INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **15** (2002) 551–563 PII: S0951-7715(02)21691-6

Statistical analysis of a semilinear hyperbolic system advected by a white in time random velocity field

Gregory Eyink[1] **and Jack Xin**[2]

[1] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA
[2] Department of Mathematics and TICAM, University of Texas at Austin, Austin, TX 78712, USA

E-mail: eyink@math.arizona.edu and jxin@math.utexas.edu

Received 6 February 2001, in final form 11 January 2002
Published 11 March 2002
Online at stacks.iop.org/Non/15/551

Recommended by L Bunimovich

Abstract

We study a system of semilinear hyperbolic equations passively advected by smooth white noise in time random velocity fields. Such a system arises in modelling non-premixed isothermal turbulent flames under single-step kinetics of fuel and oxidizer. We derive closed equations for one-point and multi-point probability distribution functions (PDFs) and closed-form analytical formulae for the one-point PDF function, as well as the two-point PDF function under homogeneity and isotropy. Exact solution formulae allow us to analyse the ensemble-averaged fuel/oxidizer concentrations and the motion of their level curves. We recover the empirical formulae of combustion in the thin reaction zone limit and show that these approximate formulae can either underestimate or overestimate average concentrations when the reaction zone is not tending to zero. We show that the averaged reaction rate slows down locally in space due to random advection-induced diffusion, and that the level curves of ensemble-averaged concentration undergo diffusion about mean locations.

Mathematics Subject Classification: 60H15, 60H30, 76V05, 76F25

1. Introduction

We are interested in statistical properties of solutions of the following passively advective semilinear hyperbolic systems:

$$u_t + v(t, x, \omega) \circ \nabla_x u = f(u), \tag{1.1}$$

where $u \in R^m$, $m \geqslant 1$; $x \in R^N$, $N \geqslant 2$; $f : R^m \to R^m$, a smooth nonlinear map; $v(t, x, \omega) = \bar{v}(x) + v'(x, t, \omega)$, where $\bar{v}$ is the mean vector field (e.g. a constant vector field),

v' the fluctuation random field, stationary Gaussian in space and white in time, with mean zero, and covariance:

$$\langle v_i'(\vec{x},t)v_j'(\vec{x}',t')\rangle = D_{ij}(\vec{x}-\vec{x}')\delta(t-t'), \tag{1.2}$$

where $D = (D_{ij})(\cdot)$ is smooth. The $\circ$ denotes the Stratonovitch sense of stochastic product. The coupling of different components of u is through the lower order term f. The specific forms of f in applications often permit invariant regions hence uniform maximum norm bounds exist on the solutions for all times.

An example of (1.1) is the following system:

$$\begin{aligned} Y_{\mathrm{F},t} + v(t,x,\omega)\circ\nabla_x Y_{\mathrm{F}} &= \kappa\Delta Y_{\mathrm{F}} - KY_{\mathrm{F}}Y_{\mathrm{O}}, \\ Y_{\mathrm{O},t} + v(t,x,\omega)\circ\nabla_x Y_{\mathrm{O}} &= \kappa\Delta Y_{\mathrm{O}} - KY_{\mathrm{F}}Y_{\mathrm{O}}, \end{aligned} \tag{1.3}$$

where $\kappa \geqslant 0$ is the diffusion constant, $K > 0$ is the reaction rate constant, Y_{F} (Y_{O}) is the concentration of fuel (oxidizer), v is a prescribed random turbulent velocity field. Such a system arises in isothermal non-premixed turbulent combustion under the single-step reaction: $Y + \nu O_x \to P$, where P is the product, ν the stoichiometric constant (see [3, chapters 2 and 6; 9, chapters 3 and 5; 17]). We shall be concerned with the inviscid regime where $\kappa = 0$. The common quantities of interest include the ensemble-averaged concentrations and reaction rate $E[Y_{\mathrm{F}}]$, $E[Y_{\mathrm{O}}]$, $E[Y_{\mathrm{F}}Y_{\mathrm{O}}]$ (see e.g. [3, section 6.7; 9, chapter 5]).

It is known that these moments of solutions as well as the underlying probability distribution functions (PDFs) do not satisfy any closed equations in general. To proceed with an exact analytical treatment, we make the white noise in time assumption á la Kraichnan on the velocity field [11, 12]. Additionally, the sample paths of the velocity field are no less smooth than Lipschitz continuity, see the recent work [20, 1] in this so-called Batchelor limit. This permits us to get closed equations for PDFs for nonlinear systems of equations such as (1.1). To understand the analytical structures of the class of stochastic PDE problems like (1.1) and make progress without resorting to closure approximations, the white in time assumption offers a unique starting point, and helps one to gain insight. This line of inquiry has been pursued in many recent works [15, 6, 2], among others. For successes of the white noise model in the context of the linear passive scalar model, see the in-depth survey by Majda and Kramer [16]. For an extension to finite time correlation from the white noise in time linear passive scalar model, see the recent work [1].

We mention in passing that similar semilinear systems also arise in bioremediation and transport in porous media [18, 19]. In [21], system (1.3) was derived in a fast reaction limit of microbial reaction when the substrate retardation factor is equal to one. However, the velocity v in that particular application is spatially random and independent of time [18]. The random dependence of velocity field in time is more relevant in turbulent combustion. A model similar to (1.3) was recently considered by Majda and Souganidis [17] for the case of two-scale periodic velocity field (oscillatory in both space and time) to understand the large-scale effect of turbulent mixing in non-premixed combustion and condensation–evaporation in cloud physics. Mean field equations were obtained in the limit of large scale separation (homogenization limit). Here we obtain the exact equations without scale separation in space but with rapidly decorrelated velocity in time.

Within the framework of the white noise model, we are able to analyse several important quantities (ensemble average of solutions and reaction rate) often encountered in practice. We also use our exact results to understand approximate formulae in combustion literature on calculations of these averaged solutions. Moreover, we have made rigorous previous formal derivations of PDF equations by Kraichnan and co-workers [12, 4, 10].

In section 2, we derive the one-point PDF ($P = P(u,x,t)$) equation and exact solution formulae on averages of solutions under the initial condition $P(u,x,0^+) = \delta_{u=u_0(x)}$, where

u_0 is a bounded non-negative determinisitic (vector) function such that the reaction $f(u)$ is integrable. In the thin reaction limit of (1.3), or $f(u) = 0$ except on a low dimensional manifold, we recover the empirical formulae in combustion ([3, section 6.7]), and show how (under) overestimation can occur with these formulae when reaction zone is not thin. Moreover, the average reaction rate $E[f(u)]$ decreases pointwise in time to zero with a much faster rate than the advection free case due to enhanced diffusion caused by random advection. With both a scalar model of Fisher type ($f(u) = u(1-u)$) and the model system (1.3), we show that the level curves of averaged solutions undergo diffusion about the mean location.

