DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **11**, Number **2**, March **2009**

doi:10.3934/dcdsb.2009.11.421

pp. **421–442**

KPP FRONTS IN A ONE-DIMENSIONAL RANDOM DRIFT

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(Communicated by Doron Levy)

Abstract. We establish the variational principle of Kolmogorov-Petrovsky-Piskunov (KPP) front speeds in a one dimensional random drift which is a mean zero stationary ergodic process with mixing property and local Lipschitz continuity. To prove the variational principle, we use the path integral representation of solutions, hitting time and large deviation estimates of the associated stochastic flows. The variational principle allows us to derive upper and lower bounds of the front speeds which decay according to a power law in the limit of large root mean square amplitude of the drift. This scaling law is different from that of the effective diffusion (homogenization) approximation which is valid for front speeds in incompressible periodic advection.

1. **Introduction.** Reaction-diffusion front propagation in random media arises in turbulent combustion ([5, 14, 19, 20, 26, 27] and references), interacting particle systems ([15, 7] and references) and population biology ([23] and references). A fundamental issue is to characterize, bound and compute the large time front speed, an upscaled quantity that depends on statistics of the random medium in a highly nonlinear manner. In [16], the authors established a variational principle and asymptotic growth laws of KPP front speeds in temporally random shear flows in multiple dimensions. See also [14, 26, 17] for related results in spatially random shear flows. In either case, the randomness in the flow appears in time or in a direction orthogonal to that of front propagation. The front speeds are enhanced due to the spatial inhomogeneity and geometric structure of the flow.

In this paper, we study a case where randomness is in the direction of front propagation, rather than orthogonal to it. We consider solutions to the KPP reaction-advection-diffusion equation:

$$u_t = \frac{1}{2}u_{xx} + b(x)u_x + f(u), \quad t > 0, \;\; x \in \mathbb{R}. \tag{1}$$

Here $f(u)$ is KPP type, $f \in C^1([0,1])$, $f(0) = f(1)$, $0 \le f(u) \le f'(0)u$ for all $u \in (0,1)$, e.g. $f(u) = u(1-u)$. The initial data $u_0(x) \in [0,1]$ is compactly supported. For the random drift $b(x,\hat{\omega}) : \mathbb{R} \times \hat{\Omega} \to \mathbb{R}$ we assume: (1) that b is a stationary random process on $\mathbb{R}$ defined over the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, Q)$ with

2000 *Mathematics Subject Classification.* Primary: 35K57; Secondary: 35K15.
Key words and phrases. fronts, random drift, large deviations, KPP.

zero mean, $E_Q[b] = 0$; (2) that $b(\cdot, \hat{\omega})$ is almost surely locally Lipschitz continuous and that translation with respect to x generates an ergodic transformation of the space $\hat{\Omega}$; (3) that the process $b(x, \hat{\omega})$ satisfies

$$E_Q \left[\sup_{x \in [-2,2]} |b(x, \hat{\omega})| \right] < \infty. \tag{2}$$

However, we do not assume that the process b is globally bounded or globally Lipschitz continuous.

We shall describe the asymptotic spreading of the solution as $t \to \infty$, and show that the solution develops into fronts propagating to the left and to the right with deterministic constant asymptotic speeds $c^*_\pm$. The speeds obey a variational principle. The fronts separate the region where $u \approx 1$ from the region where $u \approx 0$, and their propagation may be interpreted as the spreading of a chemical reaction. Based on the variational principle, we derive bounds on $c^*_\pm$ implying that the speeds decrease towards zero in the limit of large root mean square amplitude of the drift b. Fronts are slowed down to nearly motionless by the presence of the large drift which plays a role of trapping. Moreover, the front speeds behave quite differently from what is suggested by a homogenization (diffusion) approximation of the linear part of the right hand side of (1). This approximation replaces the advection-diffusion operator $\frac{1}{2} u_{xx} + b(x) u_x$ by an effective diffusion operator $\bar{\kappa} u_{xx}$, with $\bar{\kappa}$ a positive constant, and front speeds are $O(\sqrt{\bar{\kappa}})$. In periodic incompressible flows, the homogenization approximation gives the correct scaling behavior of front speeds for large drift [21]. However, we shall show that the approximation is not correct in the random setting here, even when front speeds and $\bar{\kappa}$ are both finite.

Our analysis of $u(x, t)$ involves large deviations estimates for the associated diffusion process $X^x(t)$ in the random environment. From assumption (2) and the assumption of stationarity and ergodicity, it follows that almost surely with respect to Q there is a constant $k = k(\hat{\omega})$ such that $|b(x, \hat{\omega})| \le k(1 + |x|)$ for all $x \in \mathbb{R}$. Therefore, for each $\hat{\omega} \in \hat{\Omega}$ fixed, we can define $X^x(t)$ to be the strong solution to the Itô equation:

$$X^x(t) = x + \int_0^t b(X^x(s))\, ds + W(t) \tag{3}$$

where $W(t) = W(t, \omega)$ is a one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with $W(0) = 0$, P-a.s.

The idea of analyzing the front speed via large deviation estimates for $X^x(t)$ stems from the work of Freidlin and Gärtner (see [10, 12], and Chapter VII of [9]) who studied the equation (1) under the assumption that b is uniformly bounded or when randomness appears in the nonnegative reaction f. In [9], pp. 524-525, the author remarks that this approach might be used to study fronts in one-dimensional, uniformly bounded random drift; however, we are not aware that this has been carried out for the present case. Moreover, we obtain nearly optimal asymptotic estimates of front speeds in the large drift limit. Solutions to (1) in multiple dimensions with uniformly bounded coefficients were also studied more recently by Lions and Souganidis [13] using nonlinear homogenization techniques. The hyperbolic scaling of equation (1) in the homogenization approach reveals the asymptotic behavior of the fronts when the support of the initial data is large with respect to the spatial correlation length of the drift ($\sim O(\epsilon)$). Here we fix the initial data for (1), and we consider unbounded coefficients arising naturally in stochastic processes (e.g. a Gaussian process) and derive estimates on the speed of the propagating fronts. The analysis involves techniques used recently by Comets, Gantert, and Zeitouni [6] and by Taleb [25] to describe large deviations for discrete random walks in a random environment and for continuous diffusions in random environment.

