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Periodic homogenization of G-equations and viscosity effects

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Abstract
G-equations are well-known front propagation models in combustion and are Hamilton–Jacobi type equations with convex but non-coercive Hamiltonians. Viscous G-equations arise from numerical discretization or modeling dissipative mechanisms. Although viscosity helps to overcome non-coercivity, we prove homogenization of an inviscid G-equation based on approximate correctors and attainability of controlled flow trajectories. We verify the attainability for two-dimensional mean zero incompressible flows, and demonstrate asymptotically and numerically that viscosity reduces the homogenized Hamiltonian in cellular flows. In the case of one-dimensional compressible flows, we found an explicit formula of homogenized Hamiltonians, as well as necessary and sufficient conditions for wave trapping (effective Hamiltonian vanishes identically). Viscosity restores coercivity and wave propagation.

Mathematics Subject Classification: 70H20, 76M50, 76M45, 76N20

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Front or interface propagation in fluid flows is a robust nonlinear phenomenon arising in liquid phase chemical reactions and premixed flame propagation in fluid turbulence [13, 34, 35, 41] among other applications. Mathematical models range from reaction–diffusion–advection equations to advective Hamilton–Jacobi equations (HJ) [10, 14, 17, 42, 43]. A particular HJ equation, the so called G-equation, is the most popular in the combustion science literature [16, 25, 38, 45]. The G-equation is

$$G_t + V(x,t)\cdot D_x G = s_l|D_x G| + d\Delta_x G, \tag{1.1}$$

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where G is a scalar function (the level set function of the interface), $V(x,t)$ is a prescribed flow velocity field, s_l is a positive constant (laminar front speed), $d \geqslant 0$ is a diffusion coefficient. If $d = 0$ (inviscid regime), the G-equation (1.1) is the level set equation of the interface motion law: the exterior normal velocity of the interface equals the laminar speed s_l plus the projection of the fluid velocity along the normal, see chapter 6 of [32, 33]. The viscous term $d\Delta_x G$ introduces an additional length scale; $d > 0$ is proportional to the so-called Markstein length [16, 33]. The viscous term arises from numerical discretization [32] or a simplification of curvature [16].

A fundamental problem in turbulent combustion is to study the large time front speed, or the asymptotic growth rate $\lim_{t\to+\infty} G(x,t)/t$, and analyse its dependence on the advection field V. Such a limit (if it exists) is called the turbulent front speed (s_T) [34, 43]. The large time front speed may be captured by first performing a scaling transform $G^\epsilon(x,t) = \epsilon G(x/\epsilon, t/\epsilon)$, then taking the limit $\epsilon \to 0$. The transformed equation is

$$G^\epsilon_t + V(x/\epsilon, t/\epsilon)\cdot D_x G^\epsilon = s_l|D_x G^\epsilon| + \epsilon\, d\, \Delta_x G^\epsilon, \tag{1.2}$$

which is a homogenization problem. Here $\epsilon \to 0$ plays the role of $t \to +\infty$. We shall see later that the travelling front solution of G-equation satisfies the cell problem of homogenization, and that front speed is associated with the homogenized Hamiltonian.

Homogenization of HJ equation

$$u^\epsilon_t + H\left(\frac{x}{\epsilon}, D_x u^\epsilon\right) = 0, \tag{1.3}$$

when Hamiltonian $H = H(x,p)$ is a periodic function in x (so-called periodic homogenization), was originated in [23] in the 1980s, and further developed [18, 19] to include viscous HJs and fully nonlinear equations. Besides periodicity of H in x, the Hamiltonian is usually required to be coercive:

$$|H(x,p)| \to +\infty \qquad \text{as } |p| \to +\infty, \quad \text{uniformly in } x. \tag{1.4}$$

Recently, much progress has been made in extending homogenization to stationary ergodic media for *convex and coercive* inviscid and viscous HJs [20, 21, 24, 36, 37, 39, 40].

The Hamiltonian of G-equation is $H(x,p) = -s_l|P| + V(x)\cdot P$, which is not coercive if V changes sign and has large enough amplitude as in a strong advection regime. In this paper, we study the role of viscosity (or parameter d) in periodic homogenization and qualitative properties of homogenized Hamiltonian $\overline{H}$ of G-equations. In the viscous case, $\overline{H}$ is given by the cell (corrector) problem whose solution is classical. Homogenization of inviscid G-equation is more interesting in that exact solutions of the cell problem may not exist due to the lack of coercivity. There are quite a few papers in the literature on homogenization of non-coercive HJs, see [2–7, 11] among others. However, the inviscid G-equation does not satisfy assumptions in the quoted papers. To carry out homogenization, we construct approximate solutions of the cell problem under specific conditions of the flows.

In one space dimensional compressible flows, the effective Hamiltonian is given by an explicit formula which may be zero and cause wave trapping (propagation failure). Necessary and sufficient condition of trapping is found in closed form. In particular, trapping occurs for a gradient flow with large enough amplitude. If viscosity is present however, trapping disappears for any mean zero spatial flow.

In multi-dimensional mean zero incompressible flows, we identify a sufficient condition for approximate correctors in terms of global attainability of controlled flow trajectories. This property is verifiable for two-dimensional Hamiltonian flows, and it implies homogenization. Our proof is based on a combined use of PDE and one-sided controllability. The $\overline{H}$ is coercive. We demonstrate via cellular flows that viscosity reduces $\overline{H}$ or the effective speed of front propagation.

The paper is organized as follows. In section 2, we review and present homogenization results of viscous G-equation for spatially periodic flows. In section 3, we prove homogenization of one space dimensional inviscid G-equation with spatial compressible flows, and state necessary and sufficient trapping conditions. The corresponding trapping condition of the viscous G-equation demonstrates the dramatic difference a positive viscosity makes. In section 4, we prove homogenization of inviscid G-equation in two-dimensional mean zero incompressible flows based on the global attainability property of controlled flows. We also show the viscosity effect on $\overline{H}$ in cellular flows. In section 5, we conclude with remarks on future work.