In section 3, we derive the multi-point PDF equation and analyse its solutions. By a comparison argument, we show the decay of the correlation functions for large times using the correlation functions of Kraichnan's passive scalar model. We also find the two-point PDF solution formula in closed form under isotropy and homogeneity conditions.

Section 4 contains concluding remarks. Many open issues remain for future work. One of them is to extend results here to the regime where velocity–time correlation is a finite value away from zero. The other is to study more properties of level curves of concentration variables using PDF information. For results along this line using a different approach, see Constantin and Procaccia [5].

2. One-point PDF equation and its solutions

2.1. Derivation of one-point PDF equation

Let us first recall a representation of Gaussian random velocity v, so-called proper orthogonal decomposition (see [14] among others). The velocity field $v(t, x, \omega)$ is formally the time derivative of a cylindrical Brownian motion W_t on a separable Hilbert space H, or a Gaussian process indexed by $H \times R^+$ with covariance matrix $\mathrm{cov}(W_t(h), W_s(h')) = \min(t, s)\langle h, h'\rangle_C$, $h, h' \in H$. The underlying space H carries information about the original x variable. The covariance of velocity v can be regarded as a positive definite bilinear operator acting on H. If $D = D(x, y)$ is in $L^2_{\mathrm{loc}}(\mathrm{d}x \times \mathrm{d}y)$, then there exists an orthogonal basis $\mathrm{e}^{(n)}(x) \in R^N$ of space H such that

$$D_{i,j}(x, y) = \sum_n e_i^{(n)}(x) e_j^{(n)}(y), \qquad 1 \leqslant i, j \leqslant N. \tag{2.1}$$

Now let $W^{(n)}(t, \omega) = W_t(\mathrm{e}^{(n)})$, which is a sequence of independent Wiener processes (with ω omitted below), the representation reads

$$v(x, t, \omega) = \sum_n \mathrm{e}^{(n)}(x) \frac{\mathrm{d}W^{(n)}(t)}{\mathrm{d}t}. \tag{2.2}$$

With (2.2), the original system (1.1) can be put into

$$d_t u = -\sum_n (\mathrm{e}^{(n)}(x) \cdot \nabla_x u) \circ \mathrm{d}W^{(n)}(t) + f(u)\,\mathrm{d}t \equiv -\sum_n g^{(n)}(x, u) \circ \mathrm{d}W^{(n)}(t) + f(u)\,\mathrm{d}t, \tag{2.3}$$

where $\circ$ refers to the Stratonovich sense of differential product. Equation (2.3) with Lipschitz velocity u falls into the category of stochastic flows studied in Kunita [13], solutions exist and are unique, and can be approximated by finite-dimensional stochastic ODEs. Recall a well-known conversion from Stratonovich to Ito (non-anticipating) stochastic ODEs ($X = X(t, \omega) \in R^M$, $g^{(m)} \in R^M$):

$$\mathrm{d}X = f(X)\,\mathrm{d}t + \sum_{m=1}^{M} g^{(m)}(X) \circ \mathrm{d}W^{(m)}(t), \tag{2.4}$$

where $W^{(m)}(t)$ values are independent Wiener processes, f and g are Lipschitz functions of X. Then the equivalent Ito equation is

$$\mathrm{d}X = \left[f(X) + \frac{1}{2}\sum_{m=1}^{M} g^{(m)}(X)\cdot\nabla_X g^{(m)}(X)\right]\mathrm{d}t + \sum_{m=1}^{M} g^{(m)}(X)\,\mathrm{d}W^{(m)}(t), \tag{2.5}$$

the new drift term is the so-called noise-induced drift. A similar conversion for (2.3) generates the drift term:

$$\begin{aligned}\frac{1}{2}\sum_n \int \mathrm{d}y\, g^{(n)}(y,u)\nabla_{u(y)} g^{(n)}(x,u) &= \frac{1}{2}\sum_n \int \mathrm{d}y\, \mathrm{e}^{(n)}(y)\cdot\nabla_y u(y)\mathrm{e}^{(n)}(x)\cdot\nabla_x\delta(x-y)\\ &= \frac{1}{2}\nabla_x\cdot\int \mathrm{d}y\Big[\sum_n \mathrm{e}^{(n)}(x)\times\mathrm{e}^{(n)}(y)\Big]\cdot\nabla_y u(y)\delta(x-y)\\ &= \frac{1}{2}\nabla_x\cdot\int \mathrm{d}y D(x,y)\nabla_y u(y)\delta(x-y)\quad \text{using (2.1)}\\ &= \tfrac{1}{2}\nabla_x\cdot(D(x,x)\nabla_x u(x))\\ &= \tfrac{1}{2}\nabla_x\cdot(D(0)\nabla_x u(x)).\end{aligned} \tag{2.6}$$

It follows that the Ito form of the equation is

$$d_t u = -\sum_n (\mathrm{e}^{(n)}(x)\cdot\nabla_x u)\,\mathrm{d}W^{(n)}(t) + [f(u) + \tfrac{1}{2}\nabla_x\cdot(D(0)\nabla_x u)]\,\mathrm{d}t, \tag{2.7}$$

or in PDE form it is

$$u_t + v\cdot\nabla_x u = f(u) + \tfrac{1}{2}\nabla_x\cdot(D(0)\nabla_x u), \tag{2.8}$$

The advantage of the Ito form (2.7) or (2.8) is that the stochastic advection term has mean zero.

Let $\varphi = \varphi(u): R^N \to R^N$ be a smooth nonlinear map, and $J_\varphi = J_\varphi(u)$ be its Jacobian. Multiplying system (1.1) on the left by J_φ, we find

$$(\varphi(u))_t + v\circ\nabla_x\varphi(u) = J_\varphi(u)f(u), \tag{2.9}$$

whose equivalent Ito system of equations is with the same derivation as above:

$$\varphi(u)_t + v\cdot\nabla_x\varphi(u) = \tfrac{1}{2}\nabla_x\cdot(D(0)\nabla_x\varphi(u)) + J_\varphi(u)f(u). \tag{2.10}$$

Taking the ensemble mean of (2.10), we get

$$E[\varphi(u)]_t = \tfrac{1}{2}\nabla_x\cdot(D(0)\nabla_x E[\varphi(u)]) + E[J_\varphi(u)f(u)], \tag{2.11}$$

and the last term is equal to $-\int\varphi(u)\nabla_u\cdot(f(u)P(u,x,t))\,\mathrm{d}u$. Hence it follows from (2.11) that $P(u,x,t)$ satisfies the equation

$$P_t + \nabla_u\cdot(f(u)P) = \tfrac{1}{2}\nabla_x\cdot(D(0)\nabla_x P). \tag{2.12}$$

We may summarize the results of this section in the following.

