Asymptotic Analysis of Nonlinear Schrödinger Equations and Dynamics of Vortices and Light Bullets

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Abstract

Nonlinear Schrödinger equations (NLS) in two space dimensions have been playing a central role in our understanding of dynamics of particle like wave solutions. Asymptotic results are presented for both the Ginzburg-Landau type defocusing NLS and the focusing singularly perturbed cubic NLS. The former is on dynamics of quantized infinite energy point vortices, and the latter is on dynamics of localized finite energy propagating pulses, known as the light bullets. Related numerical findings are also shown on light bullets and their interaction.

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1 Introduction

Nonlinear Schrödinger equations (NLS) in two space dimensions and related equations have served as fundamental models of more complicated physical phenomena. We present asymptotic analysis of NLS arising in two applications to improve our understanding of dynamics of localized particle like wave structures.

The first application is the dynamics of quantized vortices observed in rotating bucket experiments of superfluids (Helium II) [16], [5]. Though NLS model is a great simplification of the original Landau two fluids theory, it captures rich dynamical behaviors depending on time scales. For example, sound waves propagate much earlier than motion of vortices. We present results based on a fluid dynamical formulation of NLS, which faciliates passage to weak limits via energy estimates, energy decomposition, and properties of linear momentum. As the vortices have infinite energy in the asymptotic limit, the analysis amounts to renormalizing the diverging piece of energy and decoding the finite portion.

The second application is on finite energy particle solutions (light bullets) to Maxwell type wave equations in two space dimensions. The two space dimensional sine-Gordon (SG) equation (also known as (2+1) SG) is the simplest model and has its own bullets solutions. Numerical simulations showed that the propagation and interaction of SG bullets are similar to Maxwell bullets. Singularly perturbed cubic NLS is derived from SG, and the subsequent asymptotic analysis of such NLS provides dynamical information on the bullets as they go through focusing-defocusing competition during their propagation. It is amusing that SG has no nontrivial permanent traveling wave solutions of finite energy in two space dimensions, hence the finite energy particle like wave solutions can only be described asymptotically. Much more work needs to be done to reach the Maxwell system and the analysis of its particle like wave solutions.

The rest of the paper is organized as follows. Section 2 is on the analysis of NLS vortices and sound waves. Section 3 is on SG, its NLS approximation, and asymptotic reduction of NLS.

2 Defocusing NLS: Vortices and Sound Waves

Two dimensional defocusing NLS of Ginzburg-Landau type is:

$$iu_t = \Delta_x u + (1 - |u|^2)u, \quad x \in R^2, \tag{2.1}$$

where u is a complex scalar order parameter, used to model a mixture of regular fluid and superfluid. As $|u| \sim 0$ we have the normal fluid phase; $|u| \sim 1$ the superfluid phase. NLS (2.1) was proposed by Ginzburg-Pitaevskii (1958 [8]) simplifying Landau's original two fluids theory [11].

Superfluid, such as liquid Helium II at temperature nearly zero Kelvin, has almost no viscosity, so it can flow through narrow channels without dissipation. Liquid Helium II at temperature slightly below 2.2 K, behaves as a mixture of normal and superfluids, with superfluid the background, normal fluid embedded as particles. It becomes normal fluid when temperature is above 2.2 Kelvin, so called the lambda transition point, where Helium is entirely liquid Helium I, a regular fluid. Experimentally superfluids were discovered by P. Kapitsa in Moskow, and independently J. Allen in Toronto in 1938. Viscosity of Helium II was demonstrated to be 1500 times less than Helium I. More background and historical notes can be found in [5].

2.1 Scalings, Fluid Limit and Vortex Motion

One of the major discoveries on superfluids was quantization of vortices by L. Onsager [16], who also predicted that quantized point vortices move according to classical fluid dynamics, the Kirchhoff law. These findings can be mathematically studied in the NLS model (2.1) under certain space time limits. First the NLS quantized vortices are:

$$u = U_n(r)e^{in\theta}, \;\; n = \pm 1, \pm 2, \cdots,$$

$$U_n(0) = 0,\ U_n(+\infty) = 1,\ U_n'(r) > 0.$$

where the ground states ($n = \pm 1$) are spectrally stable [20].