Our main result on the asymptotic spreading of the solution requires two more mild assumptions. We assume that for some $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\limsup_{z\to\infty} Q\left(\int_0^z b(s,\hat{\omega})\,ds \geq \alpha_1\right) < 1 \tag{4}$$

and

$$\limsup_{z\to\infty} Q\left(\int_{-z}^0 b(s,\hat{\omega})\,ds \leq \alpha_2\right) < 1. \tag{5}$$

This is not a very restrictive assumption. For example, if $b(x,\hat{\omega})$ is square integrable and sufficiently mixing with respect to shifts in x, then b satisfies an invariance principle [3]

$$\frac{1}{\sigma\sqrt{z}}\int_0^z b(x,\hat{\omega})\,ds \to N(0,1), \quad Q-a.s., \tag{6}$$

which implies (4). In particular, all the above assumptions on b hold for a mean zero locally Lipschitz continuous Gaussian process with sufficient decay of correlation functions, while (2) follows from the Borel inequality, [1]. The first main result is:

Theorem 1.1. *Suppose that (4), (5), and the other aforementioned assumptions hold. Then there are deterministic constants $c_-^* < 0$ and $c_+^* > 0$ such that for any closed set $F \subset (-\infty, c_-^*) \cup (c_+^*, +\infty)$*

$$\lim_{t\to\infty}\sup_{c\in F} u(ct,t,\hat{\omega}) = 0$$

for almost every $\hat{\omega} \in \hat{\Omega}$. Also, for any compact set $K \subset (c_-^*, c_+^*)$,

$$\lim_{t\to\infty}\inf_{c\in K} u(ct,t,\hat{\omega}) = 1$$

for almost every $\hat{\omega} \in \hat{\Omega}$.

Our next result describes the effect of scaling the drift $b \mapsto \delta b$ where $\delta \in [0,\infty)$ is a scaling parameter. We show that the corresponding front speed $c_+^*(\delta)$ decreases to zero as the flow amplitude increases:

Theorem 1.2. *The front speed $c_+^*(\delta)$ satisfies the lower bound*

$$c_+^*(\delta) \geq \frac{1}{C}\min\left(1, \frac{f'(0)}{1+\delta M}\right).$$

where C is the constant from Lemma 2.1 and $M = E_Q\left[\sup_{x\in[-2,2]}|b(x,\hat{\omega})|\right]$. Moreover, for any $p \in (0,1)$ there is a constant $C = C(p,\hat{\omega})$ such that

$$c_+^*(\delta) \leq C\delta^{-p} \tag{7}$$

for all $\delta > 0$. Therefore $\limsup_{\delta\to\infty} c_+^*(\delta) = 0$ holds with probability one. Similar statements hold for $|c_-^*(\delta)|$.

When $b \equiv 0$, the solutions to the initial value problem develop fronts that propagate with speed equal to $c^* = 2\sqrt{\kappa f'(0)}$, where κ is the diffusion constant ($\kappa = 1/2$ in (1)). This suggests that for nonzero b (stationary and ergodic) one might estimate the front speed by $c^* \approx c^{\bar{\kappa}} = 2\sqrt{\bar{\kappa} f'(0)}$ where $\bar{\kappa} = \lim_{t\to\infty} E[|X(t)|^2]/t$ is the effective diffusivity corresponding to the random medium. For periodic incompressible two-dimensional velocity fields, Ryzhik and Zlatoš [21] have shown that the ratio $c^*/c^{\bar\kappa}$ is bounded away from zero and infinity by constants independent of the flow. In the case we consider in this paper, however, this result does not hold. In some cases, $c^{\bar\kappa} = 0$ while $c^* > 0$. In other cases $c^{\bar\kappa} > 0$, but $c^{\bar\kappa}$ and c^* scale quite differently with respect to δ.

Because the domain is one-dimensional, the field $b(x, \hat\omega)$ is a gradient field, and from the point of view of the diffusion process $X^x(t)$, the random medium creates traps along the x axis. It has been shown that for some fields b which are unbounded in x (see [4, 22, 24]) this trapping can dramatically slow down the diffusion so that $X^x(t)$ behaves asymptotically like $(\log t)^2$, rather than $\sqrt{t}$. Hence, $\bar\kappa = 0$ in this case. Nevertheless, Theorem 1.2 shows that the asymptotic front speed is nonzero; the random medium cannot trap the fronts, despite the anomalously slow diffusive behavior.

When b is uniformly bounded, one can show [18, 22] that the process $X^x(t)$ is diffusive with effective diffusivity

$$\bar\kappa = \frac{1}{E_Q[e^{-b}]E_Q[e^b]} > 0, \tag{8}$$

almost surely with respect to Q. Suppose that the distribution of b is sign-symmetric (i.e. $b \overset{L}{=} -b$). Then $E_Q[e^{-b}] = E_Q[e^b]$, so that the effective diffusivity is

$$\bar\kappa = \frac{1}{(E_Q[e^b])^2}.$$

In this case, the effective diffusivity (and $c^{\bar\kappa}$) will decrease exponentially fast as the scaling parameter δ is increased. However, the lower bound in Theorem 1.2 shows that the corresponding front speed can decrease no faster that $O(\delta^{-1})$ as δ increases. The reason for this difference is that the front speed is determined by large deviations of the diffusion process $X^x(t)$, which may not be accurately predicted by the asymptotic behavior of the variance of the process.

