the 2D flows and the travel times of the associated flow trajectories under control. Optimal linear/sublinear growth and strong bending conditions are precisely given in terms of rotation vectors and periodic orbits. A strong bending formula of s_T in the cat's-eye flow is discovered by averaging the controlled characteristics of the G-equation. The growth rate of s_T and that of the related front speeds of reaction-diffusion-advection equations (with Kolmogorov–Petrovsky–Piskunov nonlinearity) are shown to be zero or nonzero simultaneously in 2D flows, yet they differ in the three-dimensional (3D) Roberts cell flows that depend on two spatial variables. A future program will be to extend our analysis to more complex fluid flows, such as unsteady 2D flows and 3D flows with chaotic structures.

Key words. G-equation, optimal control, cat's-eye flow, turbulent flame speed, front speed, bending effect, large flow, invariant measure, periodic orbit, rotation vector, travel time of flow intensity

AMS subject classifications. 70H20, 76F25, 76M50, 76M30

DOI. 10.1137/130929990

1. Introduction.

Turbulent combustion is a complex nonlinear and multiscale phenomenon [19]. A comprehensive physical-chemical modeling requires a system of reaction-diffusion-advection (RDA) equations coupled with the Navier–Stokes equations. For theoretical understanding and efficient modeling of the turbulent flame propagation, various simplified or phenomenological models have been proposed and studied. Most notably, these models are passive scalar RDA equations and Hamilton–Jacobi (HJ) equations, as documented in the books [25, 19, 26] and research papers [1, 2, 6, 7, 10, 14, 16, 18, 21, 23, 24, 29], to name a few. One of the approaches is the level set formulation of interface motion laws in the thin interface regime. The simplest motion law is that the normal velocity of the interface (V_n) is equal to a constant s_l (the laminar speed) plus the projection of fluid velocity along the normal n. See Figure 1. The laminar speed is the flame speed when fluid is at rest. Let the flame front be the zero level set of a function $G(x,t)$; the burnt region is $G(x,t) < 0$, and the unburnt region is $G(x,t) > 0$. The normal direction pointing from the burnt region to the unburnt region is $DG/|DG|$, and the normal velocity is $-G_t/|DG|$. The motion law becomes the so-called G-equation, a well-known model in turbulent

*Received by the editors July 22, 2013; accepted for publication (in revised form) March 17, 2014; published electronically July 8, 2014.
http://www.siam.org/journals/sima/46-4/92999.html
†Department of Mathematics, University of California at Irvine, Irvine, CA 92697 (jxin@math.uci.edu, yyu1@math.uci.edu). The first author's research was partially supported by DMS-1211179. The second author's research was partially supported by NSF CAREER award DMS-1151919.

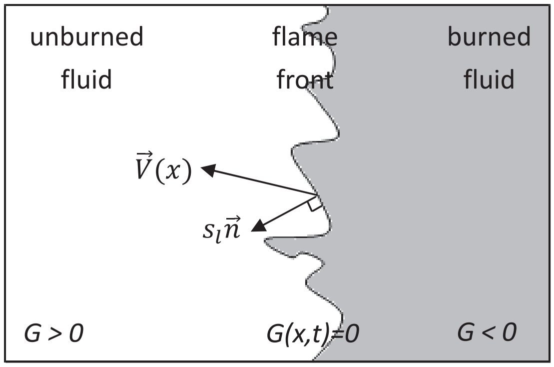

FIG. 1. *G-equation model.*

combustion [25, 19]:

$$G_t + V(x) \cdot DG + s_l|DG| = 0. \tag{1.1}$$

Chemical kinetics and Lewis number effects are all included in the laminar speed s_l which is provided by a user. Throughout this paper, the velocity field V is a smooth and periodic vector-valued function which is incompressible ($\mathrm{div}(V) = 0$) and has zero mean ($\int_{\mathbb{T}^n} V\,dx = 0$). *The prediction of the turbulent flame speed is a fundamental problem in turbulent combustion theory* [25, 21, 19]. An important project is to understand the dependence of turbulent flame speeds on flow intensity. Roughly speaking, turbulent flame speed is the average flame propagation speed under the influence of strong flow. Under the G-equation model, for a specified unit direction p, the turbulent flame speed ($s_T(p)$) is given by

$$s_T(p) = \lim_{t\to+\infty} \frac{-G(x,t)}{t} \quad \text{locally uniformly for all } x \in \mathbb{R}^n. \tag{1.2}$$

Note that the limit is independent of x. Here $G(x,t)$ is the unique viscosity solution of (1.1) with initial data $G(x,0) = p \cdot x$. According to the control interpretation, the solution $G(x,t)$ has a representation formula $-G(x,t) = \sup_\xi(-p\cdot\xi(t))$, where $\xi : [0,t] \to \mathbb{R}^n$ runs through all Lipschitz continuous curves satisfying $\xi(0) = x$ and $|\dot{\xi} + V(\xi)| \le s_l$ a.e. in $[0,t]$. This formula is the limiting case of the classical Hopf–Lax–Oleinik formula for superlinear convex HJ equations [11]. The Hamiltonian of the G-equation, $H(p,x) = s_l\,|p| + V(x)\cdot p$, is convex but not coercive. The existence of the limit (1.2) has been rigorously established in [27] and [4] independently. In homogenization theory, s_T is also the effective Hamiltonian of the following cell problem:

$$s_l|p + D\tilde{G}| + V(x)\cdot(p + D\tilde{G}) = \bar{H}(p) = s_T(p). \tag{1.3}$$

Here $\bar{H}(p)$, the effective Hamiltonian, is the unique constant such that the above equation admits periodic approximate viscosity solutions. As usual, the following inf-max formula holds:

$$s_T(p) = \inf_{\phi\in C^1(\mathbb{T}^n)} \left(\max_{\mathbb{R}^n} s_l|p + D\phi| + V(x)\cdot(p + D\phi)\right). \tag{1.4}$$

Here $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional flat torus. Now scale V to AV for $A > 0$. The cell problem becomes

$$s_l|p + D\tilde{G}| + AV(x) \cdot (p + D\tilde{G}) = s_T(p, A).$$

It is clear that $s_T = s_T(p, A)$ grows at most linearly as $A \to +\infty$. Also

$$s_T(p, A) = \lim_{t\to+\infty} \frac{-G(x,t)}{t} \quad \text{locally uniformly for all } x \in \mathbb{R}^n, \tag{1.5}$$

where

$$-G(x,t) = \sup_{\xi} \left(-p \cdot \xi(t)\right). \tag{1.6}$$

Here $\xi : [0,t] \to \mathbb{R}^n$ runs through all Lipschitz continuous curves which satisfy $\xi(0) = x$ and $|\dot{\xi} + AV(\xi)| \le s_l$ a.e. in $[0,t]$. The goal of this paper is as follows.

(1) For any unit vector $p \in \mathbb{R}^n$, identify and study properties of the limit

$$\lim_{A\to+\infty} \frac{s_T(p,A)}{A}.$$

It is not difficult to establish the existence of the limit. A much more interesting and challenging problem is to understand deep properties of the limit. The first step is to determine when the limit is zero, i.e, when the so-called *bending effect* in combustion literature occurs. A complete answer will be given here for two-dimensional (2D) flows. A future project will be to study flows with chaotic structures which are much harder to analyze [3].

(2) If $s_T(p, A)$ happens to be uniformly bounded as $A \to +\infty$ (strong bending), determine the limit

$$\lim_{A\to+\infty} s_T(p, A).$$

We will solve this problem for 2D cat's-eye flow. This is somewhat equivalent to averaging the fast control system $|\dot{\xi} + AV(\xi)| \le 1$ as $A \to +\infty$. Here we would like to mention that there are many very interesting works on the average of the random perturbation $\dot{\xi} = AV(\xi) + dW$ as $A \to +\infty$ with W as the Brownian motion (see Chapter 8 of [13], for instance).

(3) Compare the turbulent flame speeds predicted by the G-equation model with the front speeds predicted by the RDA model with Kolmogorov–Petrovsky–Piskunov (KPP) nonlinearity. Regarding the bending effect, one of our results says that these two models are qualitatively the same for 2D planar flows. However, this is no longer true if the 2D flow lies in the three-dimensional (3D) space (i.e., flows with three nonzero components that depend on two space variables).

For the reader's convenience, we give a brief review of the RDA model, which has been extensively studied in the literature; see [26] for more details and background. The passive scalar RDA equation for the temperature field is

$$\begin{aligned} &T_t + V(x) \cdot DT = d\,\Delta T + f(T), \\ &T(x,0) = T_0(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.7}$$

where T represents the reactant temperature, D is the spatial gradient operator, $V(x)$ is a prescribed fluid velocity, d is the molecular diffusion constant, and f is a KPP-type nonlinear reaction function. A prototypical example is $f(T) = T(1 - T)$. Under this model, the turbulent flame speed along any specified unit vector $p \in \mathbb{R}^n$ is defined as the minimal traveling speed and has a variational representation:

$$c_p^* = \inf_{\lambda>0} \frac{f'(0) + H^*(\lambda p)}{\lambda}, \tag{1.8}$$

where $H^*(p)$ is given by the cell problem

$$-d\Delta w + d|p + Dw|^2 + V(x) \cdot (p + Dw) = H^*(p)$$

for $w \in C^\infty(\mathbb{T}^n)$. Let us again scale V to AV; then $c_p^* = c_p^*(A)$, and the following limit holds [30]:

$$\lim_{A\to+\infty} \frac{c_p^*(A)}{A} = \sup_{\hat\sigma} \int_{\mathbb{T}^n} V \cdot p \, d\hat\sigma = c^*(p), \tag{1.9}$$

where $\hat\sigma = w^2 dx$ for $w \in \Gamma$ defined as (H^1-invariant measures)

$$\Gamma = \left\{ w \in H^1(\mathbb{T}^n) | \ V \cdot Dw = 0, \ ||w||_{L^2(\mathbb{T}^n)} = 1, \ ||Dw||_2^2 \le f'(0) \right\}.$$

We will prove the following four theorems in this paper. The proof of Theorem 1.3 is the lengthiest. Hereafter, we assume $s_l = 1$ for convenience.