Since this work was completed and submitted for review, much exciting progress has been made on homogenization of inviscid G-equation in multi-dimensions. A local attainability property of the controlled flow trajectory is utilized by two of the authors here [44] to extend periodic homogenization to any space dimensions for incompressible flows. Although global attainability in section 4 is more elegant than the local attainability [44], it may not hold in dimensions three and above. By a different method, homogenization is shown for more general periodic flows in [12]. More recently, through analysis of sub-additivity of travel time of the controlled flow trajectory, homogenization for stationary ergodic incompressible flows has been established in two space dimensions [26]. Similar results are stated in higher dimensions under sufficient conditions of travel times [26].

2. Viscous G-equation and effective Hamiltonian

Let us set $s_l = 1$ with no loss of generality, and consider 1-periodic Lipschitz continuous vector field $V = V(x)$, $u = -G$. Then (1.2) becomes

$$u_{\epsilon,t} + V\left(\frac{x}{\epsilon}\right) \cdot Du_\epsilon + |Du_\epsilon| - \epsilon d\Delta u_\epsilon = 0, \tag{2.1}$$

where d is a positive constant, V is periodic and Lipschitz continuous; $g(x)$ is uniformly continuous and grows at most linearly, $|g(x)| \leqslant C_1|x| + C_2$ for two constants C_1 and C_2. Such initial data include the affine function for initiating travelling fronts. For each $\epsilon > 0$, there exists a unique viscosity solution $u_\epsilon \in C(\mathbb{R}^n \times [0, +\infty))$ which grows at most linearly in t and x. The existence and uniqueness of u_ϵ follow from corollary 2.1 in [8]. The existence part can also be deduced from the optimal control (Lagrangian) formulation (see [20, 28] and references therein).

The formal two-scale homogenization ansatz

$$u_\epsilon(x, t) = u_0(x, t) + \epsilon\, u_1\left(x, t, \frac{x}{\epsilon}\right) + \cdots, \tag{2.2}$$

gives to leading order

$$u_{0,t} + V(y) \cdot (D_x u_0 + D_y u_1) + |D_x u_0 + D_y u_1| - d\Delta_y u_1 = 0. \tag{2.3}$$

The cell problem is as follows: given any vector $P \in \mathbb{R}^n$, find a unique number $\overline{H}(P)$ such that the equation

$$-d\Delta_y u + |P + D_y u| + (P + D_y u) \cdot V(y) = \overline{H}(P), \qquad y \in \mathbb{T}^n \tag{2.4}$$

has a periodic solution $u = u(y)$ on $\mathbb{T}^n$. If the cell problem is solvable, u_0 then formally satisfies the homogenized HJ equation

$$u_{0,t} + \overline{H}(D_x u_0) = 0. \tag{2.5}$$

Equation (2.4) of the cell problem also arrives from seeking a travelling front solution to (2.1) of the form

$$u = P \cdot x - \overline{H}t + \epsilon\, w(x/\epsilon), \tag{2.6}$$

where $w = w(y)$ is periodic. Upon substitution, we see immediately that $(w, \overline{H})$ satisfies the cell problem (2.4). If P is a unit vector, $\overline{H}(P)$ is the front speed in direction P.

The cell problem has a C^2 solution u corresponding to a unique convex function $\overline{H}(P)$. Convergence of u_ϵ to a solution of (2.5) then follows from the perturbed test function method [18]. The proofs may be adapted from [5, 19]. We skip the details and state the results below.

Theorem 2.1.

(1) Given any $P \in \mathbb{R}^n$, there exists a unique number $\overline{H}(P)$ such that the cell problem (2.4) has a periodic solution $u \in C^{2,\alpha}(\mathbb{T}^n)$, for any $\alpha \in (0, 1)$.

(2) The effective Hamiltonian $\overline{H}$ is given by the min–max formula

$$\overline{H}(P) = \min_{\phi \in C^2(\mathbb{T}^n)} \max_{\mathbb{T}^n}(-d\Delta\phi + |P + D\phi| + (P + D\phi) \cdot V), \tag{2.7}$$

and so $\overline{H}$ is convex and homogeneous of degree one in P.

(3) As $\epsilon \to 0$, u_ϵ locally uniformly converges to u which grows at most linearly in (x, t) and is the unique viscosity solution of

$$\begin{aligned} &u_t + \overline{H}(Du) = 0, \\ &u(x, 0) = g. \end{aligned} \tag{2.8}$$

Although the Hamiltonian of the G-equation $H(x, p) = |p| + V(x) \cdot p$ is non-coercive in p when V has large enough amplitude, the homogenized $\overline{H}$ may be coercive. This is the case when V is mean zero and divergence free. Integrating equation (2.4) over $\mathbb{T}^n$ and applying Jensen's inequality show that

$$\overline{H}(P) = \int_{\mathbb{T}^n} |P + D_y u|\, dy \geqslant |P|,$$

implying coercivity of $\overline{H}$.

3. Viscosity effect and wave trapping in 1D

In this section, we consider homogenization of the inviscid G-equation in one space dimensional compressible flows, and present formulae of $\overline{H}$ in closed form. Explicit formulae of inviscid $\overline{H}$ in shear flows (uni-directional incompressible flows) are known [17], where the cell problem reduces to an ordinary differential equation (ODE) with exact solution.

An interesting feature of the inviscid G-equation in one-dimensional compressible flows is that the cell problem (an ODE) may not have exact solutions. However, we show that approximate solutions to cell problem exist and this is enough to establish homogenization. If the variation of the flow is large enough, then $\overline{H}(p) \equiv 0$, implying wave front trapping by the flow, and non-coercivity of $\overline{H}$. Non-coercivity of the inhomogeneous Hamiltonian, $H(x, p) = |p| + V(x)p$, persists in the effective Hamiltonian. In contrast, we show that in the one space dimensional viscous G-equation ($d > 0$), $\overline{H}(p)$ is generically coercive. In particular, if the flow has mean equal to zero, $\overline{H}(\pm 1) > 0$, $\overline{H}(p) = \overline{H}(\mathrm{sgn}(p))|p|$, and so the effective Hamiltonian is coercive. Homogenization overcomes non-coercivity with the help of diffusivity.