Theorem 2.1. *Let $P(u,x,t)$ be the one-point probability density function of the stochastic solution $u(x,t)$ of the semilinear hyperbolic system (1.1) with white noise in time, spatially Lipschitz, incompressible, Gaussian random velocity field $v(x,t)$. Then P solves the closed equation (2.12).*

2.2. Applications

2.2.1. A model equation with Fisher nonlinearity. Let us first consider a scalar model equation of the form (1.1) with $f(u) = u(1-u)$, the so-called Fisher (or KPP, Kolmogorov–Petrovsky–Piskunov) nonlinearity. We shall assume that the advection velocity has mean equal to zero. If the advection is identically zero, then the solution is

$$u(x,t) = \frac{u_0(x)\mathrm{e}^t}{(1-u_0(x)) + u_0(x)\mathrm{e}^t}, \tag{2.13}$$

which is between zero and one if the initial datum $u_0(x)$ is so. As $t \to +\infty$, $u \to 1$ if $x \in \{x : u_0(x) > 0\}$, and $u \to 0$ if $x \in \{x : u_0(x) = 0\}$. In particular, any characteristic function is invariant in time. For the random advection, the PDF equation (2.12) reads

$$P_t + (u(1-u)P)_u = \tfrac{1}{2} D_0 \Delta_x P, \tag{2.14}$$

with initial data $P_0 = P_0(u,x) = (4\pi\sigma_u^2)^{-N/2}\exp\{-(u-u_0(x))^2/(4\sigma_u^2)\}$, where $u_0(x)$ is a bounded measurable function between zero and one, and σ_u is a small parameter. Such initial data are convenient for examining deterministic initial data by taking $\sigma_u \to 0$ limit after we find the general formula of the PDF.

Let us solve (2.14) by taking the Fourier transform in x (justified by a suitable truncation, see next subsection for details) to get

$$\hat{P}_t + u(1-u)\hat{P}_u = \left(2u - 1 - \frac{D_0|\xi|^2}{2}\right)\hat{P}, \tag{2.15}$$

which is then integrated by the method of characteristics. The characteristic curve is

$$u = \frac{u_0\mathrm{e}^t}{(1-u_0) + u_0\mathrm{e}^t}, \tag{2.16}$$

whose inverse is

$$u_0 = \frac{u}{u + (1-u)\mathrm{e}^t}. \tag{2.17}$$

So

$$\begin{aligned}\hat{P} &= \hat{P}_0(u_0,\xi)\exp\left\{2\int_0^t \frac{u_0\mathrm{e}^{t'}}{(1-u_0)+u_0\mathrm{e}^{t'}}\,\mathrm{d}t' - t - \frac{D_0}{2}|\xi|^2 t\right\}\\ &= \hat{P}_0(u_0,\xi)(1-u_0+u_0\mathrm{e}^t)^2\exp\left\{-t - \frac{D_0}{2}|\xi|^2 t\right\}\\ &= \hat{P}_0\left(\frac{u}{u+(1-u)\mathrm{e}^t},\xi\right)\frac{\mathrm{e}^{t - D_0|\xi|^2 t/2}}{(u+(1-u)\mathrm{e}^t)^2}.\end{aligned} \tag{2.18}$$

It follows by taking the inverse Fourier transform in ξ that

$$\begin{aligned}P(u,x,t) = \frac{\mathrm{e}^t(2\pi D_0 t)^{-N/2}}{(u+(1-u)\mathrm{e}^t)^2}\int_{R^N}\mathrm{d}y\,\mathrm{e}^{-|x-y|^2/(2D_0 t)}(4\pi\sigma_u^2)^{-N/2}\\ \times\exp\left(-\frac{(u/(u+(1-u)\mathrm{e}^t) - u_0(y))^2}{4\sigma_u^2}\right).\end{aligned} \tag{2.19}$$

Now we calculate all moments of solution ($f = f(u) = u^p$, $p \geqslant 1$):

$$\begin{aligned}E[f(u)](x,t) &= \int \mathrm{d}u\, f(u)P(u,x,t)\\ &= \int \mathrm{d}y\, K(t,x,y)\int \mathrm{d}u \frac{\mathrm{e}^t f(u)}{(u+(1-u)\mathrm{e}^t)^2}(4\pi\sigma_u^2)^{N/2}\\ &\quad\times\exp\left(-\frac{(u/(u+(1-u)\mathrm{e}^t) - u_0(y))^2}{4\sigma_u^2}\right),\end{aligned} \tag{2.20}$$

where K is the heat kernel with diffusion constant $D_0/2$. The inner integral in (2.20) is equal to (via the change of variable $v = u/(u + (1 - u)e^t)$)

$$\int dv\, (4\pi\sigma_u^2)^{-N/2} e^{-(v-u_0(y))^2/(4\sigma_u^2)} f\left(\frac{ve^t}{1 - v + ve^t}\right). \tag{2.21}$$

Taking the limit $\sigma_u \to 0$, we find that (2.21) converges to $f(u_0(y)e^t/(1-u_0(y)+u_0(y)e^t))$, and

$$E[f(u)] \to \int_{R^N} dy\, K(t,x,y) f\left(\frac{u_0(y)e^t}{1 - u_0(y) + u_0(y)e^t}\right). \tag{2.22}$$

In particular, if the initial datum is a front, namely, $u_0(y) = \chi(\{x : x_1 \in [0,\infty)\}) = \chi(R_+^N)$, (2.22) reduces to

$$\int_{R_+^N} dy\, K(t,x,y) = (2\pi D_0 t)^{-1/2} \int_0^\infty dy_1\, e^{-(x_1-y_1)^2/(2D_0t)} = \pi^{-1/2} \int_{x_1/\sqrt{2D_0t}}^\infty e^{-z^2}\, dz. \tag{2.23}$$

To probe the front motion, we look at the level set $X(t)$ such that $E[f(u)](t, X(t)) = c \in (0,1)$. For large t, (2.23) shows that

$$X(t) = (X_1(t), X_2, \ldots, X_N) \sim (z_0\sqrt{2D_0t}, X_2, \ldots, X_N), \tag{2.24}$$

where z_0 is the unique number so that $\int_{z_0}^\infty e^{-z^2}\, dz = \pi^{1/2}c$. The number $z_0 > 0$ (< 0) if $c > \frac{1}{2}$ ($<\frac{1}{2}$), $z_0 = 0$ if $c = \frac{1}{2}$. This implies that the random front in the average sense undergoes normal diffusion about its mean, in this case $x_1 = 0$. Of course, the random level set $\{x : u(t,x) = c \in (0,1)\}$ is more complicated and analysis of its almost sure behaviour requires more information (multi-point statistics).