Theorem 1.1.

$$\begin{aligned}\lim_{A\to+\infty} \frac{s_T(p,A)}{A} &= \max_{\sigma\in\Lambda} \int_{\mathbb{T}^n} p \cdot V(x) \, d\sigma \\ &= \max_{\dot\xi = V(\xi)} \limsup_{T\to+\infty} \frac{\xi(T)\cdot p}{T},\end{aligned}$$

where Λ is the collection of all Borel probability measures on $\mathbb{T}^n$ which are invariant under the flow $\dot\xi = V(\xi)$.

Although the proof is not very complicated, the derivation of the above formula (the first equality) is strongly motivated by Mather's minimization principle [15] and the weak KAM theory [12]. We also would like to remark that the inf-max formula (1.4) easily implies [28]

$$\lim_{A\to+\infty} \frac{s_T(p,A)}{A} = \inf_{\phi\in C^1(\mathbb{T}^n)} \max_{\mathbb{R}^n} \{(p + D\phi)\cdot V\}.$$

However this inf-max formulation is not convenient for exploring delicate information of the limit. Denote

$$c(p) = \lim_{A\to+\infty} \frac{s_T(p,A)}{A}.$$

It is obvious from Theorem 1.1 that $c(p)$ is convex and positive homogeneous of degree one as a function of p. A much more interesting project is to study deeper properties, especially for some representative flows. When $n = 2$, due to the integrability of the flow, we are able to completely characterize the function $c(p)$.

Theorem 1.2. *Assume $n = 2$. Then one of the following holds:*

(i)

$$c(p) = 0 \quad \textit{for all unit vectors } p \in \mathbb{R}^2.$$

(ii) *There exist a nonzero vector* $Q \in \mathbb{R}^2$ *and two positive constants* λ^+, λ^- *such that for any unit vector* $p \in \mathbb{R}^2$

$$c(p) = \max\{\lambda^+ p \cdot Q,\ -\lambda^- p \cdot Q\}. \tag{1.10}$$

In particular, if $p \cdot Q = 0$, *then actually the strong bending occurs:*

$$\sup_{A \geq 0} s_T(p, A) < +\infty.$$

Moreover, case (ii) *happens if and only if there exists a periodic orbit of* $\dot{\xi} = V(\xi)$ *with a nonzero rotation vector* Q.

The above theorem gives an equivalence between linear growth of $s_T(p, A)$ and existence of nonzero rotation vectors (i.e., unbounded periodic orbits). A similar conclusion is known for the RDA model c_p^* (Theorem 1.3 in [9]). Among interesting 2D flows, cellular flows belong to case (i), and cat's-eye flows belong to case (ii). It is natural to ask what $\lim_{A\to+\infty} s_T(p, A)$ is in case (ii) when $p \cdot Q = 0$, which involves averaging the fast control system $|\dot{\xi} + AV(\xi)| \leq 1$ as $A \to +\infty$. We will answer this question for the cat's-eye flow. The proof of Theorem 1.3 relies on a delicate analysis of the nice structure of this specific flow, especially the nondegeneracy of critical points. The symmetry is not essential and only serves to simplify proofs. It is not clear to us whether the limit exists for general 2D incompressible flows. For those concrete flows, we take the periodicity to be $\mathbb{R}^2/2\pi\mathbb{Z}^2$.

THEOREM 1.3. *For the cat's-eye flow* (5.1), Q *is proportional to* $(1, 1)$, *and for* $p = \pm(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$,

$$\lim_{A\to+\infty} s_T(p, A) = \frac{\pi}{2\sqrt{2}\theta(\delta)}.$$

Here $\theta \in C([0, \delta])$ *is the continuous function given later by* (5.14).

Obviously, $c(p) \geq c^*(p)$. The following result says that the G-equation model and the RDA model predict the bending effect simultaneously for 2D flows on the plane, which, however, is not true if the flow lives in the 3D space.

THEOREM 1.4. *Assume* $n = 2$. *Then for any unit vector* p

$$c(p) = 0 \quad \textit{if and only if} \quad c^*(p) = 0. \tag{1.11}$$

However, this is in general false for $n = 3$. *For* $p = (0, 0, 1)$ *and the Roberts cell flows* $V(x) = (-H_{x_2}, H_{x_1}, \cos x_1 + \cos x_2)$, *we have that*

$$c(p) = 2 \quad \textit{but} \quad c^*(p) = 0. \tag{1.12}$$

Here $H(x) = \sin x_1 \sin x_2$.

By a suitable rotation, the above flow was introduced by Roberts in the 1970s (eq. (6.2) in [20]) in the study of dynamo actions (see also [5]). The fact that $c(p) = 2$ is obtained by an invariant measure supported on an unstable periodic orbit. It will be an interesting problem to determine the exact growth law of c_p^* in the Roberts cell flow.

The subsequent sections contain results from dynamical systems as preparation (section 2), proofs of the above four theorems (sections 3–6), and concluding remarks (section 7).

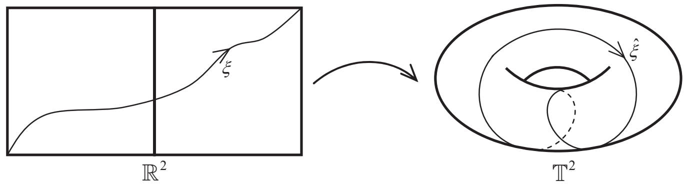

FIG. 2. *Schematic of the projection $\mathbb{R}^n \to \mathbb{T}^n$, and a periodic orbit on $\mathbb{T}^n$.*

Assumptions and notation.

(1) $\mathbb{Z}^n = \{(x_1, x_2, \ldots, x_n) | \ x_i \in \mathbb{Z}\}$. $\mathbb{T}^n$ denotes the n-dimensional flat torus, i.e, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. In computations of integrals, $\mathbb{T}^n$ is identified with the unit cube $[0,1]^n$, i.e., $\int_{\mathbb{T}^n} f\,dx = \int_{[0,1]^n} f\,dx$.

(2) $f \in C^r(\mathbb{T}^n)$ if $f \in C^r(\mathbb{R}^n)$ and it is periodic; i.e, $f(x+\vec{v}) = f(x)$ for $x \in \mathbb{R}^n$ and $\vec{v} \in \mathbb{Z}^n$. Df is the gradient of f.

(3) Throughout this paper, we assume that $s_l = 1$ and the velocity field $V \in C^\infty(\mathbb{T}^n)$ is divergence-free ($\mathrm{div}(V) = 0$) and has mean zero, i.e, $\int_{\mathbb{T}^n} V\,dx = 0$.

(4) For a curve $\xi : \mathbb{R} \to \mathbb{R}^n$, $\hat{\xi}$ is its image under the natural projection $\mathbb{R}^n \to \mathbb{T}^n$. See Figure 2 for a projected periodic orbit on the torus.

2. Preliminary. In this section, we will present some basic terminology and results in dynamical systems which are used in this paper.

DEFINITION 2.1. *An orbit $\xi : [0,\infty) \to \mathbb{R}^n$ of $\dot{\xi}(t) = V(\xi(t))$ is periodic, and $\tilde{T} > 0$ is called a period if $\xi(\tilde{T}) - \xi(0) \in \mathbb{Z}^n$. Furthermore,*

$$Q = \frac{\xi(\tilde{T}) - \xi(0)}{\tilde{T}}$$

is named the rotation vector of ξ and is independent of the choice of period. Also, if ξ is not a single point and $T_0 > 0$ is the minimum period,

$$\int_0^{T_0} |V(\xi(t))|\,dt$$

is named the unit length of the periodic orbit. Note that $Q \neq 0$ is equivalent to saying that ξ is unbounded.

DEFINITION 2.2. *$x_0 \in \mathbb{R}^n$ is called a recurrent point if along the flow $\dot{\xi} = V(\xi)$ with $\xi(0) = x_0$ there exists $T_m \to +\infty$ such that*

$$\lim_{T_m \to +\infty} d\left(\xi(T_m),\ x_0 + \mathbb{Z}^n\right) = 0.$$

DEFINITION 2.3. *A Borel probability measure σ on $\mathbb{T}^n$ is called invariant under the flow $\dot{\xi} = V(\xi)$ (or, equivalently, $\dot{\xi} = -V(\xi)$) if for any $f \in C(\mathbb{T}^n)$ and $t \geq 0$*

$$\int_{\mathbb{T}^n} f(x)\,d\sigma = \int_{\mathbb{T}^n} f(\xi_x(t))\,d\sigma.$$

Here $\xi_x : [0,\infty) \to \mathbb{R}^n$ satisfies that $\dot{\xi}_x = V(\xi_x)$ and $\xi_x(0) = x$.

The following is from the celebrated Poincaré recurrent theorem and the Birkhoff ergodic theorem for measure preserving flows.

THEOREM 2.1. *If σ is an invariant Borel probability measure on $\mathbb{T}^n$ under the flow $\dot{\xi} = V(\xi)$, then σ a.e. $x \in \mathbb{T}^n$ is recurrent, and the following generalized rotation* *vector exists:*

$$\lim_{T\to+\infty}\frac{\xi_x(T)}{T}=Q_x.$$

Here $\dot{\xi}_x=V(\xi_x)$, *and* $\xi_x(0)=x$.

Denote

$$c(p)=\max_{\sigma\in\Lambda}\int_{\mathbb{T}^n}p\cdot V(x)\,d\sigma,$$

where Λ is the collection of all Borel probability measures on $\mathbb{T}^n$ which are invariant under the flow $\dot{\xi}=V(\xi)$. The following lemma is an easy consequence of the Poincaré recurrent theorem and the Birkhoff ergodic theorem.