3.1. Wave trapping and non-coercivity in inviscid G-equation

Let us define the effective Hamiltonian $\overline{H}$ as follows:

Case I. If $\max_{\mathbb{T}^1} |V| < 1$ or $\min_{\mathbb{T}^1} |V| > 1$ (i.e. $V \neq \pm 1$), then

$$\overline{H}(p) = p \left(\int_0^1 \frac{1}{V(x) + \text{sign }(p)} \, \mathrm{d}x \right)^{-1}.$$

Case II. If $\{x \in \mathbb{T}^1 | \, V(x) = 1\} \neq \emptyset$ and $\min_{\mathbb{T}^1} V > -1$, then

(i) (one-sided wave trapping) $H(p) = 0$ for $p \leqslant 0$;
(ii) for $p > 0$,

$$\overline{H}(p) = p \left(\int_0^1 \frac{1}{V(x) + 1} \, \mathrm{d}x \right)^{-1}.$$

Case III. If $\{x \in \mathbb{T}^1 | \, V(x) = -1\} \neq \emptyset$ and $\max_{\mathbb{T}^1} V < 1$, then

(i) (one-sided wave trapping) $H(p) = 0$ for $p \geqslant 0$;
(ii) for $p < 0$,

$$\overline{H}(p) = p \left(\int_0^1 \frac{1}{V(x) - 1} \, \mathrm{d}x \right)^{-1}.$$

Case IV. If $\{x \in \mathbb{T}^1 | \, V(x) = -1\} \neq \emptyset$ and $\{x \in \mathbb{T}^1 | \, V(x) = 1\} \neq \emptyset$, then (bi-directional wave trapping)

$$\overline{H} \equiv 0.$$

The non-zero part of the $\overline{H}$ formula comes from solving the cell problem

$$|p + u'| + V(x)(p + u') = \overline{H}. \tag{3.1}$$

Since $\overline{H}(p) \neq 0$, $p + u'$ does not change sign. Hence we have that either

$$p + u' = \overline{H}/(V(x) - 1)$$

or

$$p + u' = \overline{H}/(V(x) + 1)$$

depending on the sign of p. When $\overline{H}(p) = 0$, the cell problem in general does not admit continuous viscosity solution. Instead, we construct approximate solutions explicitly below. Zero $\overline{H}$ means wave trapping by the flow. The definition of $\overline{H}$ implies that the necessary and sufficient condition for wave trapping at $p \neq 0$ is

$$\overline{H}(p) = 0 \qquad \text{iff} \quad \{x \in \mathbb{T}^1 | \, V(x) + \text{sign}(p) = 0\} \neq \emptyset. \tag{3.2}$$

We state the following theorem:

Theorem 3.1. *For any $\epsilon > 0$, there exists a function $u_\epsilon \in C^1(\mathbb{T}^1)$ such that*

$$\overline{H}(p) - \epsilon \leqslant |p + u_\epsilon'| + V(x)(p + u_\epsilon') \leqslant \overline{H}(p) + \epsilon. \tag{3.3}$$

Moreover, $\overline{H}$ satisfies the inf-max formula

$$\overline{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^1)} \max_{x \in \mathbb{T}^1} (|p + \phi'(x)| + V(x)(p + \phi'(x))). \tag{3.4}$$

Proof. We prove the case of $p = 1$. The other p numbers are similar. If $\{x \in \mathbb{T}^1 |\ V(x) = -1\} = \emptyset$, then it is easy to check that the cell problem

$$|1 + u'| + V(x)(1 + u') = \overline{H}(1)$$

has a C^1 periodic solution u satisfying $1 + u' = \overline{H}/(1 + V(x))$.

So let us look at the case $\{x \in \mathbb{T}^1 |\ V(x) = -1\} \neq \emptyset$. Without loss of generality, we assume that $V(0) = -1$. Since V is Lipschitz continuous, there exists a constant $K \geqslant 1$ such that $|V(x) + 1| \leqslant K|x|$. Then

$$c = \int_0^1 \frac{\epsilon}{|1 + V(x)| + \mathrm{e}^{\frac{-K}{\epsilon}}}\, \mathrm{d}x \tag{3.5}$$

$$\geqslant \int_0^1 \frac{\epsilon}{Kx + \mathrm{e}^{\frac{-K}{\epsilon}}}\, \mathrm{d}x \geqslant 1. \tag{3.6}$$

Now let us choose

$$u_\epsilon(x) = \int_0^x \frac{\epsilon}{c|1 + V(s)| + c\mathrm{e}^{\frac{-K}{\epsilon}}}\, \mathrm{d}s - x,$$

which satisfies inequality (3.3) with $\overline{H} = 0$.

The inf-max formula (3.4) follows from a similar argument as in theorem 2.1 of the viscous case. We omit details. □

Homogenization follows from the approximate solution of the cell problem and the perturbed test function method [19], and the result is the following theorem:

Theorem 3.2. *For $\epsilon > 0$, let u_ϵ be the unique viscosity solution of the inviscid G-equation with at most linear growth:*

$$u_{\epsilon,t} + |u_\epsilon'| + V\left(\frac{x}{\epsilon}\right) u_\epsilon' = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R},$$

$$u_\epsilon(x, 0) = g(x).$$

Then u_ϵ locally uniformly converges to the unique viscosity solution u with at most linear growth of the effective equation:

$$\begin{aligned} &u_t + \overline{H}(u') = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}, \\ &u(x, 0) = g(x). \end{aligned} \tag{3.7}$$

3.2. Viscosity and absence of wave trapping

When $d > 0$, we prove that generically wave trapping does not occur and $\overline{H}(p)$ is coercive.