2.2.2. The model combustion system. Let us consider the one-point statistics of the 2×2 combustion system (1.3) with $\kappa = 0$, $K = 1$, by studying the initial value problem of (2.12) with $f(u) = u_1u_2(-1,-1)$, and initial data:

$$P(u,x,0) = \frac{1}{(4\pi\sigma_u^2)^{N/2}} \exp\left\{-\frac{1}{4\sigma_u^2}((u_1 - u_1^0(x))^2 + (u_2 - u_2^0(x))^2)\right\},$$

the smoothed delta function located at $(u_1^0, u_2^0)(x)$. We shall find a solution formula then take the limit $\sigma_u \to 0$. To use Fourier transform in x, we need to truncate $P(u,x,0)$ at large x; this can be done by multiplying to $P(u,x,0)$ a smooth function $\psi_R(x)$ compactly supported in the ball $B_R \in R^N$. The truncated data are denoted by $P_R(u,x,0)$. Fourier transforming (2.12) in x gives

$$\hat{P}_t - (u_1u_2\hat{P})_{u_1} - (u_1u_2\hat{P})_{u_2} = \frac{-D_0|\xi|^2\hat{P}}{2}, \tag{2.25}$$

with initial data $\hat{P}_R(u,x,0)$, and $D_0 = D(0)$. Equation (2.25) is put to the form

$$\hat{P}_t - u_1u_2\hat{P}_{u_1} - u_1u_2\hat{P}_{u_2} = \left(u_1 + u_2 - \frac{D_0|\xi|^2}{2}\right)\hat{P}, \tag{2.26}$$

which we solve by the method of characteristics. The characteristic Γ equations $u_{i,t} = -u_1u_2$, $i = 1, 2$, give solutions:

$$u_1 = \frac{c_0c_1e^{c_0t}}{1 + c_1e^{c_0t}}, \qquad u_2 = -\frac{c_0}{1 + c_1e^{c_0t}}, \tag{2.27}$$

and useful relations:

$$\frac{u_1}{u_2} = -c_1e^{c_0t}, \qquad c_0 = u_1 - u_2, \qquad c_1 = -\frac{u_1}{u_2}e^{(u_2-u_1)t}. \tag{2.28}$$

Along Γ, $\hat{P}$ obeys in view of (2.26) and (2.27):

$$\hat{P}_t = \left(u_1 + u_2 - \frac{D_0|\xi|^2}{2}\right)\hat{P} = \left(\frac{2c_0c_1\mathrm{e}^{c_0t}}{1+c_1\mathrm{e}^{c_0t}} - c_0 - \frac{D_0|\xi|^2}{2}\right)\hat{P},$$

and so

$$\hat{P}(\xi,t) = \hat{P}_R\left(u_1 = \frac{c_0c_1}{1+c_1}, u_2 = \frac{-c_0}{1+c_1}, \xi\right)\frac{(1+c_1\mathrm{e}^{c_0t})^2}{(1+c_1)^2}\mathrm{e}^{-(c_0+D_0|\xi|^2)t}, \tag{2.29}$$

Using (2.28) to write (2.29), we find

$$\begin{aligned}\hat{P}(u,\xi,t) = {} & \hat{P}_R\left(\frac{(-u_1+u_2)\,u_1\mathrm{e}^{(u_2-u_1)t}}{u_2-u_1\mathrm{e}^{(u_2-u_1)t}}, \frac{(u_2-u_1)u_2}{u_2-u_1\mathrm{e}^{(u_2-u_1)t}}, \xi\right)\\ & \times\frac{(u_2-u_1)^2}{(u_2-u_1\mathrm{e}^{(u_2-u_1)t})^2}\mathrm{e}^{-(u_1-u_2+D_0|\xi|^2/2)t},\end{aligned} \tag{2.30}$$

where

$$\hat{P}_R(u,\xi) = \int \mathrm{d}x\,\psi_R(x)\,\frac{1}{(4\pi\sigma_u^2)^{N/2}}\exp\left\{-\frac{(u_1-u_1^0(x))^2}{4\sigma_u^2} - \frac{(u_2-u_2^0(x))^2}{4\sigma_u^2} - \mathrm{i}\xi\cdot x\right\}. \tag{2.31}$$

Substituting (2.31) into (2.30), and taking inverse Fourier transform, we have

$$\begin{aligned}P(u,x,t) = {} & (2\pi)^{-N}\int_{R^N}\mathrm{d}y\,\psi_R(y)\frac{1}{(4\pi\sigma_u^2)^{N/2}}\\ & \times\exp\left\{-\frac{((-u_1+u_2)\,u_1\,\mathrm{e}^{(u_2-u_1)t}/(u_2-u_1\mathrm{e}^{(u_2-u_1)t}) - u_1^0(y))^2}{4\sigma_u^2}\right.\\ & \left.-\frac{((u_2-u_1)u_2/(u_2-u_1\mathrm{e}^{(u_2-u_1)t}) - u_2^0(y))^2}{4\sigma_u^2}\right\}\\ & \times\frac{(u_2-u_1)^2}{(u_2-u_1\mathrm{e}^{(u_2-u_1)t})^2}\mathrm{e}^{(u_2-u_1)t}\int\mathrm{d}\xi\,\mathrm{e}^{-D_0|\xi|^2t/2-\mathrm{i}\xi\cdot y+\mathrm{i}\xi\cdot x},\end{aligned}$$

where the inner integral gives the heat kernel: $(2\pi D_0t)^{-N/2}\exp\{-|x-y|^2/(2D_0t)\}$. The exponential decay in y allows us to remove the truncation ψ_R by letting $R\to+\infty$. It follows that

$$\begin{aligned}P(u,x,t) = {} & \frac{(u_2-u_1)^2}{(u_2-u_1\mathrm{e}^{(u_2-u_1)t})^2}\mathrm{e}^{(u_2-u_1)t}\int_{R^N}\mathrm{d}y\,(2\pi D_0t)^{-N/2}\exp\left\{-\frac{|x-y|^2}{2D_0t}\right\}\\ & \times\frac{1}{(4\pi\sigma_u^2)^{N/2}}\exp\left\{-\frac{((-u_1+u_2)\,u_1\,\mathrm{e}^{(u_2-u_1)t}/(u_2-u_1\mathrm{e}^{(u_2-u_1)t}) - u_1^0(y))^2}{4\sigma_u^2}\right.\\ & \left.-\frac{((u_2-u_1)u_2/(u_2-u_1\mathrm{e}^{(u_2-u_1)t}) - u_2^0(y))^2}{4\sigma_u^2}\right\}.\end{aligned} \tag{2.32}$$

Note that the denominator $u_2 - u_1\mathrm{e}^{(u_2-u_1)t}$ being zero is not a singularity due to a similar term in the exponential of the Gaussian.