The paper is organized as follows. In Sections 2 and 3 we state and prove large deviations estimates for the diffusion process $X^x(t)$. These estimates may be converted into estimates on the solution $u(x,t)$ through the Feynman-Kac formula (2), and in Section 4 we prove Theorem 1.1 using this approach. In Section 5 we prove Theorem 1.2 using a representation of c^* in terms of the rate function characterizing large deviations of $X^x(t)$.

2. **Large deviations estimates.**

The proof of Theorem 1.1 is based on large deviations estimates for the associated diffusion process $X^x(t)$. These estimates are stated in Theorem 2.4. To derive these estimates, we first derive estimates for the hitting time T_r^s, which is the first time the process hits the point $x = r$ (from the right) starting from $x = s \geq r$:

$$T_r^s = \inf\{t > 0 \mid \ X^s(t) \leq r\}, \tag{1}$$

The hitting time estimates are stated in Theorem 2.3. Theorems 2.4 and 2.3 are similar to the analogous estimates in the work of Comets, Gantert, Zeitouni [6] and Taleb [25].

First, we define some auxiliary quantities that will be used in the proofs. For $\lambda \in \mathbb{R}$, let $q(r, s, \lambda)$ be the moment generating function

$$q(r,s,\lambda) = E\left[e^{\lambda T_r^s}\mathbb{I}_{T_r^s<\infty}\right], \tag{2}$$

which may be infinite for $\lambda > 0$.

Proposition 1. *Suppose that $\lambda \in \mathbb{R}$ is such that*

$$E_Q\left[|\log E[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}]|\right] < \infty. \tag{3}$$

Let $c < v$. Then, almost surely with respect to Q, the limit

$$\mu(\lambda) \triangleq \lim_{t\to\infty} \frac{1}{(v-c)t} \log q(ct, vt, \lambda) = E_Q\left[\log E[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}]\right] \tag{4}$$

holds. The convergence is uniform with respect to v and c, as v and c vary in a set that is bounded and $(v-c)$ is bounded away from zero. Moreover, $\mu(\lambda)$ is independent of v and c.

The next lemma shows that the conditions of Proposition 1 are satisfied under our assumptions on $b(x,\hat{\omega})$:

Lemma 2.1. *There is a universal constant $C > 0$ such that for all $\lambda < 0$*

$$E_Q\left[|\log E[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}]|\right] \le C(1 + |\lambda| + E_Q\left[\sup_{x\in[-2,2]} |b(x,\hat{\omega})|\right]) < \infty \tag{5}$$

Consequently, if the process $b(x,\hat{\omega})$ satisfies assumption (2), then (3) and the conclusions of Proposition 1 hold for all $\lambda \le 0$.

Now define the constant

$$\lambda_c = \sup\{\lambda \in \mathbb{R} \mid \mu(\lambda) < \infty\}. \tag{6}$$

Assumption (2) and Lemma 2.1 imply that $\lambda_c \ge 0$. For Brownian motion ($b(x) \equiv 0$), T_0^1 has a heavy tail so that $\lambda_c = 0$.

The following lemma summarizes the properties of the function $\mu(\lambda)$.

Lemma 2.2. *Under the assumption (2), the function $\mu(\lambda)$ satisfies the following properties:*

(i) $\mu(0) = 0$.
(ii) $\mu(\lambda) < 0$ *for* $\lambda < 0$.
(iii) $\mu(\lambda) \to -\infty$ *as* $\lambda \to -\infty$.
(iv) $\mu(\lambda) = +\infty$ *for* $\lambda > \lambda_c$.
(v) $\mu(\lambda)$ *is convex for* $\lambda \in (-\infty, \lambda_c)$.
(vi) *For* $\lambda < \lambda_c$, $\mu(\lambda)$ *is differentiable with*

$$\mu'(\lambda) = E_Q\left[\frac{E[T_0^1 e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}]}{E[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}]}\right] > 0 \tag{7}$$

In particular, $\mu'(0) = a_0 \triangleq E_Q\left[E[T_0^1\mathbb{I}_{T_0^1<\infty}]\right] \in (0,\infty]$.

(vii) $\mu'(\lambda)$ *is monotone increasing (strictly) for* $\lambda \in (-\infty, \lambda_c)$.

Chebychev's inequality implies that for any $\lambda < 0$, $0 < \alpha$, and $c < v$

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log P(\frac{T_{ct}^{vt}}{t} < \alpha) &= \limsup_{t\to\infty} \frac{1}{t} \log P(e^{\lambda T_{ct}^{vt}} > e^{\lambda\alpha t}) \\
&\le -\lambda\alpha + \limsup_{t\to\infty} \frac{1}{t} \log q(ct, vt, \lambda) \\
&= -\lambda\alpha + (v-c)\mu(\lambda).
\end{aligned} \tag{8}$$

Since the left hand side is independent of λ, this means that

$$\limsup_{t\to\infty} \frac{1}{t} \log P(\frac{T_{ct}^{vt}}{t} < \alpha) \leq -\sup_{\lambda<0} (\lambda\alpha - (v-c)\mu(\lambda)) . \tag{9}$$

The function on the right hand side of (9) suggests defining the function $I^+(a)$:

$$I^+(a) = \sup_{\lambda\leq\lambda_c} (a\lambda - \mu(\lambda)) , \tag{10}$$

which is the Legendre transform of the function $\mu(\lambda)$. If $a \leq a_0 = \mu'(0)$ then the supremum is achieved at some $\lambda \leq 0$. Therefore, if $\frac{\alpha}{(v-c)} \leq a_0$, the bound (9) can be written

$$\limsup_{t\to\infty} \frac{1}{t} \log P(\frac{T_{ct}^{vt}}{t} < \alpha) \leq -(v-c)I^+(\frac{\alpha}{v-c}). \tag{11}$$