Theorem 3.3. *Consider $\overline{H}$ of the homogenized one space dimensional viscous G-equation. Let $p \neq 0$. Then*

$$\overline{H}(p) = 0 \qquad \text{iff} \quad \int_0^1 V\, \mathrm{d}x + \mathrm{sign}(p) = 0. \tag{3.8}$$

Remark 3.1. Trapping condition (3.8) for the viscous G-equation is a non-local version of the pointwise trapping condition (3.2) of the inviscid G-equation.

Proof of theorem 3.3. Let us assume that $p = 1$. The other cases are similar. '$\Longrightarrow$' We first prove the necessity part. Suppose that $\overline{H}(1) = 0$. By theorem 2.1, there exists $u \in C^2(\mathbb{T}^1)$ satisfying

$$|1 + u'| + V(x)(1 + u') = \mathrm{d}u''.$$

Then $w = 1 + u'$ satisfies

$$|w| + V(x)w = \mathrm{d}w'$$

subject to

$$\int_0^1 w \,\mathrm{d}x = 1. \tag{3.9}$$

We claim that $w(x) > 0, \forall x \in \mathbb{R}^1$. In fact, according to (3.9), there exists $x_0 \in \mathbb{R}^1$ such that $w(x_0) > 0$. If our claim is not true, by periodicity of $w(x)$, we find $x_1 < x_0$ such that $w(x_1) = 0$ and $w > 0$ in (x_1, x_0). Note that

$$\mathrm{d}w' = w(1 + V) \qquad \text{in } (x_1, x_0).$$

Hence $w \equiv 0$ in (x_1, x_0). This contradicts the fact that $w(x_0) > 0$. The claim holds, and so

$$\mathrm{d}w' = w(1 + V) \qquad \text{in } \mathbb{R}.$$

So

$$w(x) = w(0)\mathrm{e}^{\int_0^x (\frac{1+V(s)}{d})\,\mathrm{d}s}.$$

Since $w(1) = w(0)$, we have

$$\int_0^1 (1 + V(s))\,\mathrm{d}s = 0,$$

or

$$\int_0^1 V(x)\,\mathrm{d}x + 1 = 0.$$

'$\Longleftarrow$' Now let us prove the sufficiency part. Assume that

$$\int_0^1 V(x)\,\mathrm{d}x = -1.$$

Then it is clear that

$$u(x) = \lambda \int_0^x \mathrm{e}^{\int_0^y (\frac{1+V(s)}{d})\,\mathrm{d}s}\,\mathrm{d}y - x$$

is a solution of

$$|1 + u'| + V(x)(1 + u') = \mathrm{d}u'',$$

if the constant λ satisfies

$$\lambda \int_0^1 \mathrm{e}^{\int_0^y (\frac{1+V(s)}{d})\,\mathrm{d}s}\,\mathrm{d}y = 1.$$

So $\overline{H}(1) = 0$. □

Corollary 3.1. *If $V(x)$ has mean zero, then the effective Hamiltonian of the viscous G-equation satisfies $\overline{H}(\pm 1) > 0$ and*

$$\overline{H}(p) = c^*(\text{sign}(p))|p|, \tag{3.10}$$

for positive constants $c^*(\pm 1)$. Coercivity is valid for $\overline{H}$.

Proof. It follows from theorem 3.3 that $\overline{H}(\pm 1) \neq 0$. Scale V to δV. By the estimates in the proof of theorem 2.1 and min–max formula, $\overline{H}_\delta = \overline{H}(\pm 1, \delta)$ is a continuous function in $\delta \in [0, 1]$, and is positive when δ is small enough such that $\delta \|V(x)\|_\infty < 1$. Theorem 3.3 implies that $\overline{H}(\pm 1, \delta) \neq 0$ for any $\delta \in [0, 1]$. By continuity in δ, $\overline{H}(\pm 1, \delta) > 0$, for any $\delta \in [0, 1]$. Setting $\delta = 1$, we obtain $\overline{H}(\pm 1) > 0$. Degree one homogeneity of $\overline{H}$ gives (3.10). □

Remark 3.2. The absence of trapping also appears in reaction–diffusion front propagation through one-dimensional spatial compressible flows [27]. When the reaction is quadratic (or Kolmogorov–Petrovsky–Piskunov) type, mean zero spatial flow (compressible flow) in one space dimension slows down the front speed. However, front speed is always positive. This is similar to corollary 3.1.

4. Two space dimensional inviscid G-equations

In this section, we formulate a sufficient condition on the two space dimensional flow field V for the existence of approximate solution of cell problem and homogenization of inviscid G-equation. In one space dimension, approximate solutions are explicitly constructed in the last section. In multi-dimensions, the approximate solvability of cell problem is more delicate. We show below that it is related to whether any point in the flow field may be connected by a controlled flow trajectory to another point where $|V|$ is small enough and coercivity holds locally ($|p|$ dominates the Hamiltonian). We also illustrate the role of viscosity on $\overline{H}$ with numerical examples.

4.1. Homogenization by controlled flows

Let $n = 2$, $\nabla \cdot V = 0$, $\int_{\mathbb{T}^2} V \,\mathrm{d}x = 0$. There exists $\mathcal{H} \in C^{1,1}(\mathbb{T}^2)$ such that

$$V = \nabla^\perp \mathcal{H}.$$

Definition 4.1 (Controlled flow trajectory). *A controlled flow trajectory associated with V is $\xi \in W^{1,\infty}([0, T]; \mathbb{R}^n)$ such that for a.e. $t \in (0, T)$*

$$\dot{\xi}(t) = \alpha(t) + V(\xi(t))$$

where control $\alpha(t) \in L^\infty([0, T]; B_1(0))$.