To inquire about vortex motion, let us consider NLS (2.1) under the scalings $x \to x/\epsilon$, $t \to t/\epsilon^2$, as $\epsilon \downarrow 0$:

$$iu_{\epsilon,t} = \Delta_x u_\epsilon + \epsilon^{-2}(1 - |u_\epsilon|^2)u_\epsilon, \qquad x \in \Omega, \tag{2.1}$$

subject to Dirichlet boundary condition:

$$u_\epsilon|_{\partial\Omega} = g(x),\ |g| = 1,\ \deg(g, \partial\Omega) = \pm n. \tag{2.2}$$

Initial data contain n one vortices, with total NLS energy:

$$E_\epsilon(u_\epsilon) = \int_\Omega e_\epsilon(u_\epsilon) \equiv \int_\Omega \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{(1 - |u_\epsilon|^2)^2}{4\epsilon^2}, \tag{2.3}$$

which is conserved in time:

$$E_\epsilon(u_\epsilon)(t) = E_\epsilon(u_\epsilon)(0) = n\pi \log\frac{1}{\epsilon} + O(1). \tag{2.4}$$

NLS can be formulated into conservation laws in the same fashion as in classical fluid dynamics [1]:

- **Conservation of mass**:

$$\partial_t |u_\epsilon|^2 = 2\nabla \cdot p(u_\epsilon), \tag{2.5}$$

where linear momentum $p(u_\epsilon) = u_\epsilon \wedge \nabla u_\epsilon$ in vector notation (or $= Im(u_\epsilon \nabla u_\epsilon^*)$).

- **Conservation of linear momentum**:

$$\partial_t p(u_\epsilon) = 2\text{div}\,(\nabla u_\epsilon \otimes \nabla u_\epsilon) - \nabla P_\epsilon, \tag{2.6}$$

where

$$P_\epsilon = |\nabla u_\epsilon|^2 + u_\epsilon \cdot \Delta u_\epsilon - \frac{|u_\epsilon|^4 - 1}{2\epsilon^2}, \tag{2.7}$$

is the pressure.

Main ideas and procedures are: (1) energy concentration and resulting weak convergence of NLS solutions; (2) energy decomposition into vortical and pressure parts; (3) take weak limit of momentum equation against divergence free test functions (i.e. project locally onto divergence free fileds); (4) calculate local first moment of velocity circulation (or vorticity), and derive motion law; (5) take weak limit of NLS solutions in the WKB regime to reveal phase (sound) wave propagation through vortices.

The following three theorems are by F-H Lin and the author [12].

Theorem 2.1 (Fluid Limit) *Consider NLS (2.1) with Dirichlet boundary condition (2.2), and initial energy (2.4) with n degree $n_j = \pm 1$ vortices. Then as $\epsilon \downarrow 0$, the energy density $e_\epsilon(u_\epsilon)$ concentrates as Radon measures in $\mathcal{M}(\Omega)$ for any fixed time $t \geq 0$:*

$$\frac{e(u_\epsilon)dx}{\pi n \log \frac{1}{\epsilon}} \rightharpoonup \sum_{j=1}^{n} \delta_{a_j(t)},$$

• *The vortices of u_ϵ converge to $a_j(t)$ moving continuously in time of $O(1)$. Vortices of u_ϵ do not move on any slower time scale $O(\lambda_\epsilon) = o(1)$.*