LEMMA 2.1. *Given a unit vector* $p\in\mathbb{R}^n$, *the following hold:*

(i) *For any orbit* $\xi:[0,\infty)\to\mathbb{R}^n$ *of* $\dot{\xi}=V(\xi)$,

$$\limsup_{T\to+\infty}\frac{p\cdot\xi(T)}{T}\le c(p). \tag{2.1}$$

In particular, if ξ *is a periodic orbit with rotation vector* Q, *then*

$$c(p)\ge p\cdot Q.$$

(ii) *There exists an orbit* $\xi:[0,\infty)\to\mathbb{R}^n$ *of* $\dot{\xi}=V(\xi)$ *such that*

$$\lim_{T\to+\infty}\frac{p\cdot\xi(T)}{T}=c(p).$$

(iii) *When* $n=2$, *there exists a periodic orbit* ξ *of* $\dot{\xi}=V(\xi)$ *with a rotation vector* Q *such that*

$$c(p)=p\cdot Q.$$

Proof. (i) For a fixed orbit ξ, we assume that $T_m\to+\infty$ as $m\to\infty$ and

$$\limsup_{T\to+\infty}\frac{p\cdot\xi(T)}{T}=\lim_{T_m\to+\infty}\frac{p\cdot\xi(T_m)}{T_m}.$$

Let σ_m be the Borel probability measure on $\mathbb{T}^n$ given by

$$\int_{\mathbb{T}^n}f\,d\sigma_m=\frac{1}{T_m}\int_0^{T_m}f(\xi)\,ds$$

for any $f\in C(\mathbb{T}^n)$. Upon a subsequence if necessary, we may assume that

$$\sigma_m\rightharpoonup\sigma\quad\text{weakly in Borel measure as } m\to\infty.$$

It is easy to see that σ is flow invariant and

$$c(p)\ge\int_{\mathbb{T}^n}p\cdot V(x)\,d\sigma=\lim_{m\to+\infty}\int_{\mathbb{T}^n}p\cdot V(x)\,d\sigma_m=\lim_{T_m\to+\infty}\frac{p\cdot\xi(T_m)}{T_m}.$$

(ii) Choose $\sigma_0\in\Lambda$ such that

$$c(p)=\int_{\mathbb{T}^n}p\cdot V(x)\,d\sigma_0. \tag{2.2}$$

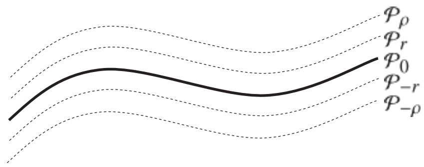

FIG. 3. *Stability of periodic orbits in two dimensions.*

Denote $\xi_x : [0, +\infty) \to \mathbb{R}^n$ as the smooth curve satisfying $\dot{\xi}_x = V(\xi_x)$ and $\xi_x(0) = x$. By Theorem 2.1, there exists a σ_0 measurable function $\bar{\psi}$ such that for σ_0 a.e. x

$$\bar{\psi}(x) = \lim_{T\to+\infty} \frac{p \cdot \xi_x(T)}{T} = p \cdot Q_x.$$

Then by the invariance of σ_0 and the dominated convergence theorem, we have that

$$\int_{\mathbb{T}^n} p \cdot V(x)\, d\sigma_0 = \int_{\mathbb{T}^n} \bar{\psi}(x)\, d\sigma_0.$$

Due to (2.1),

$$\bar{\psi}(x) \le \int_{\mathbb{T}^2} p \cdot V(x)\, d\sigma_0.$$

Hence for σ_0 a.e. x

$$\bar{\psi}(x) = \int_{\mathbb{T}^n} p \cdot V(x)\, d\sigma_0.$$

(iii) Now let us assume $n = 2$ and σ_0 from (2.2). According to the Poincaré recurrence theorem, σ_0 a.e., x is a recurrent point. Note that, for $n = 2$, if x is recurrent, ξ_x must be periodic. The reason is simple. In two dimensions, there exists a smooth periodic stream function H such that $V = (-H_{x_2}, H_{x_1})$. Then H is constant along ξ_x. Hence ξ_x must be periodic. $\square$

Stability of periodic orbits in two dimensions. Assume that $V = (-H_{x_2}, H_{x_1})$ for a smooth periodic stream function H which is constant along each orbit of $\dot{\xi} = V(\xi)$. Let $\mathcal{P}_0$ be a periodic orbit of V with a positive minimum period. Due to 2D topology, the neighborhood of $\mathcal{P}_0$ is filled with other periodic orbits; see Figure 3. This fact should be well known to experts. Since the proof is very simple, we present it here for the reader's convenience. Without loss of generality, we assume that H is constantly zero along $\mathcal{P}_0$.

THEOREM 2.2. *Assume that $H|_{\mathcal{P}_0} \equiv 0$. Then there exists $\rho > 0$ such that*

$$O_\rho = \bigcup_{|r|<\rho} \mathcal{P}_r$$

is an open set in $\mathbb{R}^2$ which contains $\mathcal{P}_0$ and

$$\min\left\{|DH(x)| : \ x \in \overline{O_\rho} = \bigcup_{|r|\le\rho} \mathcal{P}_r\right\} > 0.$$

Here $\mathcal{P}_r$ is a periodic orbit of $\{H = r\}$ with minimum periodic $T_r > 0$ and rotation vector Q_r. Moreover, T_r and Q_r are smooth functions of r.

Proof. For each $x \in \mathcal{P}_0$, let $\eta_x(t)$ be the normalized gradient flow:

$$\begin{cases} \dot{\eta}_x(t) = \frac{DH(\eta_x)}{|DH|}, \\ \eta_x(0) = x. \end{cases}$$

It is clear that there exists $\tau > 0$ such that η_x is well defined in $[-\tau, \tau]$ for all $x \in \mathcal{P}_0$ and

$$\min\{|DH(\eta_x(t))| : \ x \in \mathcal{P}_0, \ |t| \leq \tau\} > 0.$$

Note that $H(\eta_x(t))$ is strictly increasing in $[-\tau, \tau]$. Write

$$0 < 2\rho = \min\{|H(\eta_x(\pm\tau))| : \ x \in \mathcal{P}_0\}.$$

For each $r \in (-\rho, \rho)$ and $x \in \mathcal{P}_0$, choose the unique number $t_{x,r} \in (-\tau, \tau)$ such that

$$H(\eta_x(t_{x,r})) = r.$$

Then $\mathcal{P}_r = \{\eta_x(t_{x,r}) | \ x \in \mathcal{P}_0\}$. The smoothness of T_r and Q_r is obvious. Moreover, due to the 2D topology, $Q_r = \lambda_r Q$ for some $\lambda_r > 0$. ∎

3. Proof of Theorem 1.1. The second equality follows from Lemma 2.1. We need only establish the first equality. It is like a degenerate version of Mather's minimization principle, which gives a relation between the effective Hamiltonian of a superlinear convex Hamiltonian and the minimum action over Euler–Lagrangian flow invariant measures [15, 12].

Step 1. We first show that for any $A > 0$

$$\frac{s_T(p, A)}{A} \geq \max_{\sigma \in \Lambda} \int_{\mathbb{T}^n} p \cdot V(x)\, d\sigma,$$

where Λ is the collection of all Borel probability measures on $\mathbb{T}^n$ invariant under the flow $\dot{\xi} = V(\xi)$.

Note that a Borel measure σ being invariant under the flow $\dot{\xi} = V(\xi)$ is equivalent to saying that it is invariant under the reverse flow $\dot{\xi} = -V(\xi)$. By applying (ii) of Lemma 2.1 to $-p$ and $-V(x)$, there exists $\xi_0 : [0, +\infty) \to \mathbb{R}^n$ such that $\dot{\xi}_0 = -V(\xi_0)$ and

$$\lim_{T \to +\infty} \frac{-p \cdot \xi_0(T)}{T} = \max_{\sigma \in \Lambda} \int_{\mathbb{T}^n} p \cdot V(x)\, d\sigma.$$

Denote $x_0 = \xi_0(0)$. By a change of variables ($\xi(t) = \eta(At)$), the $-G(x, t)$ given by (1.6) can also be written as

$$-G(x, t) = \sup_{\eta \in \Gamma_{A,x}} -p \cdot \eta(At) \tag{3.1}$$

for $\Gamma_{A,x} = \{\eta \in W^{1,\infty}(0, At) | \ \eta(0) = x, \ |\dot{\eta} + V(\eta)| \leq \frac{1}{A} \text{ a.e}\}$. Since $\dot{\xi}_0 + V(\xi_0) = 0$, by (1.5), for fixed $A > 0$,

$$\frac{s_T(p, A)}{A} = \lim_{t \to +\infty} -\frac{G(x_0, t)}{At} \geq \lim_{t \to +\infty} \frac{-p \cdot \xi_0(At)}{At} = \max_{\sigma \in \Lambda} \int_{\mathbb{T}^n} p \cdot V(x)\, d\sigma.$$

Step 2. Next we prove that

$$\limsup_{A\to+\infty} \frac{s_T(p,A)}{A} \le \max_{\sigma\in\Lambda} \int_{\mathbb{T}^n} p\cdot V(x)\,d\sigma.$$

By (1.5) and (3.1), for fixed p, A, choose $\eta_m \in \Gamma_{A,O}$ with $\eta_m(0) = O$ (the origin) and $t_m \to +\infty$ such that

$$\frac{s_T(p,A)}{A} = \lim_{t\to+\infty} -\frac{G(O,t)}{At} = \lim_{m\to+\infty} \frac{-p\cdot\eta_m(At_m)}{At_m}. \tag{3.2}$$

Let $\mu_{m,A}$ be the probability Borel measure on $\mathbb{T}^n$ satisfying

$$\int_{\mathbb{T}^n} f(x)\,d\mu_{m,A} = \frac{1}{At_m}\int_0^{At_m} f(\eta_m(s))\,ds$$

for any $f(x) \in C(\mathbb{T}^n)$. Then for fixed A, upon a subsequence if necessary, we may assume that as $m \to +\infty$

$$\mu_{m,A} \rightharpoonup \sigma_A \quad \text{weakly in Borel measures.}$$

Since $|\dot\eta_m + V(\eta_m)| \le \frac{1}{A}$ and $\eta_m(At_m) = \int_0^{At_m} \dot\eta_m\,ds$, by (3.2), we have that

$$\left|\int_{\mathbb{T}^n} p\cdot V(x)\,d\sigma_A - \frac{s_T(p,A)}{A}\right| \le \frac{1}{A}. \tag{3.3}$$

Moreover, we claim that σ_A is nearly flow invariant; i.e, for fixed $t \ge 0$ and $f \in C(\mathbb{T}^n)$

$$\lim_{A\to+\infty}\left|\int_{\mathbb{T}^n} f(\Phi(x,t))\,d\sigma_A - \int_{\mathbb{T}^n} f(x)\,d\sigma_A\right| = 0. \tag{3.4}$$

Here $\Phi(x,t)$ denotes the flow: $\frac{d}{dt}\Phi(x,t) = -V(\Phi)$ and $\Phi(x,0) = x$. Due to the definition of $\mu_{m,A}$, we have that for fixed $t \ge 0$

$$\int_{\mathbb{T}^n} f(\Phi(x,t))\,d\sigma_A = \lim_{t_m\to+\infty} \frac{1}{At_m}\int_0^{At_m} f(\Phi(\eta_m(s),t))\,ds.$$