Lemma 4.1 (Attainability by controlled flow trajectory in 2D). *For any $x \in \mathbb{R}^2$, there exists a controlled flow trajectory $\xi : [0, T_x] \to \mathbb{R}^2$ satisfying (i) $\xi(0) = x$, $|V(\xi(T_x))| \leqslant \frac{1}{2}$; (ii) $T_x \leqslant 4 \max_{\mathbb{T}^2} |\mathcal{H}|$.*

Proof. If $|V(x)| \leqslant \frac{1}{2}$, it is obvious. Assume that $|V(x)| > \frac{1}{2}$ and choose ξ as

$$\begin{cases} \dot{\xi}(t) = \dfrac{D\mathcal{H}(\xi(t))}{|D\mathcal{H}(\xi(t))|} + V(\xi(t)), \\ \xi(0) = x. \end{cases}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\xi(t)) = |D\mathcal{H}(\xi(t))| = |V(\xi(t))|.$$

Since $\mathcal{H}$ is bounded, there must exist $t_1 > 0$ such that $|V(\xi(t))| > \frac{1}{2}$ for $t \in [0, t_1)$ and $|V(\xi(t_1))| = \frac{1}{2}$. Accordingly,

$$\mathcal{H}(\xi(t_1)) - \mathcal{H}(x) = \int_0^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\xi(t))\,\mathrm{d}t \geqslant \frac{t_1}{2}.$$

So

$$t_1 \leqslant 4 \max_{\mathbb{T}^2} |\mathcal{H}|.$$

□

Next we establish an inequality for a subsolution of the modified cell problem at two points connected by a controlled flow trajectory.

Lemma 4.2. *Suppose that $u \in C(\mathbb{T}^n)$ is a viscosity subsolution of*

$$\lambda u + |P + Du| + V(x) \cdot (P + Du) = 0 \qquad \text{in } \mathbb{R}^n.$$

Then for any controlled flow trajectory $\xi : [0, T] \to \mathbb{R}^n$,

$$u(\xi(T)) - \mathrm{e}^{-\lambda T} u(\xi(0)) \leqslant -\int_0^T P \cdot \dot{\xi} \mathrm{e}^{\lambda(s-T)}\,\mathrm{d}s.$$

Proof. For $\delta > 0$, consider the super involution

$$u_\delta(x) = \sup_{y\in\mathbb{R}^n} \left(u(y) - \frac{1}{\delta}|x-y|^2 \right) = \sup_{z\in\mathbb{R}^n} \left(u(x+z) - \frac{1}{\delta}|z|^2 \right).$$

It is clear that u_δ is periodic, semiconvex and Lipschitz continuous. Since u is bounded,

$$u_\delta(x) = \sup_{z\in B_{\tilde{C}\sqrt{\delta}}(0)} \left(u(x+z) - \frac{1}{\delta}|z|^2 \right),$$

where $\tilde{C} = \sqrt{2\max_{\mathbb{T}^n}|u|}$. Hence u_δ is a viscosity subsolution of

$$\lambda u_\delta + (1 - C_u\sqrt{\delta})|P + Du_\delta| + V(x) \cdot (P + Du_\delta) \leqslant o(1)$$

for some constant C_u which is independent of δ and $\lim_{\delta\to 0} o(1) = 0$. Since the Hamiltonian is convex in P, by mollifying u_δ, we may assume that u_δ is C^1. Suppose that $\dot{\xi}(t) = \alpha(t) + V(\xi(t))$. Let $\xi_\delta : [0, T] \to \mathbb{R}^n$ be the control satisfying

$$\begin{cases} \dot{\xi}_\delta = (1 - C_u\sqrt{\delta})\alpha + V(\xi_\delta), \\ \xi_\delta(0) = \xi(0). \end{cases}$$

Then for a.e. t

$$\begin{aligned} \frac{\mathrm{d}(P \cdot \xi_\delta + u_\delta(\xi_\delta))}{\mathrm{d}t} &= (P + Du_\delta(\xi_\delta)) \cdot ((1 - C_u\sqrt{\delta})\alpha(t) + V(\xi_\delta(t))) \\ &\leqslant (1 - C_u\sqrt{\delta})|P + Du_\delta(\xi_\delta)| + V(\xi_\delta)(P + Du_\delta(\xi_\delta)) \\ &\leqslant -\lambda u_\delta(\xi_\delta) + o(1). \end{aligned}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{\lambda t} u_\delta(\xi_\delta(t))) \leqslant -\mathrm{e}^{\lambda t} P \cdot \dot{\xi}_\delta(t) + o(1).$$

Therefore

$$u_\delta(\xi_\delta(T)) - \mathrm{e}^{-\lambda T} u(\xi_\delta(0)) \leqslant -\int_0^T P \cdot \dot{\xi}_\delta \mathrm{e}^{\lambda(s-T)}\,\mathrm{d}s + o(1).$$

Sending $\delta \to 0$, the above lemma holds. □

Standard Perron's method implies that for given $\lambda > 0$, there exists a unique periodic viscosity solution $u_\lambda \in C(\mathbb{T}^n)$ of

$$\lambda u_\lambda + |P + Du_\lambda| + V(x)\cdot(P + Du_\lambda) = 0 \qquad \text{in } \mathbb{R}^n.$$

We refer to [15] for details. By maximum principle,

$$|\lambda u_\lambda| \leqslant |P|\left(1 + \max_{\mathbb{T}^n}|V|\right).$$

The following is the key lemma of this section.

Lemma 4.3. *For any sequence $\lambda_m \to 0$ as $m \to +\infty$, there exists a subsequence $\lambda_{m_k} \to 0$ as $k \to +\infty$ such that*

$$\lim_{k\to+\infty} \lambda_{m_k} u_{\lambda_{m_k}} = c \qquad \text{uniformly in } \mathbb{R}^2$$

for some constant $c \in \mathbb{R}$.

Proof.
Step I. Clearly, within the region $W = \{x \in \mathbb{R}^2 |\ |V(x)| < 1\}$, u_λ is locally Lipschitz continuous and

$$|Du_\lambda(x)| \leqslant \frac{|P|(1 + \max_{\mathbb{T}^n}|V|)}{1 - V(x)} \qquad \text{for a.e. } x \in W.$$

Therefore there exists a subsequence $\lambda_{m_k} \to 0$ as $k \to +\infty$ such that

$$\lim_{k\to+\infty} \lambda_{m_k} u_{\lambda_{m_k}} = g(x) \qquad \text{uniformly in } W_{\frac{3}{4}} = \{x \in \mathbb{R}^2 |\ |V(x)| \leqslant \tfrac{3}{4}\}, \tag{4.1}$$

for some continuous function $g \in C(W_{\frac{3}{4}})$.