• *On the time scale $t \sim O(1)$, the linear momentum $p(u_\epsilon)$ converges weakly in $L^1([0,T]; L^1_{loc}(\Omega_a))$ to a solution v of the incompressible Euler equation:*

$$v_t = 2v \cdot \nabla v - 2\nabla P, \quad \operatorname{div} v = 0,$$

$$x \in \Omega_a \equiv \{\Omega \backslash (a_1(t), \cdots, a_n(t))\}$$

with boundary condition: $v \cdot \tau = g \wedge g_\tau$, τ the unit tangential vector on $\partial\Omega$. The function v is precisely characterized as:

$$v = \nabla(\Theta_a + h_a),$$

where:

$$\Theta_a = \sum_{j=1}^{n} \arg \left(\frac{x - a_j(t)}{|x - a_j(t)|} \right)^{n_j},$$

and h_a is harmonic on Ω satisfying the boundary condition: $h_{a,\tau} = -\Theta_{a,\tau} + g \wedge g_\tau$, on $\partial\Omega$. So h is unique up to an additive constant.

• *The NLS solution* u_ϵ *converges weakly in* $H^1(\Omega_a)$ *for each* t *as:*

$$u_\epsilon(t,x) \rightharpoonup e^{i(\Theta_a+h_a)}.$$

• *The total pressure* $2P$ *is a single-valued function on* Ω*, and is smooth on* Ω_a*. The quadratic tensor product weakly converges as:*

$$\nabla u_\epsilon \otimes \nabla u_\epsilon \rightharpoonup v\otimes v + \mu, \qquad \mathcal{M}(\Omega_a), \tag{2.8}$$

μ *a symmetric tensorial Radon defect measure of finite mass over* Ω*,* $\mathrm{div}(\mu) = \nabla P_\mu$*, on* Ω_a*,* P_μ *is a well-defined distribution function on* Ω_a*.*

Theorem 2.2 (Vortex Motion Law) *Consider the same assumptions as in Theorem 2.1, and in addition the initial NLS energy is almost minimizing, namely:*

$$E_\epsilon(u_\epsilon)(0) = n\pi \log\frac{1}{\epsilon} + \pi W(a(0)) + o(1),$$

as ϵ *goes to zero. Let* $H_j = H_j(a)$*,* $a = (a_1, \cdots, a_n)$*, denote the smooth part of* $\Theta_a + h_a$ *near each vortex, and define the renormalized energy function as:*

$$\nabla_{a_j} W(a) = 2n_j\left(-\frac{\partial H_j}{\partial x_2}(a_j), \frac{\partial H_j}{\partial x_1}(a_j)\right),$$

$j = 1, \cdots, n$*. The vortex motion obeys the classical Kirchhoff law (*$j = 1, \cdots, n$*):*

$$a_j'(t) = n_j J \nabla_{a_j} W(a) = 2\nabla H_j(a),$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and:

$$W(a) = -\sum_{l\neq j} n_l n_j \log|a_l - a_j| + \text{boundary contributions}.$$

Remarks:

(1) Total NLS energy can be decomposed into a sum of three parts:

$$\begin{aligned} E_\epsilon(u_\epsilon) &= n\pi \log \frac{1}{\epsilon} \text{ (vortex self} - \text{energy)} + \pi W(a(0))(\text{Kirchhoff energy}) \\ &\quad + O(1)(\text{excessive energy}). \end{aligned}$$

The Kirchhoff energy facilitates the vortex motion, excessive $O(1)$ energy creates the defect measure μ, a form of pressure, and may contribute to vortex motion. The function W, renormalized energy, was first given in Bethuel, Brezis and Hélein [2].

If the excessive energy is absent, or initial energy satisfies:

$$E_\epsilon(u_\epsilon)(0) = n\pi \log \frac{1}{\epsilon} + \pi W(a(0)) + o(1),$$

the linear momentum $p(u_\epsilon)$ converges strongly, defect measure $\mu = 0$.

For defect measures and weak limits of two-D Euler, see Diperna and Majda [4], P-L Lions [13]. Uniqueness of two-D Euler is unknown if initial vorticity is in L^p, $p \in [1, \infty)$, much less for bounded measure.