Let $w_m(t) = |\eta_m(s+t) - \Phi(\eta_m(s),t)|$. Then $w_m(0) = 0$ and $\frac{dw_m}{dt} \le Lw_m + \frac{1}{A}$ for $L = \max_{\mathbb{R}^n}|DV|$. Hence it is easy to show that $w_m(t) \le \frac{te^{tL}}{A}$. Therefore, for fixed t (3.4) holds. Now choose $A_k \to +\infty$ such that

$$\limsup_{A\to+\infty}\frac{s_T(p,A)}{A} = \lim_{k\to+\infty}\frac{s_T(p,A_k)}{A_k}.$$

Upon a subsequence, we may assume that

$$\sigma_{A_k} \rightharpoonup \sigma_0 \quad \text{weakly in Borel measures.}$$

Owing to (3.4), σ_0 is invariant under the flow $\dot\xi = V(\xi)$; i.e., for all $t \in \mathbb{R}$,

$$\int_{\mathbb{T}^n} f(\Phi(x,t))\,d\sigma_0 = \int_{\mathbb{T}^n} f(x)\,d\sigma_0.$$

Also, by (3.3), the following holds:

$$\lim_{k\to+\infty}\frac{s_T(p,A_k)}{A_k} = \lim_{k\to+\infty}\int_{\mathbb{T}^n} p\cdot V(x)\,d\sigma_{A_k} = \int_{\mathbb{T}^n} p\cdot V(x)\,d\sigma_0 \le \max_{\sigma\in\Lambda}\int_{\mathbb{T}^n} p\cdot V(x)\,d\sigma. \qquad \square$$

4. Proof of Theorem 1.2. We first prove a lemma which is mainly due to the zero mean of the velocity field V.

LEMMA 4.1. *Let $\omega : \mathbb{R} \to \mathbb{R}^2$ be a periodic orbit of $\dot{\omega} = V(\omega)$ with a rotation vector Q. Then there must exist another periodic orbit $\tilde{\omega}$ with a rotation vector λQ for some $\lambda < 0$.*

Proof. This is obvious if $Q = 0$. So let us assume that $Q \neq 0$. Since $n = 2$, any two periodic orbits must have parallel rotation vectors. Thanks to the incompressibility of V, the flow $\dot{\omega} = V(\omega)$ preserves the Lebesgue measure. So according to the Poincaré recurrence theorem, for a.e. x in $\mathbb{R}^2$, ξ_x ($\dot{\xi}_x = V(\xi_x)$ and $\xi_x(0) = x$) is a periodic orbit with a rotation vector proportional to Q. Hence for a.e. x,

$$\lim_{T\to+\infty} \frac{\xi_x(T) - x}{T} = \lambda_x Q$$

for some $\lambda_x \in \mathbb{R}$. Denote $\omega(0) = \bar{x}$. Then $\lambda_{\bar{x}} = 1$. Due to the stability of the periodic orbit (Theorem 2.2), when x is close to $\bar{x}$, ξ_x is also periodic and $\lambda_x > 0$. Note that for any $T > 0$

$$\begin{aligned} 0 = \int_{\mathbb{T}^2} V(x) \cdot Q\, dx &= \frac{1}{T} \int_0^T \int_{\mathbb{T}^2} V(\xi_x(t)) \cdot Q\, dxdt \\ &= \frac{1}{T} \int_{\mathbb{T}^2} \int_0^T V(\xi_x(t)) \cdot Q\, dtdx \\ &= \int_{\mathbb{T}^2} \frac{(\xi_x(T) - x) \cdot Q}{T}\, dx. \end{aligned}$$

Sending $T \to +\infty$, by the dominant convergence theorem, we derive that

$$\int_{\mathbb{T}^2} \lambda_x\, dx = 0.$$

So there must exist $\tilde{x}$ such that $\xi_{\tilde{x}}$ is periodic and $\lambda_{\tilde{x}} < 0$. ∎

We first prove (i) and (ii) in the statement of Theorem 1.2.

Case 1. If the flow $\dot{\xi} = V(\xi)$ does not have a periodic orbit with nonzero rotation vector, then (iii) in Lemma 2.1 implies that

$$c(p) = 0 \quad \text{for all unit vectors } p.$$

Case 2. Assume that there is a periodic orbit with a nonzero rotation vector Q. Then all the other rotation vectors of periodic orbits must be parallel to Q and $c(p)$ must satisfy the following properties:

(1) $c(p)$ is a nonnegative convex function and is homogeneous of positive degree one.

(2) $c(p) = 0$ if and only if $p \cdot Q = 0$. This follows from Lemma 4.1 together with (i) and (iii) in Lemma 2.1.

Owing to property (1), we have that

$$c(p) = \max_{q\in\partial c(O)} p \cdot q.$$

Here $\partial c(O)$ represents the set of subgradients of $c(p)$ at the origin $O = (0, 0)$. Thanks to property (2), vectors in $\partial c(O)$ must be parallel to Q; i.e., there exist λ^+, $\lambda^+ > 0$ such that

$$\partial c(O) = \{\lambda Q |\ \lambda \in [-\lambda^-, \lambda^+]\}.$$

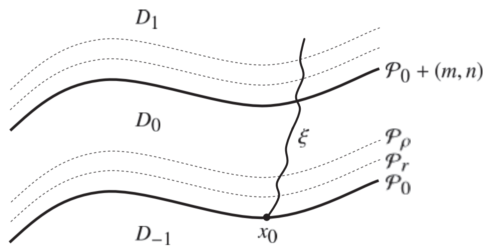

FIG. 4. *Strong bending.*

The positivity of $\lambda^{\pm}$ is due to Lemma 4.1. Hence $c(p)$ must have the form (1.10).

Next we prove the strong bending property of s_T.

LEMMA 4.2. *Suppose that* $\omega : [0, +\infty) \to \mathbb{R}^2$ *is a periodic orbit of* $\dot{\omega} = V(\omega)$ *with a nonzero rotation vector* Q*. Then*

$$\sup_{A \geq 0} s_T(p, A) < +\infty \quad \textit{if } p \cdot Q = 0.$$

Proof. Assume that $\omega(0) = x_0$. We choose a stream function H such that

$$V = \nabla^{\perp} H = (-H_{x_2}, H_{x_1}).$$

Then H is constant along ω. Without loss of generality, we assume $H(\omega(s)) \equiv 0$. Denote $\omega = \mathcal{P}_0$. According to the stability of periodic orbits (Theorem 2.2), let $\mathcal{P}_r$ be a nearby periodic orbit such that $H(\mathcal{P}_r) = r$ for $|r| < \rho$. Since $Q \in \mu\mathbb{Z}^2$ for some $\mu \in \mathbb{R}$ and $|p| = 1$, we may choose $(m, n) \in \mathbb{Z}^2$ such that $(m, n) = -(\sqrt{m^2 + n^2})p$. Without loss of generality, we assume that for $r \in [0, \rho]$, $\mathcal{P}_r$ lies in the domain D_0 bounded by $\mathcal{P}_0$ and $\mathcal{P}_0 + (m, n)$. See Figure 4. Let $\xi : [0, t] \to \mathbb{R}^2$ be a Lipschitz continuous curve such that $\xi(0) = x_0 \in \mathcal{P}_0$ and $|\dot{\xi} + AV(\xi)| \leq 1$. Suppose that $\xi(t)$ lies in the region $D_0 + k(m, n)$ for some $k \in \mathbb{N}$. Note that

$$\frac{d}{ds} H(\xi(s)) \leq |DH(\xi(s))| \quad \text{for a.e. } s \in [0, t].$$

Hence it takes ξ at least $\frac{\rho}{\max_{\mathbb{R}^2} |DH|}$ duration of time (in s) to reach $\mathcal{P}_\rho$ before it can reach $\mathcal{P}_0 + (m, n)$. So the minimum time for ξ to travel from $\mathcal{P}_0$ to $\mathcal{P}_0 + (m, n)$ is no less than $\frac{\rho}{\max_{\mathbb{R}^2} |DH|}$. Owing to periodicity, we have that

$$t \geq \frac{k\rho}{\max_{\mathbb{R}^2} |DH|}. \tag{4.1}$$

Moreover,

$$\frac{-p \cdot (\xi(t) - \xi(0))}{t} \leq \max_{x \in \mathcal{P}_0, y \in \mathcal{P}_0 + (k+1)(m,n)} (-p) \cdot (y - x) = \tau_0 + (k+1)\sqrt{m^2 + n^2}.$$

Here $\tau_0 = \max_{x \in \mathcal{P}_0, y \in \mathcal{P}_0} p \cdot (y - x)$ is finite due to the periodicity of $\mathcal{P}_0$ and $p \cdot Q = 0$. Since the above equality is true for any such ξ, by (1.6), we derive that

$$\frac{-G(x_0, t)}{t} \leq \frac{\tau_0 + (k+1)\sqrt{m^2 + n^2}}{t} + \frac{|x_0|}{t}.$$

Then by (1.5) and (4.1), we deduce that $s_T(p, A) \leq \frac{\sqrt{m^2+n^2} \max_{\mathbb{R}^2} |DH|}{\rho}$. $\square$

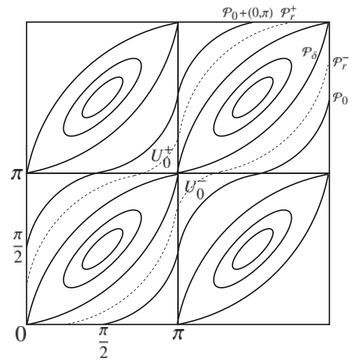

FIG. 5. *Cat's-eye flow.*

5. Proof of Theorem 1.3 for the cat's-eye flow. Throughout this section, the cat's-eye flow $V(x)$ is given by

$$V(x) = (-H_{x_2}, H_{x_1}). \tag{5.1}$$

Here the stream function is

$$H = \sin x_1 \, \sin x_2 + \, \delta \cos x_1 \cos x_2$$

for a fixed constant $\delta \in (0,1)$. When $\delta = 0$, V becomes the so-called cellular flow, and it is known that $s_T(p, A) = O(\frac{A}{\log A})$ for all unit vectors p (see [17], for instance). Hence the cellular flow belongs to case (i) in Theorem 1.2. In this section, an orbit ξ is called periodic if $\xi(\tilde{T}) - \xi(0) \in 2\pi\mathbb{Z}^2$ for some $\tilde{T} > 0$.