Step II. Note that $v_k = \lambda_{m_k} u_{\lambda_{m_k}}$ is a viscosity solution of

$$\lambda_{m_k} v_k + |\lambda_{m_k} P + Dv_k| + V(x)\cdot(\lambda_{m_k} P + Dv_k) = 0.$$

Let $\bar{v} = \limsup_{k\to\infty, y\to x} v_k$. Then $\bar{v}$ is bounded, upper semicontinuous and a viscosity subsolution of

$$|D\bar{v}| + V(x)\cdot D\bar{v} \leqslant 0 \qquad \text{in } \mathbb{R}^2.$$

As in the proof of lemma 4.2, we consider the super involution $\bar{v}_\delta$ of $\bar{v}$. Then when δ is small enough, $\bar{v}_\delta$ is a viscosity subsolution of

$$\tfrac{1}{2}|D\bar{v}_\delta| + V(x)\cdot D\bar{v}_\delta \leqslant 0 \qquad \text{in } \mathbb{R}^2.$$

Note that there is no error term o(1) on the right-hand side since the above equation does not involve the zeroth order term. Taking integration over $\mathbb{T}^2$, we derive that

$$\int_{\mathbb{T}^2} |D\bar{v}_\delta|\,\mathrm{d}x = 0.$$

Hence $\bar{v}_\delta(x) \equiv c_\delta$ for some constant $c_\delta \in \mathbb{R}$. Upon a subsequence if necessary, we may assume that $\lim_{\delta\to 0} c_\delta = c$. Since $\lim_{\delta\to0}\bar{v}_\delta = \bar{v}$, we get that

$$\bar{v}(x) \equiv c \tag{4.2}$$

for some constant $c \in \mathbb{R}$.

Step III. It is easy to see that

$$g(x) \equiv c \qquad \text{in } W_{\frac{1}{2}} = \{x \in \mathbb{R}^2 |\ |V(x)| < \tfrac{1}{2}\}.$$

Step IV. Now for any $y \in \mathbb{R}^2$, let $\xi : [0, T_y] \to \mathbb{R}^2$ be the control from lemma 4.1. Thanks to lemma 4.2,

$$u_{\lambda_{m_k}}(\xi(T_y)) - \mathrm{e}^{-\lambda_{m_k} T_y} u_{\lambda_{m_k}}(y) \leqslant -\int_0^{T_y} P \cdot \dot{\xi} \mathrm{e}^{\lambda_{m_k}(s-T)}\, \mathrm{d}s \leqslant T_y |P| \left(1 + \max_{\mathbb{T}^n} |V|\right) \leqslant C.$$

Since $|V(\xi(T_y))| \leqslant \frac{1}{2}$, thanks to (4.1), (4.2) and step III, we get for all $x \in \mathbb{R}^2$

$$\liminf_{k\to\infty, y\to x} \lambda_{m_k} u_{\lambda_{m_k}}(y) \geqslant c = \limsup_{k\to\infty, y\to x} \lambda_{m_k} u_{\lambda_{m_k}}(y).$$

So

$$\lim_{k\to\infty} \lambda_{m_k} u_{\lambda_{m_k}}(x) = c \qquad \text{uniformly in } \mathbb{R}^2.$$

□

Lemma 4.4.

$$\lim_{\lambda\to 0} \lambda u_\lambda = -\overline{H}(P), \qquad \text{uniformly in } \mathbb{R}^2,$$

where $\overline{H}(P)$ is a constant. As a function of P, it is Lipschitz continuous, convex and homogeneous of degree one.

Proof. By lemma 4.3, we find a subsequence $\lambda_m \to 0$ as $m \to +\infty$ such that

$$\lim_{m\to+\infty} \lambda_m u_{\lambda_m} = c \qquad \text{uniformly in } \mathbb{R}^2.$$

We show that

$$\lim_{\lambda\to 0} \lambda u_\lambda = c \qquad \text{uniformly in } \mathbb{R}^2.$$

If not, owing to lemma 4.3, then there exists another subsequence $\lambda_m^{'} \to 0$ as $m \to +\infty$ such that

$$\lim_{m\to+\infty} \lambda_m^{'} u_{\lambda_m'} = c' \neq c \qquad \text{uniformly in } \mathbb{R}^2.$$

Without loss of generality, we assume that $c' > c$. Choose $c' > t_2 > t_1 > c$. Then when m is sufficiently large, u_{λ_m} is a viscosity subsolution of

$$|P + Du_{\lambda_m}| + V(x) \cdot (P + Du_{\lambda_m}) \geqslant -t_1$$

and $u_{\lambda_m'}$ is a viscosity supersolution of

$$|P + Du_{\lambda_m'}| + V(x) \cdot (P + Du_{\lambda_m'}) \leqslant -t_2.$$

This is impossible if we consider the place where $u_{\lambda_m} - u_{\lambda_m'}$ attains minimum via a double variable method [15]. Let us denote $\overline{H}(P) = -c$.

Next we prove that $\overline{H}(P)$ is Lipschitz continuous. In fact, fix λ, let u_P and u_Q be unique periodic viscosity solutions of the following two equations, respectively:

$$\lambda u_P + |P + Du_P| + V(x) \cdot (P + Du_P) = 0$$

and

$$\lambda u_Q + |Q + Du_Q| + V(x) \cdot (Q + Du_Q) = 0.$$

Then it is clear that $\tilde{u}_P = u_P + \frac{|P-Q|(1+\max_{\mathbb{T}^n}|V|)}{\lambda}$ is a viscosity supersolution of

$$\lambda \tilde{u}_P + |Q + D\tilde{u}_P| + V(x) \cdot (Q + D\tilde{u}_P) \geqslant 0.$$

Hence comparison principle implies that $\lambda u_Q \leqslant \lambda u_P + |P - Q|(1 + \max_{\mathbb{T}^n} |V|)$. Sending $\lambda \to 0$, we get that

$$|\overline{H}(P) - \overline{H}(Q)| \leqslant |P - Q|\left(1 + \max_{\mathbb{T}^n} |V|\right).$$

Next we prove that $\overline{H}$ is convex. Using super involution as in the proof of lemma 4.2, it is not hard to prove that $\tilde{u} = \frac{u_P + u_Q}{2}$ is a viscosity subsolution of

$$\lambda\tilde{u} + \left|\frac{P+Q}{2} + D\tilde{u}\right| + V(x) \cdot \left(\frac{P+Q}{2} + D\tilde{u}\right) \leqslant 0.$$

Hence comparison principle implies that

$$\lambda\tilde{u} \leqslant \lambda u_{\frac{P+Q}{2}}.$$

Convexity follows after sending $\lambda \to 0$.