The PDF formula (2.32) can be further simplified by taking $\sigma_u \to 0$ for any finite time t. This is most conveniently done when we calculate the average quantities, $E[u_1\,u_2]$, $E[u_1]$, $E[u_2]$, in the limit $\sigma_u \to 0$.

$$\begin{aligned}E[u_1\,u_2] = {} & \int u_1\,u_2 P(u,x,t)\,\mathrm{d}u_1\,\mathrm{d}u_2\\ & \to \int \mathrm{d}y\,(2\pi D_0t)^{-N/2}\exp\left\{-\frac{|x-y|^2}{2D_0t}\right\}u_1^0(y)\,u_2^0(y)\,J(u_1^0,u_2^0,t),\end{aligned} \tag{2.33}$$

where $J = J(v_1, v_2, t) = \det(\partial(u_1, u_2)/\partial(v_1, v_2))$, and the mapping $(v_1, v_2) \to (u_1, u_2)$ at any time t is the inverse of

$$v_1 = \frac{(-u_1 + u_2)u_1 \mathrm{e}^{(u_2-u_1)t}}{u_2 - u_1 \mathrm{e}^{(u_2-u_1)t}}, \qquad v_2 = \frac{(u_2 - u_1)u_2}{u_2 - u_1 \mathrm{e}^{(u_2-u_1)t}}. \tag{2.34}$$

Notice that (v_1, v_2) of (2.34) satisfies the system: $v_{i,t} = v_1 v_2$, $i = 1, 2$, with initial data $(v_1, v_2)(0) = (u_1, u_2)$. So the inverse map is the solution of the system: $u_{i,t} = -u_1 u_2$ with initial data (v_1, v_2), or

$$u_1 = \frac{(v_1 - v_2)v_1 \mathrm{e}^{(v_1-v_2)t}}{v_1 \mathrm{e}^{(v_1-v_2)t} - v_2}, \qquad u_2 = \frac{(v_1 - v_2)v_2}{v_1 \mathrm{e}^{(v_1-v_2)t} - v_2}, \tag{2.35}$$

and the Jacobian is

$$J = \exp\left\{-\int_0^t (u_1 + u_2)(s)\,\mathrm{d}s\right\} = \frac{(v_1 - v_2)^2 \mathrm{e}^{(v_1-v_2)t}}{(v_1 \mathrm{e}^{(v_1-v_2)t} - v_2)^2}. \tag{2.36}$$

From (2.33) and (2.36), we have

$$E[u_1 u_2] = \int \mathrm{d}y\,(2\pi D_0 t)^{-N/2} \mathrm{e}^{-|x-y|^2/(2D_0 t)}\, \frac{u_1^0(y)\, u_2^0(y)(u_1^0(y) - u_2^0(y))^2\, \mathrm{e}^{(u_1^0(y)-u_2^0(y))t}}{(u_1^0(y)\mathrm{e}^{(u_1^0(y)-u_2^0(y))t} - u_2^0(y))^2}, \tag{2.37}$$

and the total average reaction rate is

$$\int_{R^N} E[u_1\, u_2]\,\mathrm{d}x = \int_{R^N} \frac{u_1^0(y)\, u_2^0(y)(u_1^0(y) - u_2^0(y))^2\, \mathrm{e}^{(u_1^0(y)-u_2^0(y))t}}{(u_1^0(y)\mathrm{e}^{(u_1^0(y)-u_2^0(y))t} - u_2^0(y))^2}\,\mathrm{d}y. \tag{2.38}$$

It is natural to compare (2.37) and (2.38) with the deterministic case when say $v = 0$. Indeed, we solve two equations: $u_{i,t} = -u_1 u_2$, to find

$$u_1 = \frac{\mathrm{d}u_1^0(x)\,\mathrm{e}^{dt}}{u_1^0(x)\mathrm{e}^{dt} - u_2^0(x)}, \qquad u_2 = \frac{\mathrm{d}u_2^0(x)}{u_1^0(x)\mathrm{e}^{dt} - u_2^0(x)}, \tag{2.39}$$

where $d = d(x) = u_1^0(x) - u_2^0(x)$, $u_i^0 \geqslant 0$, $i = 1, 2$. The total reaction rate is

$$\int_{R^N} \mathrm{d}x \frac{(u_1^0(x) - u_2^0(x))^2 u_1^0(x) u_2^0(x) \mathrm{e}^{(u_1^0(x)-u_2^0(x))t}}{(u_1^0(x)\mathrm{e}^{(u_1^0(x)-u_2^0(x))t} - u_2^0(x))^2}, \tag{2.40}$$

which agrees exactly with (2.38)! This turns out to be true more in general on a system like (1.1). For any smooth incompressible velocity field v, the integral $\int_{R^n} g(u)\,\mathrm{d}x$ (g is a nonlinear function so that $g(u) \in L^1(R^n)$) is independent of v by writing the solution in characteristic variables and noting that the change of variables from x to the characteristic variables has unit Jacobian. However, the local (pointwise) average reaction rate $E[u_1 u_2]$ decays faster by a factor $\mathrm{O}(t^{-N/2})$ due to advection-induced diffusion.

Similarly, we find the formulae on $E[u_i]$, $i = 1, 2$:

$$\begin{aligned} E[u_1] &= \int_{R^N} \mathrm{d}y \frac{u_1^0(y)(u_1^0(y) - u_2^0(y))\mathrm{e}^{(u_1^0(y)-u_2^0(y))t}}{u_1^0(y)\mathrm{e}^{(u_1^0(y)-u_2^0(y))t} - u_2^0(y)} K(x, y, t), \\ E[u_2] &= \int_{R^N} \mathrm{d}y \frac{(u_1^0(y) - u_2^0(y))u_2^0(y)}{u_1^0(y)\mathrm{e}^{(u_1^0(y)-u_2^0(y))t} - u_2^0(y)} K(x, y, t), \end{aligned} \tag{2.41}$$

which are convolutions of the deterministic solutions with zero advection (2.39) with K the heat kernel of diffusion coefficient $D_0/2$. If the initial data $u_1^0 = \chi(\{x \in R^N : x_1 > 0\})$, $u_2^0(x) = 1 - u_1^0(x)$, namely the front data, then in the large time limit (2.41) behaves like (2.23) and the level curves of $E[u_i] = c$, $i = 1, 2$ behave like (2.24). In other words, fronts undergo diffusion about the mean location $x_1 = 0$ in the average sense.

To summarize, we state the following proposition.