The cat's-eye flow has critical points of both elliptic type ($\{\frac{\pi}{2}v| \ v \in (2\mathbb{Z}+1)^2\}$) with maximum (minimum) values of 1 (-1) and hyperbolic type ($\{\pi v| \ v \in \mathbb{Z}^2\}$) with value $\pm\delta$. All the level curves of H (streamlines) are periodic except $\{|H| = \delta\}$, which is heteroclinic (i.e., connecting two hyperbolic critical points).

Let $\mathcal{P}_0 \subset \mathbb{R}^2$ be the periodic orbit of $\{H = 0\}$ which passes through $x_0 = (0, -\frac{\pi}{2})$. It is easy to check that $\mathcal{P}_0 = \mathcal{P}_0 + (\pi, \pi)$ (i.e., these two are the same curves). Hence its rotation vector Q must be parallel to $(1, 1)$, and the cat's-eye flow belongs to case (ii) of Theorem 1.2. The main goal of this section is to identify the limit

$$\lim_{A\to+\infty} s_T(p, A)$$

for $p = \pm(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$. Let D_0 be the region bounded by $\mathcal{P}_0$ and $\mathcal{P}_0 + (0, \pi)$. We also denote (see Figure 5)

$$U_0^+ = D_0 \cap \{0 \le H \le \delta\} \cap \{x_2 \ge x_1\}$$

and

$$U_0^- = D_0 \cap \{0 \le H \le \delta\} \cap \{x_2 \le x_1\}.$$

It is clear that $H \in [0, 1]$ in D_0. For each $r \in [0, \delta)$, the set $\{H = r\}$ consists of two periodic orbits. In order to differentiate them, we write

$$\mathcal{P}_r^+ : \text{the periodic orbit of } \{H = r\} \text{ in } U_0^+$$

and

$$\mathcal{P}_r^- : \text{the periodic orbit of } \{H = r\} \text{ in } U_0^-.$$

Note that $\mathcal{P}_0^- = \mathcal{P}_0$ and $\mathcal{P}_0^+ = \mathcal{P}_0 + (0, \pi)$. Due to the symmetry, $\mathcal{P}_r^+$ and $\mathcal{P}_r^-$ have the same minimum period and unit length, which we denote as T_r and L_r, respectively. We also let $\mathcal{P}_\delta$ be $\{H = \delta\} \cap D_0$. For $x \in \mathbb{R}^2$, denote $u_A(x)$ as the least time needed to reach $\mathcal{P}_0 + (0, \pi)$ through admissible control trajectories:

$$u_A(x) = \inf_\xi \{t|\ \xi(0) = x,\ \xi(t) \in \mathcal{P}_0 + (0, \pi),\ |\dot{\xi}(t) + AV(\xi)| \leq 1 \text{ a.e.}\}.$$

In the following, *an admissible control trajectory* refers to a Lipschitz continuous curve $\xi : [0, T] \to \mathbb{R}^2$ satisfying $|\dot{\xi}(t) + AV(\xi)| \leq 1$ for a.e. $s \in [0, T]$. For the reader's convenience, we first present the main idea for proving Theorem 1.3.

Sketch of the proof. Due to (1.5) and (1.6), we can show that

$$s_T(p, A) \approx \frac{\pi}{\sqrt{2}u_A(x)} \quad \text{for } x \in \mathcal{P}_0 \text{ when } A \gg 1.$$

Thus we just need to determine $\lim_{A \to +\infty} u_A$. The proof consists of two parts:

• Show that $\{u_A\}_{A \geq 1}$ are uniformly bounded and equicontinuous as $A \to +\infty$ so that we may apply the Arzela–Ascoli theorem. This is done in Lemmas 5.1 and 5.3–5.5. Due to a simple triangle inequality, it suffices to prove that any two points $x \in \mathcal{P}_{r_1}^+$ (or $\mathcal{P}_{r_1}^-$) and $y \in \mathcal{P}_{r_2}^+$ (or $\mathcal{P}_{r_2}^-$) can be connected through an admissible control trajectory within a short time when $r_1 \approx r_2$ and $A \gg 1$. The argument goes like this: in order to go from x to y, we can first cross the orbit from $\mathcal{P}_{r_1}^+$ to $\mathcal{P}_{r_2}^+$, then travel at most one period of $\mathcal{P}_{r_2}^+$. The amount of time needed to cross the orbit is controlled by $|r_1 - r_2|$, and the total travelling time within one period of closed streamlines is at most $\frac{C \log A}{A}$.

• Intuitively, it is clear that u_A will become constant along each streamline as $A \to \infty$. That is, $\lim_{A \to +\infty} u_A(x) = \theta(H(x))$ for $x \in U_0^+$ and some function $\theta : [0, \delta] \to [0, \infty)$, which implies that $u_A(x) \to 2\theta(\delta)$ for $x \in \mathcal{P}_0$. To finish the proof, we need to derive the dynamics of θ (Lemmas 5.2 and 5.6).

Now we will write out all the details. It is not hard to prove that $u_A \in C(\mathbb{R}^2)$. Nevertheless, for our purposes, we need only show that u_A is a real-valued function, i.e., any point can reach $\mathcal{P}_0 + (0, \pi)$ through a suitable admissible control trajectory within finite time.

LEMMA 5.1. *For all $x \in \mathbb{R}^2$, $u_A(x) < \infty$ and $u_A(x + (\pi, \pi)) = u_A(x)$. Moreover, if x is connected to y by an admissible control trajectory ($\xi(0) = x$, $\xi(t) = y$), then the triangle inequality holds:*

$$u_A(x) \leq u_A(y) + t. \tag{5.2}$$

Proof. The lower-semicontinuity of u_A and the triangle inequality are obvious. The equality $u_A(x + (\pi, \pi)) = u_A(x)$ follows from the fact that $V(x + (\pi, \pi)) = V$ and $\mathcal{P}_0 = \mathcal{P}_0 + (\pi, \pi)$ (i.e., the same curve). Now let us fix $\bar{x} \in \mathbb{R}^2$ and prove that $u_A(\bar{x})$ is finite. Without loss of generality, we may assume that $\bar{x}$ is below $\mathcal{P}_0 + (0, \pi)$ on the plane. Since $\mathcal{P}_0$ is a periodic orbit with a nonzero rotation vector proportional to (1,1), there exists $M > 0$ such that

$$\mathcal{P}_0 + (0, \pi) \subset \{x_2 - x_1 < M\}.$$

Then $\bar{x} \in \{x_2 - x_1 < M\}$ as well. Taking integration on both sides of (1.3) and by Jensen's inequality, it is easy to see that $s_T(p, A) \geq s_l = 1$ (enhancement). Hence if we choose $p = \frac{1}{\sqrt{2}}(1, -1)$, according to (1.5) and (1.6), there must exist an admissible control trajectory $\xi_0 : [0, \alpha_0] \to \mathbb{R}^2$ such that $\xi_0(0) = \bar{x}$ and

$$\xi_0(\alpha_0) \cdot (-1, 1) > M.$$

Therefore ξ_0 must intersect $\mathcal{P}_0 + (0, \pi)$ at some $t < \alpha_0$. So $u_A(x_0) < \alpha_0$. ∎

The next lemma is true for any periodic smooth stream function H. Suppose that r is not a critical value of H. Let $\mathcal{P}_r$ be a periodic orbit of $\{H = r\}$ with a minimum period $T_r > 0$ and $\mathcal{P}_{r+\beta}$ be the nearby periodic orbit $\{H = r + \beta\}$ for $|\beta| < \rho$ as in Theorem 2.2 (stability of periodic orbits). Write

$$t_A(r, \beta) = \inf_{\xi}\{t |\ \xi(0) \in \mathcal{P}_r,\ \xi(t) \in \mathcal{P}_{r+\beta},\ |\dot{\xi} + AV(\xi)| \leq 1 \text{ a.e.}\} \tag{5.3}$$

as the minimum time required to travel from $\mathcal{P}_r$ to $\mathcal{P}_{r+\beta}$ through all possible admissible control trajectories. Considering the curves $\dot{\xi}(t) = -AV(\xi(t)) \pm \frac{DH(\xi(t))}{|DH|}$ and $\xi(0) = x \in \mathcal{P}_r$, it is easy to deduce that $t_A(r, \beta) \leq \frac{\rho}{\min_{\overline{O_\rho}} |DH|} < \infty$. Here $\overline{O_\rho} = \cup_{|\beta| \leq \rho} \mathcal{P}_{r+\beta}$ and $\min_{\overline{O_\rho}} |DH| > 0$ (nondegeneracy).

LEMMA 5.2. *For $|\beta| \leq \rho$,*

$$\limsup_{A \to +\infty} \left| \frac{|\beta|}{t_A(r, \beta)} - \frac{L_r}{T_r} \right| \leq C_{r,\rho} |\beta|.$$

Here L_r and T_r are the unit length and minimum period of $\mathcal{P}_r$, respectively. $C_{r,\rho}$ is a constant depending only on r and ρ.

Proof. Without loss of generality, we assume $\beta > 0$. Throughout the proof, C represents various constants which depend only on r and ρ.