Finally, it is clear that for $s > 0$, $u_s = s u_\lambda$ is a viscosity solution of

$$\lambda u_s + |sP + Du_s| + V(x) \cdot (sP + Du_s) = 0.$$

So $\overline{H}(P)$ is homogeneous of positive degree one. □

It is not clear whether the $\overline{H}(P)$ is given by the inf-max formula as in the one-dimensional case (theorem 5.1). Regardless, convexity and degree one homogeneity of $\overline{H}$ are established.

According to lemma 4.4, we can find approximate viscosity solution of the cell problem, i.e. for any $\tau > 0$, there exists a viscosity solution $u_\tau \in C(\mathbb{T}^2)$ of

$$\overline{H}(P) - \tau \leqslant |P + Du_\tau| + V(x) \cdot (P + Du_\tau) \leqslant \overline{H}(P) + \tau.$$

By the perturbed test function method [18, 19], we have the following theorem:

Theorem 4.1 (Homogenization). *Consider a two-dimensional spatially periodic flow field $V(x)$, which is mean zero and divergence free. For $\epsilon > 0$, let u_ϵ be the unique viscosity solution of the inviscid G-equation with at most linear growth:*

$$u_{\epsilon,t} + |Du_\epsilon| + V\left(\frac{x}{\epsilon}\right) \cdot Du_\epsilon = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}^2$$
$$u_\epsilon(x, 0) = g(x).$$

Then u_ϵ locally uniformly converges to the unique viscosity solution u with at most linear growth of the effective equation:

$$\begin{aligned} &u_t + \overline{H}(Du) = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}^2 \\ &u(x, 0) = g(x), \end{aligned} \tag{4.3}$$

where $\overline{H}$ is Lipschitz continuous, convex and homogeneous of degree one.

A non-trapping (coercivity) result holds as in the viscous case.

Corollary 4.1 (Coercivity of $\overline{H}$). *The effective Hamiltonian in a mean zero divergence free two space dimensional periodic flow satisfies*

$$\overline{H}(P) \geqslant |P|.$$

Proof. For any $\tau > 0$, by lemma 4.4, there exists a viscosity subsolution $u_\tau \in C(\mathbb{T}^2)$ of

$$|P + Du_\tau| + V(x) \cdot (P + Du_\tau) \leqslant \overline{H}(P) + \tau.$$

By considering super involution of u_τ as in the proof of lemma 4.2, given $\delta' \in (0, 1)$, there exists a viscosity subsolution $\tilde{u} \in W^{1,\infty}(\mathbb{T}^2)$ of

$$(1 - \delta')|P + D\tilde{u}| + V(x) \cdot (P + D\tilde{u}) \leqslant \overline{H}(P) + \tau.$$

Taking integration on both sides, we have by Jensen's inequality that

$$\overline{H}(P) + \tau \geqslant (1 - \delta') \int_{\mathbb{T}^2} |P + D\tilde{u}| \, \mathrm{d}x \geqslant (1 - \delta')|P|.$$

Sending $\tau \to 0$ then $\delta' \to 0$, we have

$$\overline{H}(P) \geqslant |P|.$$

□

Remark 4.1. A general control system is called 'uniform exact controllable' [4] if there exists $T > 0$ such that any two points can be connected by a controlled trajectory within time $\leqslant T$. It was proved in [4] that uniform exact controllability implies homogenization. See also [2] for generalization to two-player differential games. However, the uniform exact controllability is either false or hard to verify for the control system associated with the inviscid G-equation. The main novelty of this section is that, due to the special structure of the G-equation, a point only needs to reach the region where $|V| < 1$ via a controlled flow trajectory to achieve homogenization when the flow field is mean zero and divergence free. The relation between ergodicity and controllability to a subset is also discussed in [4] (attractor) and [6] (target) with very restrictive assumptions which do not apply to the G-equation.

4.2. Viscosity effect on $\overline{H}$ in 2-D

Although homogenization proofs show the enhancement of $\overline{H}(P)$ over the laminar speed $|P|$ in two-dimensional incompressible flows, they do not reveal the qualitative and quantitative effects of viscosity. In one space dimensional compressible flows, viscosity arrests trapping and promotes transport. In two-dimensional incompressible flows, viscosity may slow down the effective transport.

Let us consider cellular flow with amplitude A:

$$V(x_1, x_2) = A(\sin(2\pi x_1)\cos(2\pi x_2), -\sin(2\pi x_1)\cos(2\pi x_2)). \tag{4.4}$$

corresponding to Hamiltonian $\mathcal{H}(x_1, x_2) = (A/2\pi)\sin(2\pi x_1)\sin(2\pi x_2)$. The effective Hamiltonian $\overline{H} = \overline{H}(A, d)$ is a function of two variables (A, d). It is known [1, 28, 31] that the inviscid $\overline{H}$ grows with A like

$$\overline{H}(A, 0) \sim O(A/\log A), \qquad A \gg 1, \tag{4.5}$$

and with enough viscosity $d \gg 1$ (independent of A), the viscous $\overline{H}$ is slowed down significantly as [28]

$$\overline{H}(A, d) \leqslant O(\sqrt{\log A}), \qquad A \gg d \gg 1. \tag{4.6}$$