(2) The fluid formulation of NLS was known to Madelung [14] in a different form, who identified $|u|^2$ as fluid density, $\nabla\theta = \nabla \arg u$ as fluid velocity, and defined the linear momentum $p = \rho\nabla\theta$. The fluid formulation has been used to analyze WKB limit of NLS with no vortices in the solutions and before shock formation, Grenier [10].

(3) Local nature of our approach extends results to zero Neumann case, also the whole plane and periodic case if vortices have zero sum. Colliander and Jerrard [3] proved the vortex dynamics on the periodic domain in the absence of excessive energy with a different approach.

2.2 Sound Wave Propagation

If the time scaling is instead $t \to t/\epsilon$, we are in the compressible WKB regime and sound waves propagate through steady vortices. The rescaled NLS is:

$$iv_{\epsilon,t} = \epsilon\Delta v_\epsilon + \epsilon^{-1}(1 - |v_\epsilon|^2)v_\epsilon. \tag{2.9}$$

Theorem 2.3 (Scattering of Sound Waves) *Suppose that the initial data:*

$$v_\epsilon(0,x) \rightharpoonup e^{ih(x)} \prod_{j=1}^{n} \frac{x-a_j}{|x-a_j|},$$

weakly in $H^1(\Omega_a)$*,* $h(x) \in H^1(\Omega)$*, and* $\frac{|v_\epsilon(0,x)|-1}{\epsilon} \to 0$ *in* $L^2(\Omega')$*, for any compact subset* Ω' *of* Ω_a*. Then there is no vortex motion at later time and:*

$$v_\epsilon(t,x) \rightharpoonup e^{ih(t,x)} \prod_{j=1}^{n} \frac{x-a_j}{|x-a_j|}, \tag{2.10}$$

where the phase function $h(t,x) \in H^1(\Omega)$ *and is the weak solution of finite energy of the following initial-boundary value problem of the linear wave equation:*

$$\begin{aligned} &h_{tt} - 2\Delta h = 0, \quad x \in \Omega, \\ &h(t,x) = h(x), \quad x \in \partial\Omega, \\ &h(0,x) = h(x), \quad h_t(0,x) = 0. \end{aligned} \tag{2.11}$$

WKB regime is a precursor of vortex motion, during which the excessive energy gets redistributed through propagation of sound (phase) waves. At the time scale of ϵ^{-2}, most excessive energy is outgoing to infinity (in the whole plane case), leaving $o(1)$ amount near vortices. This is suggested by the above result on weak limit. In other words, it is expected that in this case defect measure does not change the Kirchhoff law of vortex motion.

3 Modified Focusing NLS and SG Bullets

3.1 Light Bullets in Maxwell and SG

Light bullets are spatially localized self-supporting traveling light pulses in two or three space dimensions, able to maintain their overall shapes over a long distance. They are extensions of the familiar envelope solitons in one space dimension to two or three space dimensions, however, may not be exact permenant wave solutions as in one space dimension. This is why asymptotic solutions become useful.

Goorjian and Silverberg [9] performed direct numerical simulation of light bullets from the Maxwell system with cubic nonlinearity (so called Kirr medium):

$$\begin{aligned}
\vec{D}_{tt} &= -\frac{1}{\mu_0}\nabla \times (\nabla \times \vec{E}), \\
\vec{D} &= \epsilon_0[\epsilon_\infty \vec{E} + \chi^{(3)}(\vec{E}\cdot\vec{E})\vec{E} + \vec{P}], \\
\vec{P}_{tt} + \omega_0^2 \vec{P} &= \omega_0^2(\epsilon_s - \epsilon_\infty)\vec{E},
\end{aligned} \tag{3.12}$$

in the transverse electric regime, namely the electric field $\vec{E} = (0, E(t,x,z), 0))$, similarly for the electric displacement D and the polarization P. In this regime the system is de-vectorized, and two dimensional in space. The physical parameters are μ_0 and ϵ_0 are vaccum permeability and permitivity resp.; ϵ_s and ϵ_∞ are low and high frequency linear relative permittivities resp.; ω_0 is the medium resonance frequency. The function $\chi^{(3)}(\cdot)$ depends on the medium, and is such that the nonlinearity is cubic in $\vec{E}$. The propagation of a single bullet, and interaction of two Maxwell light bullets are shown in Figure 1 and Figure 2, see [9] for more details.