Proposition 2.1. *Consider the* 2×2 *model combustion system (1.3) obeying conditions on the velocity field in theorem 1; and deterministic, non-negative initial data* $(u_1^0, u_2^0)(x) \in (L^\infty(R^N))^2$. *Then* $E[u_1]$, $E[u_2]$, $E[u_1u_2]$ *are expressed in closed analytical form. The total average reaction rate* $E[\|u_1u_2\|_{L^1(R^N)}](t)$ *is given by the closed-form formula (2.40), the same as the formula when the random velocity field is absent. However,* $E[u_1u_2](x,t)$ *is smaller, due to advection-induced diffusion, than* $u_1u_2(x,t)$ *in the absence of random advection. The mean concentration fronts understood in the sense of level curves of* $E[u_1]$, $E[u_2]$ *diffuse about the mean position with diffusion constant* $D(0)$.

We remark that if the mean velocity $\bar{v}$ is present, then the PDF equation (2.12) has on the left-hand side a term $\bar{v} \cdot \nabla_x P$. If $\bar{v}$ is a constant vector, this term can be removed by going to the comoving frame $\xi = x - \bar{v}t$, and the same results on reaction rates as above can be obtained.

Now we draw a connection between formula (2.41) and an empirical procedure in combustion ([3, pp 247–8]). Let $Z = u_1 - u_2$, then Z satisfies the passive scalar equation (without reaction), its PDF denoted by $P(Z)$ is Gaussian with kernel $K(x,y,t)$, in view of (2.12). If reaction occurs in a thin zone, i.e. either $u_1 = 0$, $u_2 \neq 0$ or $u_1 \neq 0$, $u_2 = 0$, or $u_1 = u_2 = 0$, then when $Z < 0$, we have $Z = -u_2$, $u_1 = 0$; when $Z > 0$, $Z = u_1$, $u_2 = 0$. It follows that

$$\begin{aligned} E[u_1] &= \textstyle\int_{Z>0} Z P(Z)\,\mathrm{d}Z, \\ E[u_2] &= -\textstyle\int_{Z<0} Z P(Z)\,\mathrm{d}Z. \end{aligned} \tag{2.42}$$

This provides a way to approximate the average of solutions based on the PDF function of the (virtual inert) passive scalar Z.

In the case of initial data, $u_1^0 = \chi(\{x \in R^N : x_1 < 0\})$, $u_2^0(x) = 1 - u_1^0(x)$, we are in the thin domain regime. We have from (2.41) that

$$E[u_1] = \int_{u_1^0>0} u_1^0 K(x,y,t) = \int_{Z>0} Z P(Z)\,\mathrm{d}Z,$$

similarly for $E[u_2]$, hence recovering the empirical formula (2.42).

If the initial data are $u_1^0 = \chi(\{x \in R^N : x_1 < 0\})$, $u_2^0 = \epsilon$ if $x_1 \in (-\delta, 0)$, $u_2^0 = 0$ if $x_1 \leqslant -\delta$, $u_2^0 = 1$ if $x_1 > 0$, we have a reaction zone of thickness $\delta > 0$. Then $E[u_1]$ will differ from the thin reaction zone case above by

$$\int_{x_1 \in (-\delta,0)} \left(\frac{(u_1^0 - \epsilon)\mathrm{e}^{(u_1^0-\epsilon)t}}{u_1^0 \mathrm{e}^{(u_1^0-\epsilon)t} - \epsilon} - 1 \right) u_1^0\, K(x,y,t),$$

which is negative for fixed t, and small enough ϵ. The approximation (2.42) will overestimate $E[u_1]$.

Now if $u_2^0 = \chi(\{x \in R^N : x_1 > 0\})$, $u_1^0 = 1$ if $x_1 < 0$, $u_1^0 = \epsilon$, if $x_1 \in (0, \delta)$, $u_1^0 = 0$, if $x_1 \geqslant \delta$, then we also have a reaction zone of thickness δ. The difference of $E[u_1]$ from the thin reaction zone case is

$$\int_{x_1 \in (0,\delta)} \frac{\epsilon(\epsilon - u_2^0)\mathrm{e}^{(\epsilon-u_2^0)t}}{\epsilon \mathrm{e}^{(\epsilon-u_2^0)t} - u_2^0} K,$$

which is positive for small ϵ. Hence the approximation (2.42) will underestimate $E[u_1]$.

3. Multi-point PDF equations and solutions

We derive the multi-point PDF equation by imbedding $n \geqslant 2$ points $x_1, \ldots, x_n \in R^N$ into a vector

$$X = (x_1, \ldots, x_n) \in R^{Nn}, \tag{3.1}$$

and corresponding values of solutions into

$$\Theta(X,t) = (u(x_1,t),\ldots,u(x_n,t)) \in R^{mn}. \tag{3.2}$$

Let also

$$V(X,t) = (v(x_1,t),\ldots,v(x_n,t)), \tag{3.3}$$

and

$$F(\Theta) = (f(u(x_1,t)),\ldots,f(u(x_n,t))) \in R^{mn}. \tag{3.4}$$

Then the n equations ($u_i = u(x_i,t)$),

$$u_{i,t} + v(x_i,t,\omega)\cdot\nabla_{x_i}u_i = f(u_i),$$

can be written into the system as

$$\Theta_t + V\cdot\nabla_X\Theta = F(\Theta), \tag{3.5}$$

which is of the same form as the original equation for u. It follows that the one-point PDF equation of Θ, or n-point PDF equation of u, is ($P = P(\Theta, X, t)$):

$$P_t + \nabla_\Theta\cdot((f(u_1),\ldots,f(u_n))P) = \tfrac{1}{2}\nabla_X\cdot(D_n\nabla_X P), \tag{3.6}$$

where D_n is the $n\times n$ block matrix $(D(x_k - x_l))$ with each block an $N\times N$ covariance matrix of original velocity v.

We shall need a structure assumption on the original velocity covariance matrix $D = D(x)$, $x = (x_1,\ldots,x_N) \in R^N$:

$$D_{ij}(x) = D(0)\delta_{ij} - d_{ij}(x) = D(0)\delta_{ij} - D_1|x|^2\left[\delta_{ij} + \frac{2}{d-1}\left(\delta_{ij} - \frac{x_ix_j}{|x|^2}\right)\right], \tag{3.7}$$

valid for the domain of small $|x|$, also known as the Batchelor regime, [11, 2]. Equation (3.6) can be written as

$$P_t + \sum_{i=1}^{n}\nabla_{u_i}\cdot(f(u_i)P) = \left[\frac{D_0}{2}\left(\sum_{i=1}^{n}\nabla_{x_i}\right)^2 + \frac{1}{2}\sum_{j\neq k} d_{\alpha\beta}(x_j - x_k)\partial_{x_j^\alpha}\partial_{x_k^\beta}\right]P \equiv M_nP, \tag{3.8}$$

where x_j^α refers to the αth component of x_j, and repeated indices mean summation. The operator M_n is degenerate elliptic when acting on L^2 functions of x. If P is translation invariant, the first term on the right-hand side of (3.8) vanishes.