Step 1. We first show that

$$\limsup_{A \to +\infty} \frac{\beta}{t_A(r, \beta)} \leq \frac{L_r}{T_r} + C\beta. \tag{5.4}$$

As an abbreviation, we write $t_A = t_A(r, \beta)$. Note that for any ξ which satisfies that $|\dot{\xi} + AV(\xi)| \leq 1$, we have that $\frac{d}{dt} H(\xi(t)) \leq |DH|$. Therefore it is obvious that

$$t_A \geq \frac{\beta}{\max_{\mathbb{R}^2} |DH|}. \tag{5.5}$$

Assume that for some $N \in \mathbb{N}$

$$NT_r < At_A \leq (N + 1)T_r. \tag{5.6}$$

Choose an admissible control trajectory ξ such that $\xi(0) \in \mathcal{P}_r$ and $\xi(t_A) \in \mathcal{P}_{r+\beta}$, i.e., the minimum can be obtained in the definition of t_A. The existence of such ξ follows immediately from compactness argument. Minimality of t_A and the 2D topology imply that

$$\xi([0, t_A]) \subset \bigcup_{0 \leq s \leq \beta} \mathcal{P}_{r+s}. \tag{5.7}$$

Due to the nondegeneracy $\min_{\overline{O_\rho}} |DH| > 0$, it is easy to see that

$$\max_{0 \leq s \leq \beta} d(\mathcal{P}_{r+s}, \mathcal{P}_r) \leq C\beta. \tag{5.8}$$

Write $w(t) = \xi(\frac{t}{A})$; then $|\dot{w} + V(w(t))| \leq \frac{1}{A}$. Note that

$$\frac{dH(w(t))}{dt} \leq \frac{|DH(w(t))|}{A}.$$

Then

$$\begin{aligned}\beta &\leq \int_0^{At_A} \frac{|DH(w(t))|}{A}\,dt \\ &= \frac{1}{A}\left(\int_{NT_r}^{At_A} |DH(w(t))|\,dt + \sum_{k=0}^{N-1}\int_{kT_r}^{(k+1)T_r} |DH(w(t))|\,dt\right).\end{aligned}$$

We claim that for $0 \leq k \leq N+1$ (here $(N+1)T_r$ is set to be At_A)

$$\int_{kT_r}^{(k+1)T_r} |DH(w(t))|\,dt \leq L_r + C\left(\beta + \frac{1}{A}\right). \tag{5.9}$$

In fact, choose $x_0 \in \mathcal{P}_r$ such that $|x_0 - w(kT_r)| = d(w(kT_r), \mathcal{P}_r) \leq C\beta$. The inequality is due to (5.7) and (5.8). Let $\eta(t)$ be the smooth curve satisfying $\eta(kT_r) = x_0$ and $\dot{\eta}(t) = -V(\eta)$. For $t \in [kT_r, (k+1)T_r]$, a simple calculation shows that

$$|\eta(t) - w(t)| \leq C\left(\beta + \frac{1}{A}\right).$$

Hence the above claim (5.9) holds. Therefore

$$\beta \leq \frac{N+1}{A}\cdot\left(L_r + C\left(\beta + \frac{1}{A}\right)\right).$$

Due to (5.5) and (5.6), we get (5.4).

Step 2. Next we show that

$$\liminf_{A\to+\infty} \frac{\beta}{t_A} \geq \frac{L_r}{T_r} - C\beta. \tag{5.10}$$

For an arbitrary $x \in \mathcal{P}_r$, let $\xi(t)$ satisfy that $\xi(0) = x$ and

$$\dot{\xi}(t) = -AV(\xi(t)) + \frac{DH(\xi(t))}{|DH|}.$$

Assume that T' is the first time ξ reaches $\mathcal{P}_{r+\beta}$. Then $T' \geq t_A$, $H(\xi(t))$ is strictly increasing before T', and $\xi([0,T']) \subset \cup_{0\leq s\leq\beta}\mathcal{P}_{r+s}$. For $w(t) = \xi(\frac{t}{A})$, it is clear that $\frac{dH(w(t))}{dt} = \frac{|DH(w(t))|}{A}$. Hence

$$\begin{aligned}\beta &\geq \int_0^{At_A} \frac{|DH(w(t))|}{A}\,dt \\ &\geq \frac{1}{A}\sum_{k=0}^{N-1}\int_{kT_r}^{(k+1)T_r} |DH(w(t))|\,dt \quad \text{by (5.6)} \\ &\geq \frac{N}{A}\cdot\left(L_r - C\left(\beta + \frac{1}{A}\right)\right).\end{aligned}$$

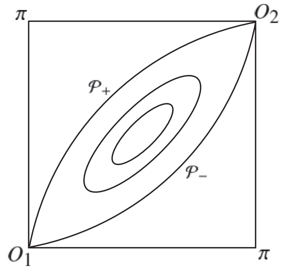

Fig. 6. *A single cat's-eye.*

Derivation of the last inequality is similar to (5.9). Hence (5.10) holds. ∎

In the next three lemmas, we will prove that $u_A(x)$ is equicontinuous as $A \to +\infty$.

Lemma 5.3. *There exists a constant C depending only on δ such that*

$$\sup_{x,y\in\mathcal{P}_\delta} |u_A(x) - u_A(y)| \leq \frac{C\log A}{A}.$$

Proof. Throughout the proof, C represents constants which depends only on δ. Due to $\mathcal{P}_\delta = \mathcal{P}_\delta + (\pi,\pi)$ and $u_A(x + (\pi,\pi)) = u_A(x)$, we need only verify the above equality for $x, y \in [0,\pi]^2 \cap \mathcal{P}_\delta$. Now let $O_1 = (0,0)$ and $O_2 = (\pi,\pi)$ be two corners of the cat's-eye which lies within $[0,\pi]^2$ (see Figure 6). Denote $\mathcal{P}_-$ as the lower eyelid, i.e.,

$$\mathcal{P}_- = \{x = (x_1,x_2) \in [0,\pi]^2 : \ H(x) = \delta, \ x_2 \leq x_1\}.$$

Assume that $\mathcal{P}_-$ is represented as the graph $x_2 = g(x_1)$. Using Taylor expansions, it is easy to derive that

$$g'(0) = \frac{1-\sqrt{1-\delta^2}}{\delta} \quad \text{and} \quad g'(\pi) = \frac{1+\sqrt{1-\delta^2}}{\delta}.$$

Now define the flow $\xi : [0,t_0] \to \mathcal{P}_-$ as follows:

$$\begin{cases} \dot\xi(t) = -AV(\xi(t)) - \frac{V}{|V|}, \\ \xi(0) = O_1, \quad \xi(t_0) = O_2. \end{cases}$$

Here $\lim_{\mathcal{P}_-\ni x\to O_1} \frac{-V}{|V|} = \frac{(1,g'(0))}{\sqrt{1+(g'(0))^2}}$ and $\lim_{\mathcal{P}_-\ni x\to O_2} \frac{-V}{|V|} = \frac{(1,g'(\pi))}{\sqrt{1+(g'(\pi))^2}}$ hold. We claim that

$$u_A(\xi(t_1)) - u_A(\xi(t_2)) \leq \frac{C\log A}{A} \quad \text{for all } 0 \leq t_1 \leq t_2 \leq t_0. \tag{5.11}$$

In fact, write $\xi(t) = (x_1(t), x_2(t))$. An easy computation shows that there exists a constant $\tau > 0$ which depends only on δ such that for all $x \in \mathcal{P}_-$

$$H_{x_2}(x) \geq \tau \sin x_1 \quad \text{and} \quad \frac{H_{x_2}(x)}{|V|} \geq \tau.$$

Accordingly, $\dot x_1(s) \geq \tau A \sin(x_1(s)) + \tau$ for $s \in [0,t_0]$. Hence

$$t_0 \leq \int_0^\pi \frac{1}{\tau(A\sin x_1 + 1)}\, dx_1 = \frac{C\log A}{A}.$$

Then (5.11) follows from the triangle inequality (5.2). We can also establish a similar version for points on the upper eyelid. Therefore Lemma 5.3 holds. ∎

LEMMA 5.4. *For any $\epsilon \in (0,\delta)$, there exists a constant C_ϵ which depends only on ϵ and δ such that for $r_1, r_2 \in (0, \delta-\epsilon)$*

$$\sup_{x\in\mathcal{P}^+_{r_1},\, y\in\mathcal{P}^+_{r_2}} |u_A(x)-u_A(y)| \le C_\epsilon\left(\frac{1}{A}+|r_1-r_2|\right)$$

and

$$\sup_{x\in\mathcal{P}^-_{r_1},\, y\in\mathcal{P}^-_{r_2}} |u_A(x)-u_A(y)| \le C_\epsilon\left(\frac{1}{A}+|r_1-r_2|\right).$$

Proof. We will just prove

$$\sup_{x\in\mathcal{P}^-_{r_1},\, y\in\mathcal{P}^-_{r_2}} |u_A(x)-u_A(y)| \le C_\epsilon\left(\frac{1}{A}+|r_1-r_2|\right). \tag{5.12}$$

The other inequality is similar. Denote

$$0<\tau_\epsilon = \min\{|DH(x)|:\ x\in\mathcal{P}^-_r,\ 0\le r\le \delta-\epsilon\}$$

and

$$J_\epsilon = \max_{0\le r\le\delta-\epsilon} |T_r|.$$

Here T_r represents the minimum period of $\mathcal{P}^-_r$. Clearly, it takes at most $\frac{J_\epsilon}{A}$ duration of time for the fast flow $\dot{\xi} = -AV(\xi(t))$ to travel through one period of $\mathcal{P}^-_r$. The periodicity $\mathcal{P}^-_r = \mathcal{P}^-_r + (\pi,\pi)$ and $u_A(x+(\pi,\pi)) = u_A(x)$ together with the triangle inequality (5.2) imply that for $r\in[0,\delta-\epsilon]$

$$\sup_{x,y\in\mathcal{P}^-_r} |u_A(x)-u_A(y)| \le \frac{J_\epsilon}{A}. \tag{5.13}$$

Now we assume that $r_1<r_2$ and $x\in\mathcal{P}^-_{r_1}$. Consider the following control trajectory:

$$\begin{cases}\dot{\xi}(t) = -AV(\xi) + \frac{DH(\xi)}{|DH(\xi)|},\\ \xi(0)=x.\end{cases}$$

Note that $H(\xi(t))$ is increasing and before H reaches $\delta-\epsilon$

$$\frac{dH(\xi(t))}{dt} = |DH| \ge \tau_\epsilon.$$

So it takes at most $\frac{r_2-r_1}{\tau_\epsilon}$ duration of time for ξ to reach $\mathcal{P}^-_{r_2}$. Hence by the triangle inequality (5.2)

$$\inf_{y\in\mathcal{P}^-_{r_2}} (u_A(x)-u_A(y)) \le \frac{r_2-r_1}{\tau_\epsilon} \quad \text{for all } x\in\mathcal{P}^-_{r_1}.$$

Moreover, by 2D topology

$$\sup_{y\in\mathcal{P}^-_{r_2}} (u_A(x)-u_A(y)) \ge 0 \quad \text{for all } x\in\mathcal{P}^-_{r_1}.$$

Together with (5.13), we conclude that (5.12) holds by choosing $C_\epsilon = J_\epsilon + \frac{1}{\tau_\epsilon}$. ∎

LEMMA 5.5. *There exists a constant $C > 0$ depending only on δ such that*

$$\sup_{x\in\mathcal{P}_r^{\pm},y\in\mathcal{P}_\delta} |u_A(x) - u_A(y)| \le C\left(\sqrt{\delta - r} + \frac{\log A}{A}\right)$$

for $r \in [0, \delta)$.