The author [21] discovered that the (2+1) sine-Gordon (SG) equation

$$u_{tt} - c^2 \Delta_{x,y}\, u + \sin\, u = 0, \tag{3.13}$$

also have similar bullets solutions. Here c is a constant. In Figure 3 and Figure 4, we show the SG single bullet propagation and interaction. More details will be presented in [18]. The advantage of SG is that it is much simpler than the Maxwell system, and that a new class of focusing NLS can be cleanly derived to build asymptotic solutions and explain the origin of light bullets. SG is to Maxwell bullets what KdV is to water wave solitons.

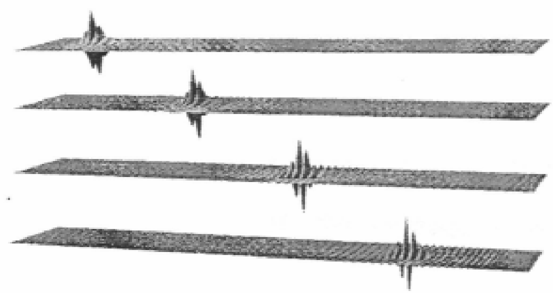

Fig. 4. Electric field of a pulse undergoing dispersion, diffraction, and nonlinear refraction after 155 fs, 310 fs, 465 fs, and 620 fs of propagation. Initial electric-field amplitude, $E_0 = 2.9 \times 10^{10}$ V/m; initial pulse width, 0.9 μm (FWHM); initial pulse duration, 18 fs (FWHM); pulse propagation distance, 28 Rayleigh ranges.

Figure 1: 2D Maxwell single bullet [9].

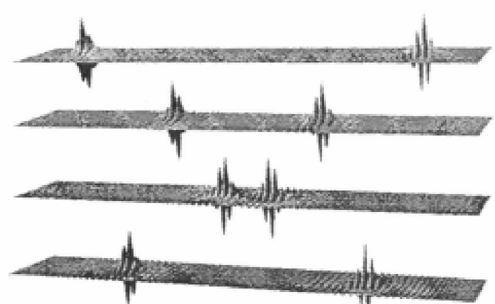

Fig. 7. Electric field of two counterpropagating pulses after 155 fs, 310 fs, 465 fs, and 620 fs of propagation. Initial pulse parameters are given in Fig. 4.

Figure 2: 2D Maxwell bullet interaction [9].

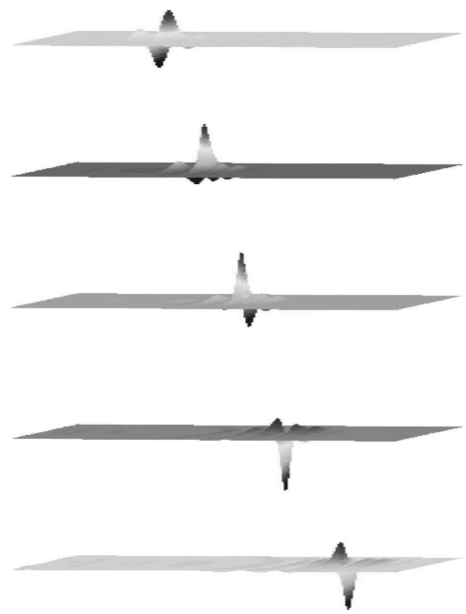

Figure 3: 2-D SG single bullet, $t = [0, 5, 15, 30, 45]$.