The proof of (4.6) is based on the viscous cell problem, equation (2.4). Numerical computation of $\overline{H}(A, d)$ for d above a moderately small value ($d \geqslant 0.1$) can be done by the finite difference method on (2.4). $\overline{H}$ is approximated by the iteration scheme

$$-d\Delta_h u_{k+1} + V(y) \cdot D_h u_{k+1} = H_k(P) - |P + D_h u_k| - V(y) \cdot P,$$

$$H_k(P) = \langle |P + D_h u_k| \rangle, \tag{4.7}$$

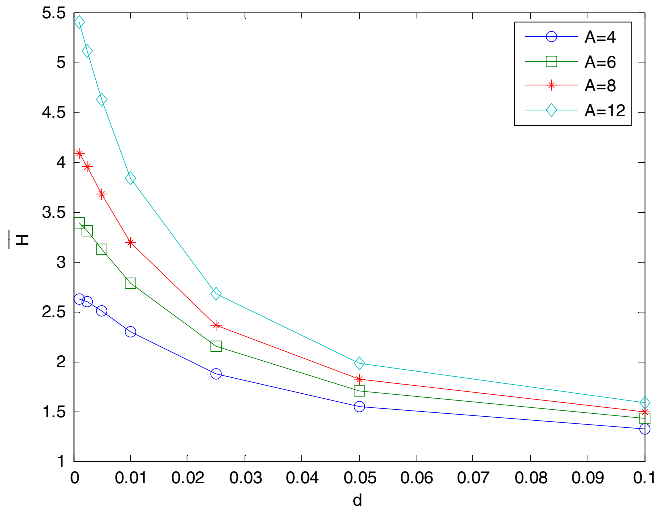

Figure 1. $\overline{H}$ is monotone decreasing as d increases for a range of A values.

where Δ_h (D_h) is the central (upwind) difference approximation of Laplacian and gradient operators, and the bracket denotes the integral average over the periodic cell $[0, 1]^2$. For $d < 0.1$, we integrate the time-dependent G-equation with affine data:

$$u_t + V(x) \cdot Du = |Du| + d\,\Delta u, \qquad u(x, 0) = P \cdot x,$$

whose solution is written as $u = w(x, t) + P \cdot x$, and $w(x, t)$ is periodic in x. The w equation is

$$w_t + V(x) \cdot (Dw + P) = |Dw + P| + d\Delta w, \qquad w(x, 0) = 0. \tag{4.8}$$

In our computation, $P = (1, 0)$. Equation (4.8) is discretized semi-implicitly over the periodic cell and integrated over a long time interval to extract the linear growth rate of solution which is $\overline{H}$. The scheme is explicitly upwind in the nonlinear term, and implicit Euler in Δw and the advection term $V(x) \cdot Dw$. The iterative scheme (4.7) converges much faster at $d \geqslant 0.1$ than the large time method (4.8) which can also handle smaller d values. The two methods give the same result at $d = 0.1$. Figure 1 shows that $\overline{H}$ is monotone decreasing in small values of d for $A = 4, 6, 8, 12$. Figure 2 shows that $\overline{H}$ is growing sublinearly in $A \in [0, 50]$ for $d = 0.05, 0.1, 0.2, 0.4$. Both properties of $\overline{H}$ remain for future analytical study.

5. Conclusions

G-equations are HJ models for studying front propagation in fluid flows, especially in turbulent combustion. We compared results on the periodic homogenization of viscous and inviscid G-equations which have convex yet non-coercive Hamiltonians. In the case of the inviscid G-equation, the cell (corrector) problem may not have exact solution due to non-coercivity of the Hamiltonian. However, homogenization suffices with an approximate corrector whose existence depends on connectivity of a controlled flow trajectory. When the flow field is mean zero and incompressible, we only need such connectivity from a point to the region where $|V| < 1$ and coercivity holds locally. We verify it for two-dimensional incompressible flows. The effective Hamiltonians of both the viscous and inviscid G-equations are coercive. For cellular flows, we demonstrate both asymptotically and numerically that viscosity reduces $\overline{H}$ or travelling front speed. In one space dimensional compressible flows, trapping may occur

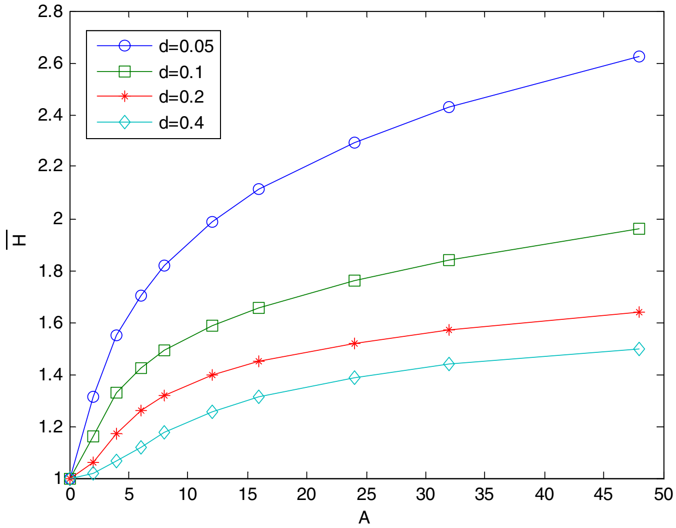

Figure 2. Sublinear growth of $\overline{H}$ in A for a range of d values.

and $\overline{H}$ vanishes (non-coercivity persists). Necessary and sufficient conditions of trapping are found for both the viscous and inviscid G-equations. If the flow field has mean zero and large enough amplitude, wave trapping occurs in the inviscid G-equation but not in the viscous G-equation. Viscosity restores coercivity in $\overline{H}$.

In future work, we shall continue to study $\overline{H}$ of the slightly viscous G-equation in concrete flows and compare with enhanced propagation and diffusion in quadratic HJ or linear transport equations [1, 29, 30]. We also plan to address similar issues for flows in three space dimensions.

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