If one neglects reactions, then the last equation is of the same form as the equation for the Lagrangian n-particle transition probability density $P_n^{t,s}(X;Y)$, which gives the probability to find n Lagrangian particles at positions y_k at time s if they were at positions x_k at time $t < s$, for $k = 1,\ldots,n$ (see [8, p 21]). There is a relation between this object and our multi-point probability density of the scalar amplitudes, $P(\Theta; X, t)$:

$$P(\Theta; X, t) = \int \mathrm{d}Y\, P_n^{0,t}(X;Y)P(\Theta; Y, 0). \tag{3.9}$$

This relation is, in fact, just the integral solution of our equation (3.8). However, it does not hold in the general context of spatially non-Lipschitz but only Hölder continuous velocities that was considered in [8]. In fact, it holds only in the context of the 'Batchelor regime' that we consider, where velocities are Lipschitz in space. Indeed, a simple consequence of our equation is that the ensemble-average 'scalar energy' $\frac{1}{2}\int \mathrm{d}x\,\langle\theta^2(x,t)\rangle$ is conserved in time (without reactions), whereas this is not true in the case of non-Lipschitz velocities.

A particular case of interest of equation (3.8) holds for $n = 2$, if the velocity field is also rotationally invariant:

$$P_t + \nabla_{u_1}\cdot(f(u_1)P) + \nabla_{u_2}\cdot(f(u_2)P) = D_2r^{N-1}(r^{N+1}P_r)_r. \tag{3.10}$$

Equation (3.10) is the rigorous justification of a similar equation deduced by Kraichnan, (5.10) in [12]. He considered a single scalar with no nonlinear reaction. His equation was written for the probability density $Q(\Delta, k, t)$ of the two-point difference $\Delta = u - v$ of the scalar at wavenumber k. Kraichnan's proposed equation reads

$$Q_t = D_2[(k(kQ_k)_k + NkQ_k] \tag{3.11}$$

in the special case where molecular diffusion vanishes. We obtain this equation by changing variables in formula (3.12) (written for the case of $m = 1$ and $f = 0$) from u and v to Δ and $U = (u + v)/2$, by integrating out the second variable U, and by then Fourier transforming from space variable r to wavenumber k.

For the system (1.3), if we consider rotationally invariant (isotropic) solutions for the two-point PDF P, then (3.10) becomes

$$P_t + (-u_1u_2P)_{u_1} + (-u_1u_2P)_{u_2} + (-v_1v_2P)_{v_1} + (-v_1v_2P)_{v_2} = D_2r^{N-1}(r^{N+1}P_r)_r, \tag{3.12}$$

where D_2 is a positive constant; we have used (u_1, u_2) to denote the solutions at the first point, (v_1, v_2) at the second point, and their separation distance is r. Assuming that the initial data and solutions decay rapidly at space infinities, we multiply (3.8) when $n = 2$ by $u_1u_2v_1v_2$, integrate over (u, v), to get ($E_2 = E[u_1u_2v_1v_2]$ for short)

$$(E_2)_t = D_2r^{N-1}(r^{N+1}E_2)_r - \int u_1u_2v_1v_2\,(u_1 + u_2 + v_1 + v_2)\,P(u, v, x, t)\,\mathrm{d}u\,\mathrm{d}v, \tag{3.13}$$

implying that $0 \leqslant E_2 \leqslant G = G(r, t)$, where G is the solution of

$$G_t = D_2r^{1-N}(r^{N+1}G_r)_r, \tag{3.14}$$

which is the equation for the two-point correlation function of Kraichnan's passive scalar model in the free decay regime (see [20, 7]).

To solve either (3.14) or (3.12), let us make the change of variables $\xi = N\log r$, then the operator

$$D_2r^{1-N}(r^{N+1}\partial_r\cdot)_r = D_2N^2(\partial_{\xi\xi} + \partial_\xi)\cdot. \tag{3.15}$$

Solution to (3.14) is then ($D_3 = D_2N^2$)

$$\begin{aligned} G &= (4\pi D_3t)^{-1/2}\int \mathrm{d}\eta\,\exp\left\{-\frac{(\xi - \eta + D_3t)^2}{4D_3t}\right\}G_0(\mathrm{e}^{\eta/N}),\\ &= (4\pi D_3t)^{-1/2}\int_0^\infty \exp\left\{-\frac{(N\log(r/s) + D_3t)^2}{4D_3t}\right\}G_0(s)\,Ns^{-1}\,\mathrm{d}s, \end{aligned} \tag{3.16}$$

which shows exponential decay rate on any compact set of r away from zero for fast decaying $G_0(s)$. $G(0, t)$ is however conserved due to the smoothness of G near $r = 0$, or not decaying to zero within a neighbourhood of $\mathrm{O}(c_0^{\sqrt{D_3t}}\mathrm{e}^{-D_3t/N})$ to leading order, c_0 a constant >1. The behaviour of G implies that $\int_{(x,y)\in\Omega\times\Omega} E[u_1(x)u_2(x)u_1(y)u_2(y)]\,\mathrm{d}x\,\mathrm{d}y$ converges to zero exponentially as $t \to \infty$, $\Omega \in R^N$ a compact domain. Using L^2 decay property of the semigroup generated by the operator M_n [14], the decay of all correlator functions follow without isotropy.

We solve (3.12) in (u, v, ξ, t) by combining Fourier transform in ξ and method of characteristics in (u, v, t) as before. The result is

$$\begin{aligned} P(u, v, \xi, t) = {}& \frac{(u_2 - u_1)^2}{(u_2 - u_1\mathrm{e}^{(u_2-u_1)t})^2}\frac{(v_2 - v_1)^2}{(v_2 - v_1\mathrm{e}^{(v_2-v_1)t})^2}\mathrm{e}^{(u_2-u_1+v_2-v_1)t}\\ &\times\int_{R^1}\mathrm{d}y(4\pi D_3t)^{-1/2}\exp\left\{-\frac{|\xi - y + D_3t|^2}{4D_3t}\right\}\\ &\times P_0\left(\frac{(-u_1 + u_2)\,u_1\,\mathrm{e}^{(u_2-u_1)t}}{u_2 - u_1\mathrm{e}^{(u_2-u_1)t}}, \frac{(u_2 - u_1)u_2}{u_2 - u_1\mathrm{e}^{(u_2-u_1)t}},\right.\\ &\qquad\left.\frac{(-v_1 + v_2)\,v_1\,\mathrm{e}^{(v_2-v_1)t}}{v_2 - v_1\mathrm{e}^{(v_2-v_1)t}}, \frac{(v_2 - v_1)u_2}{v_2 - v_1\mathrm{e}^{(v_2-v_1)t}}, \mathrm{e}^{y/N}\right), \end{aligned}$$