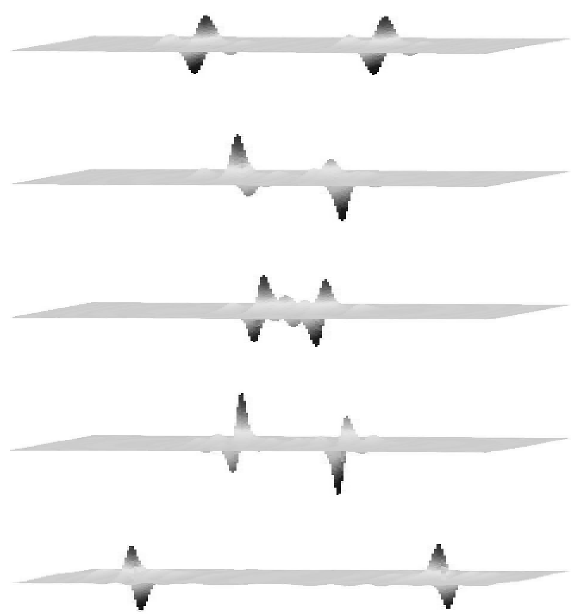

Figure 4: 2-D SG bullet interaction, $t = [0, 4, 16, 28, 48]$.

3.2 NLS Approximation of SG

The SG envelope approximation is:

$$u = \epsilon A(\epsilon(x-\nu t), \epsilon y, \epsilon^2 t) e^{i(kx-\omega(k)t)} + c.c. + R, \tag{3.14}$$

where $\omega = \omega(k) = \sqrt{1+c^2k^2}$, the dispersion relation; $\nu = \omega'(k)$ the group velocity, R remainder; $X = \epsilon(x-\nu t)$, $Y = \epsilon y$, $T = \epsilon^2 t$. For ϵ small the approximate solutions contain multiple scales: the envelope A varies on slow spatial-temporal scales, and the carrier wave $e^{i(kx-\omega(k)t)}$ is on faster scale.

Substitution of (3.14) into SG gives the remainder equation:

$$\begin{aligned} R_{tt} - c^2\Delta_{xy}R + R = e^{i(kx-\omega t)}\Big[&\epsilon^3(-2i\omega)A_T + \epsilon^3(\nu^2-c^2)A_{XX} - \epsilon^3 c^2 A_{YY} \\ &-2\epsilon^4\nu A_{XT} + \epsilon^5 A_{TT} + \sum_{j=1}^{\infty}(-1)^j[(2j+1)!]^{-1}\cdot\epsilon^{2j+1}\cdot|A|^{2j}\cdot A\cdot\binom{2j+1}{j+1}\Big] \\ &+c.c. + F_1'(\epsilon A, e^{i(kx-\omega t)}) + F_2(\epsilon A\cdot e^{i(kx-\omega t)}, R) \\ &+R\cdot\sum_{j=1}^{\infty}(-1)^j[(2j)!]^{-1}(\epsilon A e^{i(kx-\omega t)} + c.c.)^{2j} \end{aligned} \tag{3.15}$$

where the forcing terms: (1) F_1' containes cubic and higher powers of ϵA, multiplied by $e^{ij(kx-\omega t)}$, $j \neq \pm 1$, all nonresonant; (2) F_2 contains quadratic and higher powers of R multiplied by powers of $\epsilon A e^{i(kx-\omega t)}$.

The forcing terms multiplied by $e^{i(kx-\omega t)}$ and their complex conjugate (c.c.) are resonant and lead to growth of remainder R. Removing all resonant terms in the bracket, we derive a new NLS equation:

$$\begin{aligned} (-2i\omega)A_T + \epsilon^2 A_{TT} &= \frac{c^2}{\omega^2}A_{XX} + c^2 A_{YY} + 2\epsilon\nu A_{XT} \\ &+\epsilon^{-3}\sum_{j=1}^{\infty}(-1)^{j+1}[(j+1)!j!]^{-1}\cdot\epsilon^{2j+1}|A|^{2j}A, \end{aligned} \tag{3.16}$$

which reduces to the celebrated cubic focusing NLS if we ignore terms with small ϵ coefficients. However, since solutions of cubic focusing NLS can blow up in finite time, these perturbation terms play important roles when solutions are focusing.