Proof. We will just verify for $x \in \mathcal{P}_r^-$. The proof for $x \in \mathcal{P}_r^+$ is similar. Due to the 2D topology, it is obvious that

$$\sup_{y\in\mathcal{P}_\delta} (u_A(x) - u_A(y)) \ge 0 \quad \text{for any } x \in \mathcal{P}_r^-.$$

Due to the nondegeneracy of critical points, it is easy to see that there exists $C > 0$ such that

$$|DH(x)| \ge C\sqrt{\delta - H} \quad \text{if } H(x) \in [0, \delta].$$

For $x \in \mathcal{P}_r^-$, consider the following control trajectory in the region $H \in [0, \delta]$:

$$\begin{cases} \dot\xi(t) = -AV(\xi) + \frac{DH(\xi)}{|DH(\xi)|}, \\ \xi(0) = x. \end{cases}$$

Since $\frac{dH(\xi(t))}{dt} = |DH| \ge C\sqrt{\delta - H}$, it takes at most $O\left(\sqrt{\delta - r}\right)$ amount of time for ξ to reach $\mathcal{P}_\delta$. Hence by the triangle inequality (5.2), we deduce that

$$\inf_{y\in\mathcal{P}_\delta} (u_A(x) - u_A(y)) \le C\sqrt{\delta - r} \quad \text{for any } x \in \mathcal{P}_r^-.$$

Combining this with Lemma 5.3, the above lemma holds. ∎

LEMMA 5.6. *The following limits hold:*

(i)

$$\lim_{A\to+\infty} u_A(x) = \theta(H(x)) \quad \textit{uniformly for } x \in U_0^+.$$

(ii)

$$\lim_{A\to+\infty} u_A(x) = 2\theta(\delta) - \theta(H(x)) \quad \textit{uniformly for } x \in U_0^-.$$

Here $\theta = \theta(r) \in C([0, \delta])$ is given by

$$\begin{cases} \theta'(r) = \frac{T_r}{L_r} \quad \textit{for } r \in [0, \delta), \\ \theta(0) = 0. \end{cases} \tag{5.14}$$

T_r and L_r are minimum period and unit length of $\mathcal{P}_r^-$, respectively.

Proof. We will just prove (i). The proof of (ii) is similar. According to previous lemmas, $u_A(x)$ is equicontinuous in the sense that for any $\epsilon > 0$ there exist $d_\epsilon > 0$ and $A_\epsilon > 0$ such that when $A \ge A_\epsilon$

$$|x - y| \le d_\epsilon \Longrightarrow |u_A(x) - u_A(y)| \le \epsilon \quad \text{for all } x, y \in U_0^+ \cup U_0^-.$$

Hence upon a subsequence if necessary, we may assume that

$$\lim_{A\to+\infty} u_A(x) = u(x) \quad \text{uniformly in } U_0^+ \cup U_0^-.$$

Then it is clear that $u(x)$ is continuous and is constant along any connected level curve of H (periodic or heteroclinic). So for $x \in U_0^+$, there exists $\theta \in C([0,\delta])$ such that $\theta(0) = 0$ and

$$u(x) = \theta(H(x)).$$

Since it takes $\frac{T_s}{A}$ amount of time for the fast flow $\dot{\xi} = -AV(\xi)$ to travel one period of $\mathcal{P}_s$, by the triangle inequality (5.2), we have that for fixed $r \in [0,\delta)$, $\beta \in [0,\delta - r)$, $y \in \mathcal{P}_r^+$, and $x \in \mathcal{P}_{r-\beta}^+$

$$t_A(r,-\beta) - \frac{T_{r-\beta}}{A} \le u_A(y) - u_A(x) \le t_A(r,-\beta) + \frac{T_{r-\beta} + T_r}{A}.$$

See (5.3) for the definition of $t_A(r,\beta)$. The dynamics (5.14) follows from Lemma 5.2 by sending $A \to +\infty$ on the above inequality. So the limit is unique. ∎

Proof of Theorem 1.3. By symmetry, we need only prove the limit for $p = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. We first show that

$$\limsup_{A\to+\infty} s_T(p,A) \le \frac{\pi}{2\sqrt{2}\theta(\delta)}. \tag{5.15}$$

Let $\xi : [0,T] \to \mathbb{R}^2$ satisfy $x_0 = \xi(0) \in \mathcal{P}_0$ and $|\dot{\xi} + AV(\xi)| \le 1$ a.e. We may also assume that

$$\xi(T) \in D_0 + k(0,\pi) \quad \text{for some } k \in \mathbb{Z}.$$

Recall that D_0 is the domain bounded by $\mathcal{P}_0$ and $\mathcal{P}_0 + (0,\pi)$. Then

$$(\xi(T) - \xi(0)) \cdot (-p) \le \tau_0 + (|k| + 1)\frac{\pi}{\sqrt{2}}$$

for $\tau_0 = \max_{x,y\in\mathcal{P}_0}(y - x) \cdot p$. Also, by symmetry and periodicity

$$T \ge |k| \inf_{x\in\mathcal{P}_0} u_A.$$

Therefore

$$\limsup_{T\to+\infty} \frac{-\xi(T)\cdot p}{T} \le \frac{\pi}{\sqrt{2}\inf_{x\in\mathcal{P}_0} u_A}.$$

Since this is true for any such ξ, (5.15) follows from (1.5)–(1.6) and Lemma 5.6. Note that (ii) of Lemma 5.6 implies that

$$\lim_{A\to+\infty} u_A(x) = 2\theta(\delta) \quad \text{uniformly for } x \in \mathcal{P}_0. \tag{5.16}$$

Next we prove that

$$\liminf_{A\to+\infty} s_T(p,A) \ge \frac{\pi}{2\sqrt{2}\theta(\delta)}. \tag{5.17}$$

According to the definition of u_A, for each $m \in \mathbb{N}$, there exists a Lipschitz continuous curve $\xi_m : [0,T_m] \to \mathbb{R}^2$ such that $\xi_m(0) \in \mathcal{P}_0$, $\xi_m(T_m) \in \mathcal{P}_0 + m(0,\pi)$, $|\dot{\xi}_m + AV(\xi_m)| \le 1$ for a.e. t and

$$0 < T_m \le m \sup_{x\in\mathcal{P}_0} u_A.$$

Then by (1.5) and (1.6)

$$s_T(p,A) \geq \limsup_{m\to+\infty} -\frac{\xi_m(T_m)\cdot p}{T_m} \geq \lim_{m\to+\infty} \frac{-\tau_0 + \frac{m\pi}{\sqrt{2}}}{m\sup_{\mathcal{P}_0} u_A} = \frac{\pi}{\sqrt{2}\sup_{\mathcal{P}_0} u_A}.$$

Hence (5.17) follows from (5.16). ∎

6. Proof of Theorem 1.4. Throughout this section, $V = (-H_{x_2}, H_{x_1})$. Here H is a general smooth and periodic function on $\mathbb{R}^2$. We first prove (1.11). The proof follows immediately from the following lemma, which was first proved in [9] for more general domains. Since the proof is simple and short in our setting, we present it here for the reader's convenience.

LEMMA 6.1. *We suppose that $\xi : \mathbb{R} \to \mathbb{R}^2$ is a periodic orbit of $\dot{\xi} = V(\xi)$ with a nonzero rotation vector Q. Then*

$$c^*(p) > 0 \quad \textit{if and only if } p\cdot Q \neq 0.$$

Proof. Since $c(p) = 0$ if and only if $p\cdot Q = 0$, together with $0 \leq c^*(p) \leq c(p)$, we have that $c^*(p) = 0$ if $p \cdot Q = 0$. Therefore we just need to show that

$$p\cdot Q \neq 0 \Longrightarrow c^*(p) > 0.$$

Due to Lemma 4.1, we may assume that $p\cdot Q > 0$. Without loss of generality, let $H(\xi(t)) \equiv 0$. Using notation from Theorem 2.2, we denote $\xi = \mathcal{P}_0$ and $\mathcal{P}_r$ as nearby periodic orbits for $|r| \leq \rho$ with nonzero rotation vector Q_r and minimum period T_r. Note that $Q_r = \lambda_r Q$ for some $\lambda_r > 0$. Set

$$\mathcal{M} = \mathop{\cup}_{|r|\leq\rho} \mathcal{P}_r.$$

Apparently, there exists a constant $C_0 > 0$ such that when $T > C_0$

$$\frac{(\xi_x(T) - x)\cdot p}{T} > 0 \quad \text{for all } x \in \mathcal{M}. \tag{6.1}$$

Here $\dot{\xi}_x = V(\xi_x)$ and $\xi_x(0) = x$. For $K > 0$, we define $w \in W^{1,\infty}(\mathbb{R}^n)$ as follows: for $\mathbb{Z}^2 + \mathcal{P}_r = \{\vec{v} + x |\ \vec{v} \in \mathbb{Z}^2,\ x \in \mathcal{P}_r\}$

$$\begin{cases} w|_{\mathbb{Z}^2+\mathcal{P}_r} = K + 1 - \frac{4r^2}{\rho^2} & \text{for } |r| \leq \frac{\rho}{2}, \\ w = K & \text{elsewhere in } \mathbb{R}^2. \end{cases}$$

Clearly w is periodic and $Dw \cdot V \equiv 0$. Also, if we choose K large enough,

$$||Dw||_2^2 \leq f'(0)||w||_2^2.$$

Denote $\tilde{\mathcal{M}} = \cup_{|t|\leq\rho}\tilde{\mathcal{P}}_r \subset \mathbb{T}^2$ as the natural projection of $\mathcal{M}$ to $\mathbb{T}^2$. Since $\operatorname{div}(V) = 0$ and $\tilde{\mathcal{M}}$ is invariant, for all $s > 0$,

$$\begin{aligned}\int_{\tilde{\mathcal{M}}} V(x)\cdot pw^2(x)\,dx &= \int_{\tilde{\mathcal{M}}} V(\xi_x(s))\cdot pw^2(\xi_x(s))\,dx \\ &= \int_{\tilde{\mathcal{M}}} w^2(x)V(\xi_x(s))\cdot p\,dx.\end{aligned}$$