The properties of μ imply that $I^+(a)$ satisfies

(i) $I^+(a) > 0$ for $a \in (0, a_0)$, where $a_0 \triangleq \mu'(0)$.
(ii) $I^+(a)$ is convex and decreasing in a for $a \in (0, a_0)$.
(iii) $\lim_{a\to 0^+} I^+(a) = +\infty$, and $\lim_{a\to(a_0)^-} I^+(a) = 0$.
(iv) If $a_0 < \infty$, then $I^+(a_0) = 0$, and $I^+(a) \geq 0$ for $a \in (a_0, \infty)$.
(v) If $\lambda_c = 0$ and $a_0 < \infty$, then $I^+(a) = 0$ for $a \in (a_0, \infty)$.

Now we state the main results of this section:

Theorem 2.3. *Suppose $b(x, \hat\omega)$ satisfies (2). Almost surely with respect to Q, the following estimates hold. For any $v, c \in \mathbb{R}$ with $c < v$ and any closed set $G \subset (0, (v-c)a_0)$*

$$\limsup_{t\to\infty} \frac{1}{t} \log P\left(\frac{T_{ct}^{vt}}{t} \in G\right) \leq -(v-c) \inf_{a\in G} I^+(\frac{a}{v-c}), \tag{12}$$

and for any open set $F \subset (0, (v-c)a_0)$,

$$\liminf_{t\to\infty} \frac{1}{t} \log P\left(\frac{T_{ct}^{vt}}{t} \in F\right) \geq -(v-c) \inf_{a\in F} I^+(\frac{a}{v-c}). \tag{13}$$

Remark 1. If assumption (2) holds for $b(x, \hat\omega)$, then it also holds for the reflected process $b(-x, \hat\omega)$. Therefore, analogous bounds apply to the first hitting times when the initial point is to the left of the terminal point:

$$T_r^s = \inf\{t > 0 \mid \ X^s(t) \geq r\}, \tag{14}$$

for $s \leq r$. In this way we obtain two functions I^+ and I^- which may not be equal in general. Nevertheless, I^- satisfies

(i) $I^-(a) = \sup_{\lambda\leq\lambda_c^-} (\lambda a - \mu^-(\lambda))$, $\mu^-(\lambda) \triangleq E_Q \left[\log E[e^{\lambda T_0^{-1}} \mathbb{I}_{T_0^1<\infty}]\right]$
(ii) $I^-(a) > 0$ for $a \in (0, a_0^-)$, where $a_0^- \triangleq (\mu^-)'(0)$.
(iii) $I^-(a)$ is convex and decreasing in a for $a \in (0, a_0^-)$.
(iv) $\lim_{a\to 0^+} I^-(a) = +\infty$, and $\lim_{a\to(a_0^-)^-} I^-(a) = 0$.
(v) If $a_0^- < \infty$, then $I(a_0^-) = 0$, and $I^-(a) \geq 0$ for $a \in (a_0^-, \infty)$.
(vi) If $\lambda_c^- = 0$ and $a_0^- < \infty$, then $I^-(a) = 0$ for $a \in (a_0^-, \infty)$.

Theorem 2.4. *Suppose $b(x, \hat\omega)$ satisfies (2). Almost surely with respect to Q, the following estimates hold. Let $v \in \mathbb{R}$, $\kappa \in (0, 1]$. For any closed set $G \subset [(a_0^+)^{-1}, \infty)$*

$$\limsup_{t\to\infty} \frac{1}{\kappa t} \log P\left(\frac{vt - X^{vt}(\kappa t)}{\kappa t} \in G\right) \leq -\inf_{c\in G} cI^+(\frac{1}{c}), \tag{15}$$

and for any open set $F \subset [(a_0^+)^{-1}, \infty)$,

$$\liminf_{t\to\infty} \frac{1}{\kappa t} \log P\left(\frac{vt - X^{vt}(\kappa t)}{\kappa t} \in F\right) \geq -\inf_{c\in F} cI^+(\frac{1}{c}). \tag{16}$$

For any closed set $G \subset (-\infty, -(a_0^-)^{-1}]$

$$\limsup_{t\to\infty} \frac{1}{\kappa t} \log P\left(\frac{vt - X^{vt}(\kappa t)}{\kappa t} \in G\right) \leq -\inf_{c\in G} |c| I^-(\frac{1}{|c|}), \tag{17}$$

and for any open set $F \subset (-\infty, -(a_0^-)^{-1}]$,

$$\liminf_{t\to\infty} \frac{1}{\kappa t} \log P\left(\frac{vt - X^{vt}(\kappa t)}{\kappa t} \in F\right) \geq -\inf_{c\in F} |c| I^-(\frac{1}{|c|}). \tag{18}$$

We define $(a_0^{\pm})^{-1} = 0$ *if* $a_0^{\pm} = \infty$.

3. **Proof of large deviations estimates.** In this section we prove the large deviations estimates in the preceding section.