Remainder equation:

$$
\begin{aligned}
& R_{tt} - c^2 \Delta_{x,y} R + (1 - \sum_{j=1}^{\infty} (-1)^j [(2j)!]^{-1} (\epsilon A e^{i(kx-\omega t)} + c.c.)^{2j}) R \\
& = -F_1'(\epsilon A, e^{i(kx-\omega t)}) - F_2(\epsilon A \cdot e^{i(kx-\omega t)}, R), \qquad (3.17)
\end{aligned}
$$

F' is non-resonant and F_2 is quadratic in R. For $\epsilon|A| \ll 1$ and smooth, (3.17) is a nonlinear Klein-Gordon with quadratic nonlinearities. Ozawa, Tsutaya, and Y. Tsutsumi [17] showed that global solutions exist and decay without forcing. If A is well-behaved (bounded and smooth), one expects R to remain small.

The new NLS equation (3.16), the SGNLS for short, on the other hand offers qualitative information on SG pulse evolution. As it is a singular perturbation of the cubic focusing NLS, we give a quick summary of the related results and tools.

3.3 Cubic NLS and its Singular Perturbations

The cubic focusing NLS in two-D:

$$
-iu_t = \Delta_x u + |u|^2 u, \quad x \in R^2, \qquad (3.18)
$$

is extensively studied, see the recent book by C. Sulem, and P-L Sulem [19]. It conserves the following integrals:

$$
N = \frac{1}{2\pi} \int_{R^2} |u|^2 \, dx,
$$

total mass (power in optics),

$$
H = \frac{1}{2\pi} \int_{R^2} |\nabla u|^2/2 \, - |u|^4/4 \;\; dx,
$$

total energy (Hamiltonian).
It has the ground state $u = e^{it} R(r)$, $r = |x|$, and R obeys:

$$
\Delta R - R + R^3 = 0, \; R'(0) = 0, \; R(\infty) = 0. \qquad (3.19)
$$

$H(R) = 0$, $N(R) = 1.86 = N_c$, the critical mass.
Variance identity:

$$\frac{d^2}{dt^2} \int_{R^2} dx \, |x|^2 |u|^2 = (2\pi) H(0), \tag{3.20}$$

gives a sufficient condition for finite time blowup (collapse) in H^1 norm: $H(0) < 0$. A necessary condition for blowup (collapse) is: $N \geq N_c$.
The explicit self-similar blowup solution is:

$$u = \frac{1}{T-t} R(\frac{r}{T-t}) \exp\{i \frac{1 - r^2/4}{T-t}\}. \tag{3.21}$$

Other blowup solutions have rates between $(T-t)^{\frac{1}{2}-}$ and $(T-t)^{\frac{1}{2}+}$ depending on initial data, and require more refined analysis .

- Perturbed cubic NLS:

$$-iu_t = \Delta_x u + |u|^2 u + \epsilon F(u, u_t, \nabla u, \cdots), \tag{3.22}$$

may not have blowup. A formal singular perturbation theory of focusing solutions has been developed by G. Fibich and G. Papanicolaou [6], [7].