So for all $T > 0$

$$\begin{aligned}\int_{\tilde{\mathcal{M}}} V(x) \cdot pw^2(x)\,dx &= \frac{1}{T}\int_0^T \int_{\tilde{\mathcal{M}}} V(\xi_x(s)) \cdot pw^2(\xi_x(s))\,dxds \\ &= \frac{1}{T}\int_{\tilde{\mathcal{M}}} w^2(x) \int_0^T \dot{\xi}_x(s) \cdot p\,dsdx \\ &= \frac{1}{T}\int_{\tilde{\mathcal{M}}} w^2(x)(\xi_x(T) - x) \cdot p\,dx.\end{aligned}$$

Since $w \geq K$, thanks to (6.1) and the invariance of $\tilde{\mathcal{M}}$, we have that when $T > C_0$,

$$\frac{1}{T}\int_{\tilde{\mathcal{M}}} w^2(x)(\xi_x(T) - x) \cdot p\,dx > \frac{K^2}{T}\int_{\tilde{\mathcal{M}}} (\xi_x(T) - x) \cdot p\,dx = K^2 \int_{\tilde{\mathcal{M}}} V \cdot p\,dx.$$

Then according to (1.9),

$$c^*(p) \geq \frac{1}{||w||^2_{L^2}} \int_{\mathbb{T}^2} V(x) \cdot pw^2\,dx > \frac{K^2}{||w||^2_{L^2}} \int_{\mathbb{T}^2} V(x) \cdot p\,dx = 0.$$

Next we prove (1.12). For $x \in \mathbb{R}^2$, denote $\xi_x : [0, +\infty) \to \mathbb{R}^3$ as the flow starting from $(x, 0)$, i.e., $\dot{\xi}_x = V(\xi_x)$ and $\xi_x(0) = (x, 0)$. We claim that for $p = (0, 0, 1)$

(i)

$$\limsup_{T\to+\infty} \frac{\xi_x(T) \cdot p}{T} \leq 2;$$

(ii) for $x = 2\pi(m, n)$ and $t \in \mathbb{R}$

$$\xi_x(t) \equiv (x, 2t);$$

(iii) if $H(x) = \sin x_1 \sin x_2 \neq 0$, then

$$\lim_{T\to+\infty} \frac{\xi_x(T) \cdot p}{T} = 0.$$

In fact, (i) and (ii) are obvious. We need only establish (iii). Due to symmetry, it suffices to show (iii) for $\bar{x} \in (0, \pi) \times (0, \pi)$. Assume that $\xi_{\bar{x}}(t) = (x_1(t), x_2(t), x_3(t))$. Since $(\pi, \pi) - \bar{x}$ is also on the the same streamline as $\bar{x}$, there exists $T_0 > 0$ such that

$$(x_1(t + T_0), x_2(t + T_0)) = (\pi - x_1(t),\ \pi - x_2(t)) \quad \text{for all } t \geq 0.$$

Then $x_1(2T_0) = x_1(0)$, $x_2(2T_0) = x_2(0)$, and

$$x_3(2T_0) - x_3(0) = \int_0^{2T_0} g(x_1(t), x_2(t))\,dt = 0$$

for $g(x) = \cos x_1 + \cos x_2$. The last equality is due to $g((\pi, \pi) - x) = -g(x)$. Hence $\xi_{\bar{x}}$ is a periodic orbit with zero rotation vector. So (iii) holds.

Since the set $\{H = 0\}$ has Lebesgue measure 0, using formula (1.9), statement (iii) implies that $c^*_p = 0$. Theorem 1.1 together with statements (i) and (ii) says that $c(p) = 2$. $\square$

7. Concluding remarks. We have presented a systematic analysis of asymptotic properties of turbulent flame speeds in planar and nonplanar 2D steady flows in the G-equation.

One of the most studied examples of steady incompressible periodic 3D flows is the Arnold–Beltrami–Childress (ABC) flow:

$$V(x) = (C\cos x_2 + A\sin x_3,\ B\sin x_1 + A\cos x_3,\ B\cos x_1 + C\sin x_2).$$

Here A, B, C are three constant parameters. Such flows are steady solutions of the Euler equation. If one of the parameters is zero, the flow becomes one of those integrable Roberts cell flows. If all three parameters are nonzero, the flow is nonintegrable and contains a mixture of chaotic regions and regular islands [8]. Numerical simulation shows that s_T grows linearly along any direction, i.e., $c_p > 0$ for all unit p in this case; see [22] for similar findings on RDA models. A future program of ours is to rigorously establish this behavior. We would like to mention that the linear growth of Lagrangian particle trajectory was found in [3] for a special class of unsteady 2D cellular flows. This then implies the linear growth of s_T. In contrast, the speed growth is sublinear (bending) for steady 2D cellular flows. The additional enhancement in the unsteady case is due to the chaotic structure which results from the time dependence (so-called chaotic advection) or Lagrangian chaos.

Acknowledgments. We would like to thank our colleague Anton Gorodetski for many helpful discussions and suggestions about dynamical systems and our former student Yu-yu Liu for drawing all of the figures in this paper. We also want to thank Alexei Novikov for pointing out those related works in random perturbations.

REFERENCES

[1] M. Abel, M. Cencini, D. Vergni, and A. Vulpiani, *Front speed enhancement in cellular flows*, Chaos, 12 (2002), pp. 481–488.

[2] B. Audoly, H. Berestycki, and Y. Pomeau, *Réaction diffusion en écoulement stationnaire rapide*, C. R. Acad. Sci. Paris Ser. IIb, 328 (2000), pp. 255–262.

[3] R. Camassa and S. Wiggins, *Chaotic advection in a Rayleigh-Bénard flow*, Phys. Rev. A, 43 (1990), pp. 774–797.

[4] P. Cardaliaguet, J. Nolen, and P.E. Souganidis, *Homogenization and enhancement for the G-equation*, Arch. Ration. Mech. Anal., 199 (2011), pp. 527–561.

[5] S. Childress and A.D. Gilbert, *Stretch, Twist, Fold: The Fast Dynamo*, Springer, Berlin, Heidelberg, 1995.

[6] P. Clavin and F. Williams, *Theory of premixed-flame propagation in large-scale turbulence*, J. Fluid Mech., 90 (1979), pp. 598–604.

[7] P. Constantin, A. Kiselev, A. Oberman, and L. Ryzhik, *Bulk burning rate in passive-reactive diffusion*, Arch. Ration. Mech. Anal., 154 (2000), pp. 53–91.

[8] T. Dombre, U. Frisch, J.M. Greene, M. Hénon, A. Mehr, and A.M. Soward, *Chaotic streamlines in the ABC flows*, J. Fluid Mech., 167 (1986), pp. 353–391.

[9] M. El Smaily and S. Kirsch, *The speed of propagation for KPP reaction-diffusion equations within large drift*, Adv. Differential Equations, 16 (2011), pp. 361–400.

[10] P. Embid, A. Majda, and P. Souganidis, *Comparison of turbulent flame speeds from complete averaging and the G-equation*, Phys. Fluids, 7 (1995), pp. 2052–2060.

[11] L.C. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, AMS, Providence, RI, 1998.

[12] A. Fathi, *Weak KAM Theorem in Lagrangian Dynamics*, Cambridge Stud. Adv. Math., Cambridge University Press, Cambridge, UK, to appear.

[13] M.I. Freidlin and A.D. Wentzell, *Random Perturbations of Dynamical Systems*, 3rd ed., Grundlehren Math. Wiss. 260, Springer, Heidelberg, 2012.

[14] A. Majda and P. Souganidis, *Large scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales*, Nonlinearity, 7 (1994), pp. 1–30.

[15] J. Mather, *Variational construction of connecting orbits*, Ann. Inst. Fourier (Grenoble), 43 (1993), pp. 1349–1386.

[16] J. Nolen and J. Xin, *Asymptotic spreading of KPP reactive fronts in incompressible space-time random flows*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), pp. 815–839.
[17] J. Nolen, J. Xin, and Y. Yu, *Bounds on front speeds for inviscid and viscous G-equations*, Methods Appl. Anal., 16 (2009), pp. 507–520.
[18] S. Osher and R. Fedkiw, *Level Set Methods and Dynamic Implicit Surfaces*, Appl. Math. Sci. 153, Springer, New York, 2003.
[19] N. Peters, *Turbulent Combustion*, Cambridge University Press, Cambridge, UK, 2000.
[20] G.O. Roberts, *Dynamo action of fluid motions with two-dimensional periodicity*, Phil. Trans. R. Soc. Lond. A, 271 (1972), pp. 411–454.
[21] P.D. Ronney, *Some open issues in premixed turbulent combustion*, in Modeling in Combustion Science, Lecture Notes in Phys. 449, J.D. Buckmaster and T. Takeno, eds., Springer-Verlag, Berlin, Heidelberg, 1995, pp. 1–22.
[22] L. Shen, J. Xin, and A. Zhou, *Finite element computation of KPP front speeds in 3D cellular and ABC flows*, Math Model. Nat. Phenom., 8 (2013), pp. 182–197.
[23] G. Sivashinsky, *Cascade-renormalization theory of turbulent flame speed*, Combust. Sci. Tech., 62 (1988), pp. 77–96.
[24] G. Sivashinsky, *Renormalization concept of turbulent flame speed*, in Numerical Combustion, Lecture Notes in Phys. 351, Springer, Berlin, 1989, pp. 131–148.
[25] F.A. Williams, *Turbulent combustion*, in The Mathematics of Combustion, Frontiers Appl. Math. 2, J.D. Buckmaster, ed., SIAM, Philadelphia, 1985, pp. 97–131.
[26] J. Xin, *An Introduction to Fronts in Random Media*, Surv. Tutor. Appl. Math. Sci. 5, Springer, New York, 2009.
[27] J. Xin and Y. Yu, *Periodic homogenization of inviscid G-equation for incompressible flows*, Commun. Math. Sci., 8 (2010), pp. 1067–1078.
[28] J. Xin and Y. Yu, *Sharp asymptotic growth laws of turbulent flame speeds in cellular flows by inviscid Hamilton-Jacobi models*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 30 (2013), pp. 1049–1068.
[29] V. Yakhot, *Propagation velocity of premixed turbulent flames*, Combust. Sci. Tech., 60 (1988), pp. 191–241.
[30] A. Zlatos, *Sharp asymptotics for KPP pulsating front speed-up and diffusion enhancement by flows*, Arch Ration. Mech. Anal., 195 (2010), pp. 441–453.