Proof of Proposition 1. Let $r < s < t$. By the Markov property of X,

$$E\left[e^{\lambda T_r^t} \mathbb{I}_{T_r^t<\infty}\right] = E\left[e^{\lambda T_s^t} \mathbb{I}_{T_s^t<\infty}\right] E\left[e^{\lambda T_r^s} \mathbb{I}_{T_r^s<\infty}\right] \tag{1}$$

so that $\log q(r,t,\lambda)$ is an additive process:

$$\log q(r,t,\lambda) = \log q(r,s,\lambda) + \log q(s,t,\lambda). \tag{2}$$

Suppose that $c > 0$ is a real number and $0 \leq k < n$ are integers. Then

$$\begin{aligned} E_Q\left[\log q(ck, cn, \lambda)\right] &= E_Q\left[\sum_{j=1}^{n-k} \log q(cj, c(j+1), \lambda)\right] \\ &= (n-k) E_Q\left[\log q(0,c,\lambda)\right], \end{aligned} \tag{3}$$

since $E_Q \log E[q(cj, c(j+1), \lambda)] = E_Q \log E[q(0,c,\lambda)]$ by the stationarity of the process b. Therefore, the ergodic theorem [2] implies that the limit

$$\begin{aligned} \mu(\lambda) &\triangleq \lim_{n\to\infty} \frac{1}{cn} \log q(0, cn, \lambda) \\ &= \frac{1}{c} E_Q\left[\log E[e^{\lambda T_0^c} \mathbb{I}_{T_0^c<\infty}]\right] = E_Q\left[\log E[e^{\lambda T_0^1} \mathbb{I}_{T_0^1<\infty}]\right] \end{aligned} \tag{4}$$

holds Q-a.s., provided that $E_Q\left[\log E[e^{\lambda T_0^1} \mathbb{I}_{T_0^1<\infty}]\right]$ is finite.

Now extend the convergence to continuous time. Suppose that $t \in [n, n+1]$. By the additive property of $\log q$,

$$\begin{aligned} \log q(0, ct, \lambda) &= \log q(0, cn, \lambda) + \log q(cn, ct, \lambda) \\ &= \log q(0, c(n+1), \lambda) - \log q(ct, c(n+1), \lambda). \end{aligned} \tag{5}$$

If $\lambda < 0$, then $\log q(cn, ct, \lambda) < 0$ and $\log q(ct, c(n+1), \lambda) < 0$, so that (5) implies

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{ct} \log q(0, ct, \lambda) &\leq \limsup_{t\to\infty} \frac{1}{ct} \log q(0, cn, \lambda) \\ &= \limsup_{n\to\infty} \frac{n}{t} \frac{1}{cn} \log q(0, cn, \lambda) = \mu(\lambda), \end{aligned}$$

and (5) implies

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{ct}\log q(0,ct,\lambda) &\geq \liminf_{t\to\infty}\frac{1}{ct}\log q(0,c(n+1),\lambda)\\ &= \liminf_{n\to\infty}\frac{n+1}{t}\frac{1}{c(n+1)}\log q(0,cn,\lambda)=\mu(\lambda).\end{aligned}$$

A similar argument applies to the case $\lambda > 0$, since $\log q(cn, ct, \lambda)$ and $\log q(ct, c(n+1), \lambda)$ are both positive in this case.

Suppose $c \in [c_0, c_1] \subset (0,\infty)$ and $\epsilon > 0$. Then there is a $t_\epsilon > 0$ such that for $t > t_\epsilon$,

$$\left|\frac{1}{c_0 t}\log q(0,c_0 t,\lambda)-\mu(\lambda)\right| \leq \epsilon \tag{6}$$

Let $\hat{t} = t_\epsilon c/c_0$. Then $\hat{t} \geq t_\epsilon$ so that

$$\epsilon \geq \left|\frac{1}{c_0\hat{t}}\log q(0,c_0\hat{t},\lambda)-\mu(\lambda)\right| = \left|\frac{1}{ct}\log q(0,ct,\lambda)-\mu(\lambda)\right|$$

This proves that the convergence to $\mu(\lambda)$ is uniform over $c \in [c_0, c_1]$.

Arguments similar to the above show that for $c < 0$,

$$\lim_{t\to\infty}\frac{1}{|c|t}\log q(ct,0,\lambda)=\mu(\lambda) \tag{7}$$

almost surely with respect to Q and uniformly with respect to $c \subset [c_0, c_1] \subset (-\infty, 0)$.

If $v > c \geq 0$, the additivity of $q(r,t,\lambda)$ implies that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{(v-c)t}\log q(ct,vt,\lambda)\\ &\quad=\lim_{t\to\infty}\frac{1}{(v-c)t}\left(\log q(0,vt,\lambda)-\log q(0,ct,\lambda)\right)\\ &\quad=\frac{v}{v-c}\mu(\lambda)-\frac{c}{v-c}\mu(\lambda)=\mu(\lambda)\end{aligned}\tag{8}$$

For $c < 0 < v$,

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{(v-c)t}\log q(ct,vt,\lambda)\\ &\quad=\lim_{t\to\infty}\frac{1}{(v-c)t}\left(\log q(ct,0,\lambda)+\log q(0,vt,\lambda)\right)\\ &\quad=\frac{|c|}{v-c}\mu(\lambda)+\frac{v}{v-c}\mu(\lambda)=\mu(\lambda)\end{aligned}\tag{9}$$

A similar argument holds for $c < v \leq 0$. This concludes the proof of Proposition 1. $\square$

Proof of Lemma 2.1. Since we are assuming $\lambda \leq 0$, we will need an upper bound on T_0^1, which may not be integrable. For convenience in our notation, we prove the result for T_1^0 (the hitting time to $x = 1$ starting from $x = 0$) rather than T_0^1.

Fix $L > 0$, and define the random variable

$$M_L = 1 + \sup_{x\in[-2L,2L]} |b(x,\hat{\omega})|.$$

Let $x_0 = 0$, so that the process $X(s)$ satisfies

$$X(s) = \int_0^s b(X(\tau))\,d\tau + W(s).$$

Let $t_L = L/M_L > 0$. Suppose that

$$W(s) \geq -L, \quad \forall s \in [0, t_L] \tag{10}$$

and that

$$\sup_{s \in [0,t_L]} W(s) \geq 2L. \tag{11}$$

We claim that

$$T_L^0 \leq t_L \tag{12}$$

under assumptions (10) and (11).