3.4 Saturating Nonlinearity and Global Well-Posedness

The SGNLS (3.16) contains a special nonlinearity, call it $NL(\epsilon, A)$, with the following properties:

- If $\epsilon |A| \ll 1$,

$$NL(\epsilon, A) \sim \frac{1}{2} |A|^2 A + O(\epsilon^2 |A|^5).$$

- If $\epsilon |A| \gg 1$, $NL(\epsilon, A) \sim$

$$-\epsilon^{-2} A \cdot \left\{ -1 + (\epsilon |A|)^{-3/2} \exp\left[\{ \pm i \, (2\epsilon |A| + O(1)) \right] \right\}, \tag{3.23}$$

asymptotic to linear. Due to the saturating nonlinearity, SGNLS has global smooth solutions for each $\epsilon > 0$ and is globally well-posed just like SG itself [21].

3.5 Modulation Analysis

To reveal the role of perturbative terms in SGNLS, formal asymptotic analysis provides information on the dynamical behavior of pulse solutions near collapse. As [6], [7], we proceed under the following hypotheses: (1) The initial power is slightly above critical $N_c = 1.86$, by a small amount β_0; (2) The focusing part of the solution is close to the asymptotic profile:

$$u_s(t,x,y,\cdot) \sim \frac{1}{L(t,\cdot)} V(\zeta,\xi,\eta,\cdot) \exp\left[i\zeta(t,\cdot) + i\frac{L_t}{L}\frac{r^2}{4}\right],$$

where $\xi = \frac{x_1}{L}$, $\eta = \frac{x_2}{L}$, $V = R + O(\beta,\epsilon)$, $r = |x|$, R the cubic NLS ground state profile,

$$\zeta_t = \frac{1}{L^2}, \quad \beta = -L^3 L_{tt}, \tag{3.24}$$

and "$\cdot$" referring to dependence on ϵ.

(3) $|F| \ll |\Delta A|$, and $|A|^2 A$.

Under these conditions, the parameters (L,β), L the pulse width, obey a closed system of two ODE's. The reduced ODE for L is:

$$(y_t)^2 = -\frac{4H_0}{M}(y_M - y)(y - y_m)/y, \qquad y = L^2, \tag{3.25}$$

where:

$$H_0 \sim H(0) + \frac{\epsilon^2 C_1}{4L^4(0)},$$

$$C_1 = (2N_c)\left(\frac{1}{\omega^2} - \frac{\alpha_0}{18}\right), \ \alpha_0 \in (5,6),$$

$$y_M = \frac{M\beta_0}{-H_0}\left(1 + O\left(\frac{\epsilon^2 H_0}{\beta_0^2}\right)\right), \tag{3.26}$$

$$y_m = \frac{\epsilon^2 C_1}{4M\beta_0}\left[1 + O\left(\frac{\epsilon^2 H_0}{\beta_0^2}\right)\right]. \tag{3.27}$$

It follows that if $C_1 > 0$ ($\omega^2 < \frac{18}{\alpha_0}$), and $H(0) < 0$, L (pulse width) oscillates between $\sqrt{y_m}$ and $\sqrt{y_M}$, with period:

$$2\sqrt{My_M/(-H_0)} \int_0^{\pi/2} \left[1 - \left(1 - \frac{y_m}{y_M}\right)\sin^2\theta\right]^{1/2} d\theta. \tag{3.28}$$

This oscillatory pulse dynamics has been observed in numerical simulation of SG solutions, even for initial data away from the asymptotic regime. The asymptotic analysis also helps in selecting initial data according to (3.14). For example, a single pulse SG initial data is:

$$u(x,y,0) = A_0 e^{-0.04(x^2+y^2)} \sin mx, \tag{3.29}$$

$$u_t(x,y,0) = -\omega A_0 e^{-0.04(x^2+y^2)} \cos kx, \tag{3.30}$$

with $A_0 > 0.7$, $k \geq 2$. The $k = 2$ is used in Figure 3 and Figure 4.

There are many open problems for SG. One is how to justify the formal asymptotics, the other is to perform accurate numerical calculation of pulses for much smaller ϵ to compare the approximate solution by NLS and the SG solution quantitatively. Furthermore, a similar study on the three dimensional SG pulses will be exciting. Most challenging is the analysis of pulse dynamics of Maxwell systems.

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