where $P_0 = P_0(u, v, r)$ are the initial PDF data. Back to the original variable r, we have the two-point PDF formula:

$$P(u, v, r, t) = \frac{(u_2 - u_1)^2}{(u_2 - u_1 e^{(u_2-u_1)t})^2} \frac{(v_2 - v_1)^2}{(v_2 - v_1 e^{(v_2-v_1)t})^2} e^{(u_2-u_1+v_2-v_1)t}$$
$$\times \int_0^\infty N\, s^{-1}\, ds (4\pi D_3 t)^{-1/2} \exp\left\{-|N \log\left(\frac{r}{s}\right) + \frac{D_3 t|^2}{4D_3 t}\right\}$$
$$\times P_0\Bigg(\frac{(-u_1 + u_2)\, u_1\, e^{(u_2-u_1)t}}{u_2 - u_1 e^{(u_2-u_1)t}}, \frac{(u_2 - u_1)u_2}{u_2 - u_1 e^{(u_2-u_1)t}},$$
$$\frac{(-v_1 + v_2)\, v_1\, e^{(v_2-v_1)t}}{v_2 - v_1 e^{(v_2-v_1)t}}, \frac{(v_2 - v_1)u_2}{v_2 - v_1 e^{(v_2-v_1)t}}, s\Bigg). \tag{3.17}$$

Summarizing the results of this section, we state the following theorem.

Theorem 3.1. *Let $P(u_1, \ldots, u_n, x, t)$ be the n-point probability density function of the solution $u(x, t)$ of system (1.1) with advection velocity satisfying the conditions of theorem 1. Then P obeys the closed equation (3.6). If the separations $r_{ij} = |x_i - x_j|$ are all small for $i, j = 1, \ldots, n$, then the equation takes the special form (3.8). If, in addition, v is rotationally invariant, the two-point PDF $P(u_1, u_2, x, t)$ obeys the simplified equation (3.12) and the closed-form analytical solution for P is given by (3.17) for the case of deterministic initial data of the system (1.3).*

4. Concluding remarks

Under the white noise assumption of incompressible advection velocity field, PDF equations are closed for the solution of the semilinear hyperbolic system (1.1). Exact formulae for one-point and multi-point PDFs (under isotropy) on solutions of the isothermal non-premixed turbulent flame system (1.3) can be derived. From exact formulae, one is able to recover empirical formulae in combustion on averaged solutions in the thin reaction zone limit, and analyse the effect of finite domain zones on the empirical formulae. It is found that they can either overestimate or underestimate averaged solutions. It is also found that fronts understood as level curves of averaged concentration variables, diffuse about the mean position in the ensemble average sense. The front diffusion about mean location is known in the distribution sense in the context of Burgers' equation and other one-dimensional convex scalar conservation laws (see [22]). It is conceivable that this picture is valid for turbulent combustion fronts when the white noise assumption is relaxed. A future study with the assistance of numerical computation appears feasible. It will also be interesting to study more geometric properties of random level curves using higher order PDFs, such as its roughness and dimensions in the large time limit for front data in several space dimensions. See [5] for results in this direction with a different approach.

Acknowledgments

G E thanks the DOE for partial support of this work through grant LDRD-ER 2000047 at the Los Alamos National Laboratory. The work of JX was partially supported by NSF and ARO, and the visiting professorship at the Research Institute of Electronic Sciences of Hokkaido University during his visit there in June of 2000. JX especially thanks his host Professor Yasumasa Nishiura for his hospitality and for many helpful discussions.

References

[1] Balkovsky E and Fouxon A Universal long-time properties of Lagrangian statistics in the Batchelor regime and their application to the passive scalar problem *Phys. Rev.* E **60** 4164–75
[2] Bernard D, Gawędzki K and Kupiainen A 1998 Slow modes in passive advection *J. Stat. Phys.* **90** 519–69
[3] Borghi R and Destriau M 1998 *Combustion and Flames, Chemical and Physical Principles* (Paris: Éditions Technip)
[4] Chen H, Chen S and Kraichnan R 1989 Probability distribution of a stochastically advected scalar field *Phys. Rev. Lett.* **63** 2657–60
[5] Constantin P and Procaccia I 1994 The geometry of turbulent advection: sharp estimates for the dimensions of level sets *Nonlinearity* **7** 1045–54
[6] Constantin P 1998 Scaling exponents for active scalars *J. Stat. Phys.* **90** 571–95
[7] Eyink G and Xin J 2000 Self-similar decay in the Kraichnan model of a passive scalar *J. Stat. Phys.* **100** 679–741
[8] Gawędzki K Easy turbulence *Preprint* chao-dyn/9907024 at http://www.lanl.gov
[9] Linan A and Williams F 1993 *Fundamental Aspects of Combustion (Oxford Engineering Science Series 34)* (Oxford: Oxford University Press)
[10] Kimura Y and Kraichnan R 1993 Statistics of an advected passive scalar *Phys. Fluids* A **5** 2264–77
[11] Kraichnan R H 1968 Small-scale structure of a scalar field convected by turbulence *Phys. Fluids* **11** 945–63
[12] Kraichnan R H 1974 Convection of a passive scalar by a quasi-uniform random straining field *J. Fluid Mech.* **64** 737–62
[13] Kunita H 1990 *Stochastic Flows and Stochastic Differential Equations* (Cambridge: Cambridge University Press)
[14] Le Jan Y and Raimond O 1999 Integration of Brownian vector fields, archived as *math. Preprint* PR/9909147
[15] Majda A J 1993 Explicit inertial range renormalization theory in a model for turbulent diffusion *J. Stat. Phys.* **73** 515–42
[16] Majda A J and Kramer P R 1999 Simplified models for the turbulent diffusion: theory, numerical modelling and physical phenomena *Phys. Rep.* **314** 237–574
[17] Majda A J and Souganidis P 2000 The effect of turbulence on mixing in prototype reaction–diffusion systems *Comm. Pure Appl. Math.* **53** 1284–304
[18] Miralles-Wilhelm F, Gelhar L and Kapoor V 1997 Stochastic analysis of oxygen-limited biodegradation in three-dimensional heterogeneous aquifers *Water Resour. Res.* **33** 1251–63
[19] Oya S and Valocchi A 1997 Characterization of traveling waves and analytical estimation of pollutant removal in one-dimensional subsurface bioremediation modeling *Water Resour. Res.* **33** 1117–27
[20] Son D T 1999 Turbulent decay of a passive scalar in the Batchelor limit: exact results from a quantum-mechanical approach *Phys. Rev.* E **59** R3811–4
[21] Xin J and Hyman J M 2000 Relaxation, stability and oscillation of biodegradation fronts *SIAM J. Appl. Math.* **61** 472–505
[22] Xin J 2000 Front propogation in heterogeneous media *SIAM Rev.* **42** 161–230