To establish the claim, suppose that $T_L^0 > t_L$, and suppose that the process exits the left end of $[-2L, 2L]$ before time t_L. Using the definition of M and $X(s)$ and assumption (10) we observe that at the first hitting time $T_{-2L}^0 \leq t_L < T_L^0$

$$\begin{aligned} X(T_{-2L}^0) &\geq -T_{-2L}^0(M_L - 1) - L \\ &> -T_{-2L}^0 M_L - L \\ &= -T_{-2L}^0 \frac{L}{t_L} - L \geq -2L. \end{aligned}$$

However, the inequality $X(T_{-2L}^0) > -2L$ is a contradiction. Therefore, $X(s) \geq -2L$ for $s \in [0, t_L]$. Then we see that if $T_{-2L}^0 > t_L$, $|b(X(s))| \leq M_L$ for $s \in [0, t_L]$, which implies that

$$\begin{aligned} \sup_{s \in [0,t_L]} X(s) &\geq \sup_{s \in [0,t_L]} W(s) + \inf_{s \in [0,t_L]} \int_0^s b(X(\tau))\, d\tau \\ &\geq \sup_{s \in [0,t_L]} W(s) - M_L t_L \\ &= \sup_{s \in [0,t_L]} W(s) - L \geq 2L - L = L. \end{aligned}$$

Therefore, $T_L^0 \leq t_L$, if $T_{-2L}^0 > t_L$. Since we always have $T_L^0 \leq T_{-2L}^0$, this establishes the claim that $T_L^0 \leq t_L$.

Now if we define A_L to be the set of paths satisfying assumptions (10) and (11)

$$A_L(\hat{\omega}) = \left\{ \omega \mid \ W(s) > -L, \ \forall s \in [0, t_L]; \quad \sup_{s; \in [0,t_L]} W(s) \geq 2L \right\}, \tag{13}$$

then for $\lambda < 0$,

$$\begin{aligned} E\left[e^{\lambda T_L^0} \mathbb{I}_{T_L^0 < \infty}\right] &\geq E\left[e^{\lambda T_L^0} \chi_{A_L} \mathbb{I}_{T_L^0 < \infty}\right] \\ &\geq e^{\lambda t_L} P\left(A_L\right). \end{aligned} \tag{14}$$

The term $P(A_L)$ may be bounded by using the reflection principle, as follows. Using h_r to denote the first hitting time of the Wiener process to the level $r \in \mathbb{R}$, we have:

$$\begin{aligned} P\left(A_L\right) &= P\left(h_{2L} \leq t_L; \ \ h_{-L} > t_L\right) \\ &= P\left(h_{2L} \leq t_L\right) - P\left(h_{2L} \leq t_L; \ \ h_{-L} < t_L\right) \\ &\geq 2P(W(t_L) \geq 2L) - P\left(W(t_L) \leq -3L\right). \end{aligned}$$

Therefore,

$$\begin{aligned} P\left(A_L\right) &> 2\left[P(W(t_L) \geq 2L) - P(W(t_L) \leq -3L)\right] \\ &= \frac{2}{\sqrt{2\pi}} \int_{2\sqrt{LM_L}}^{3\sqrt{LM_L}} e^{-\frac{y^2}{2}}\, dy \geq \frac{2\sqrt{LM_L}}{\sqrt{2\pi}} e^{-\frac{9LM_L}{2}}. \end{aligned}$$

Now returning to (14) and using the fact that $M_L \geq 1$, $\lambda < 0$, we see that

$$E\left[e^{\lambda T_L^0}\mathbb{I}_{T_L^0<\infty}\right] \geq e^{\lambda L/M_L}\frac{2\sqrt{LM_L}}{\sqrt{2\pi}}e^{-\frac{9LM_L}{2}} \geq e^{\lambda L}\frac{2\sqrt{L}}{\sqrt{2\pi}}e^{-\frac{9LM_L}{2}}$$

This implies that

$$E_Q\left[\log E\left[e^{\lambda T_L^0}\mathbb{I}_{T_L^0<\infty}\right]\right] \quad \geq \quad C(\lambda - 1 - E_Q\left[M_L\right])$$

for some constant $C > 0$. The lemma now follows by using $L = 1$ and the stationarity of $b(x, \hat{\omega})$. ☐

Proof of Lemma 2.2. Properties (i) - (iv) are clear. Property (iv) follows from Hölder's inequality which implies that

$$E[e^{\frac{1}{2}(\lambda_1+\lambda_2)T_0^1}\mathbb{I}_{T_0^1<\infty}] \leq E[e^{\lambda_1 T_0^1}\mathbb{I}_{T_0^1<\infty}]^{1/2}E[e^{\lambda_2 T_0^1}\mathbb{I}_{T_0^1<\infty}]^{1/2}. \tag{15}$$

Now we prove (vi). Let $\lambda < \lambda_c$. For $T_1 < \infty$, $|h| \in (0, |\lambda_c - \lambda|/3)$ the convexity of e^x implies that

$$|\frac{e^{(\lambda+h)T_0^1} - e^{(\lambda)T_0^1}}{h}| \leq T_0^1 e^{(\lambda+|h|)T_0^1} \tag{16}$$

There is a constant C depending only on λ_c and λ such that $0 \leq T_1 \leq Ce^{T_1|\lambda_c-\lambda|/3}$. Therefore,

$$0 \leq T_1 e^{(\lambda+h)T_0^1} \leq Ce^{(\lambda+|h|+|\lambda_c-\lambda|/3)T_0^1} \leq Ce^{(\lambda_c-|h|)T_1} \tag{17}$$

Because the term $e^{(\lambda_c-|h|)T_1}\mathbb{I}_{T_0^1<\infty}$ is integrable, by definition of λ_c, the dominated convergence theorem implies that

$$\frac{d}{d\lambda}E\left[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right] = E\left[T_0^1 e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right],$$

for all $\lambda < \lambda_c$. Now the chain rule implies that

$$\frac{d}{d\lambda}\log E\left[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right] = \frac{E\left[T_0^1 e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right]}{E\left[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right]} \tag{18}$$

almost surely with respect to Q.

The function $\log q(0, 1, \lambda)$ is convex with respect to λ, Q-a.s. Therefore,

$$\frac{1}{h}\left(\log E[e^{(\lambda+h)T_1}\mathbb{I}_{T_0^1<\infty}] - \log E[e^{\lambda T_1}\mathbb{I}_{T_0^1<\infty}]\right) \tag{19}$$

is a monotonically increasing sequence of functions (of $\hat{\omega}$) if $h < 0$ increases to zero, and it is a monotonically decreasing sequence if $h > 0$ decreases to zero. The sequence is always non-negative. Therefore, the fact that expression in (19) is integrable with respect to Q implies that the expression in (18) is also integrable with respect to Q. Then, the monotone convergence theorem and dominated convergence theorem can be applied to show that for $\lambda < \lambda_c$

$$\mu'(\lambda) = \frac{d}{d\lambda}E_Q\left[\log E\left[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right]\right] = E_Q\left[\frac{E\left[T_0^1 e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right]}{E\left[e^{\lambda T_0^1}\mathbb{I}_{T_0^1<\infty}\right]}\right].$$

☐

The following lemma is an immediate consequence of the above analysis and will be used in the proof of Theorem 2.3:

Lemma 3.1. *For $c < v$, let $\mu_t(\lambda)$ be defined by*

$$\mu_t(\lambda) = \frac{1}{(v-c)t} \log q(ct, vt, \lambda) \tag{20}$$

Then almost surely with respect to Q, $\mu_t(\lambda)$ is differentiable for all $\lambda < \lambda_c$ and

$$\lim_{t\to\infty} \frac{d}{d\lambda}\mu_t(\lambda) = \mu'(\lambda) = E_Q \left[\frac{E\left[T_0^1 e^{\lambda T_0^1} \mathbb{I}_{T_0^1<\infty}\right]}{E\left[e^{\lambda T_0^1} \mathbb{I}_{T_0^1<\infty}\right]} \right] \tag{21}$$

Proof. The fact that $\mu_t(\lambda)$ is differentiable in λ follows from the analysis in the proof of Lemma 2.2. The function $\mu_t(\lambda)$ is convex in λ for all t. Moreover, $\mu(\lambda)$ is convex and differentiable. Since $\mu_t(\lambda) \to \mu(\lambda)$, Q-a.s., it follows that $\mu_t'(\lambda) \to \mu'(\lambda)$. □

The proofs of Theorems 2.3 and 2.4 follow the ideas in [6] and [25]. Here we just sketch the arguments.

Proof of Theorem 2.3. From (9) we already know that for any $\alpha > 0$,

$$\limsup_{t\to\infty} \frac{1}{t} \log P(\frac{T_{ct}^{vt}}{t} < \alpha) \quad\leq\quad -\sup_{\lambda\leq 0} (\lambda\alpha - (v-c)\mu(\lambda))$$

If $\alpha/(v-c) < a_0$, then the supremum on the right is obtained at a point $\lambda < 0$, so that this is equivalent to

$$\limsup_{t\to\infty} \frac{1}{t} \log P(\frac{T_{ct}^{vt}}{t} < \alpha) \quad\leq\quad -(v-c)I^+(\frac{\alpha}{v-c})$$

This bound and the fact that $I^+(\cdot)$ is non-increasing on $(0, a_0)$ proves (12) for $G \subset (0, (v-c)a_0]$.

The proof of the lower bound (13) follows the change of measure method, as in [8, 6, 25, 28]. Let $u \in (0, (v-c)a_0)$ and $\delta > 0$. Let $B_\delta(u)$ denote the δ-ball centered at u. Since $u < (v-c)a_0$, Lemma 2.2 implies that there is a $\lambda_u < \lambda_c$ such that

$$\mu'(\lambda_u) = E_Q \left[\frac{E(T_0^1 e^{\lambda_u T_0^1} \mathbb{I}_{T_0^1<\infty})}{E(e^{\lambda_u T_0^1} \mathbb{I}_{T_0^1<\infty})} \right] = \frac{u}{(v-c)}. \tag{22}$$

At this point λ_u, the supremum in the Legendre transform is achieved:

$$I^+(\frac{u}{v-c}) = \sup_{\lambda\leq 0} \left(\lambda \frac{u}{v-c} - \mu(\lambda) \right) = \lambda_u \frac{u}{v-c} - \mu(\lambda_u) \tag{23}$$

Then define the measure $P^{u,t}$ by

$$\frac{dP^{u,t}}{dP} = \frac{1}{S_{u,t}} e^{\lambda_u T_{ct}^{vt}} \mathbb{I}_{T_{ct}^{vt}<\infty}, \qquad S_{u,t} = E\left[e^{\lambda_u T_{ct}^{vt}} \mathbb{I}_{T_{ct}^{vt}<\infty}\right] \tag{24}$$

so that

$$P\left(\frac{T_{ct}^{vt}}{t} \in B_\delta(u)\right) \geq e^{-\lambda_u ut - \delta t|\lambda_u|} P^{u,t}\left(\frac{T_{ct}^{vt}}{t} \in B_\delta(u)\right) E\left[e^{\lambda_u T_{ct}^{vt}} \mathbb{I}_{T_{ct}^{vt}}\right].$$

As in [6] (p. 77), one can show that

$$\liminf_{t\to\infty} \frac{1}{t} \log P^{u,t}\left(\frac{T_{ct}^{vt}}{t} \in B_\delta(u)\right) = 0. \tag{25}$$

Using this, (25), and Proposition 1 we find that

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{T_{ct}^{vt}}{t}\in B_\delta(u)\right) &\geq -\lambda_u u-\delta|\lambda_u|+(v-c)\mu(\lambda_u)\\ &= (v-c)\sup_\lambda\left(\lambda_u\frac{u}{v-c}-\mu(\lambda_u)\right)-\delta|\lambda_u|\\ &= (v-c)I^+(\frac{u}{v-c})-\delta|\lambda_u|. \qquad (26)\end{aligned}$$

This implies the lower bound (13). ☐

Proof of Theorem 2.4. We can use Theorem 2.3 to derive the large deviation bounds on the velocity variables, $(x-X^x(t))/t$. We will prove only (15) and (16) since the proofs of (17) and (18) follow the same argument using the function $I^-(a)$ described in Remark 1.

The proof follows the method in [6] and [25] (see section 5). First, for $c\geq 0$,

$$P\left(\frac{vt-X^{vt}(\kappa t)}{\kappa t}>c\right) \leq P\left(\frac{T_{(v-c\kappa)t}^{vt}}{t}<\kappa\right). \qquad (27)$$

Applying Theorem 2.3, we conclude that

$$\limsup_{t\to\infty}\frac{1}{\kappa t}\log P\left(\frac{vt-X^{vt}(\kappa t)}{\kappa t}>c\right)\leq -c\inf_{a\in(0,\kappa)}I^+\left(\frac{a}{c\kappa}\right)=-cI^+\left(\frac{1}{c}\right)$$

whenever $c\geq(a_0)^{-1}$. This proves the upper bound.

Now let $u\geq(a_0)^{-1}$.

$$\begin{aligned}P\left(\frac{vt-X^{vt}(\kappa t)}{\kappa t}\in B_\delta(u)\right) &= P\left(X^{vt}(\kappa t)\in B_{\kappa t\delta}((v-\kappa u)t)\right)\\ &\geq P\left(T_{(v-\kappa u)t}^{vt}\in((1-\epsilon)\kappa t,\kappa t)\right)-P\left(A^C\right),\end{aligned}$$

where $\epsilon\in(0,1)$ and A is the set

$$A=\left\{\omega \mid \sup_{(1-\epsilon)\kappa t\leq s\leq\kappa t}|X^{vt}(s)-ktu|<\kappa t\delta\right\}.$$

By combining the method of [25] (Section 5) with large deviation bounds for hitting times in the opposite direction (which follow from Remark 1), one can show that

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log P\left(A^C\right)=-\infty. \qquad (28)$$

This implies that

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{vt-X^{vt}(\kappa t)}{\kappa t}\in B_\delta(u)\right)\\ &\qquad\geq \liminf_{\epsilon\to 0}\liminf_{t\to\infty}\frac{1}{t}\log P\left(T_{(v-\kappa u)t}^{vt}\in((1-\epsilon)\kappa t,\kappa t)\right)\\ &\qquad\geq \liminf_{\epsilon\to 0}(-\kappa uI^+(\frac{1}{u}))=-\kappa uI^+(\frac{1}{u}), \qquad (29)\end{aligned}$$

where the last inequality follows from Theorem 2.3. ☐

4. **Front propagation.** In this section we prove Theorem 1.1. We first make use of assumption (4) and (5) in the following proposition. Recall that the constants $a_0^{\pm}$ are defined as

$$\begin{aligned} a_0^+ &= (\frac{d}{d\lambda}\mu^+)(0) = E_Q\left[E[T_0^1 \mathbb{I}_{T_0^1<\infty}]\right], \\ a_0^+ &= (\frac{d}{d\lambda}\mu^-)(0) = E_Q\left[E[T_0^{-1} \mathbb{I}_{T_0^{-1}<\infty}]\right]. \end{aligned}$$

Proposition 2. *Under assumptions (4) and (5), $a_0^+ = a_0^- = +\infty$. Consequently, $\lambda_c^+ = \lambda_c^- = 0$.*

As a consequence of Proposition 2, the large deviation estimates of Theorem 2.4 hold along the entire real line (i.e. when the sets G and F are subsets of $\mathbb{R}$). Moreover, the functions $I^+(a)$ and $I^-(a)$ are positive for all $a \in (0,\infty)$. For now, we postpone the proof of Proposition 2.

Define nonrandom constants $c_+^* > 0$ and $c_-^* < 0$ by the equations

$$(c_+^*)I^+(1/c_+^*) = f'(0), \quad \text{and} \quad (|c_-^*|)I^-(1/|c_-^*|) = f'(0). \tag{1}$$

Then by the properties of $I(a)$, $cI^+(1/c) > f'(0)$ for all $c > c_+^*$ and $|c|I^-(1/|c|) > f'(0)$ for all $c < c_-^*$ and

Remark 2. Notice that $c_-^* \neq -c_+^*$, in general, since the law of the process $b(x,\hat{\omega})$ may not be invariant with respect to space reversal.

Proof of Theorem 1.1. The solution $u(x,t)$ may be represented by the Feynman-Kac formula

$$u(x,t) = E\left[e^{\int_0^t \zeta(X^x(s),t-s)\,ds} u_0(X^x(t))\right], \tag{2}$$

where

$$\zeta(y,s) = \frac{f(u(y,s))}{u(y,s)}. \tag{3}$$

Since $f(u)$ is the KPP-type nonlinearity, $\zeta(y,s) \le f'(0)u(y,s) \le f'(0)$.

First, we prove the upper bound in Theorem 1.1. Let $F \subset (c_+^*,\infty)$. A bound for the general case follows in the same manner. Without loss of generality, suppose that $u_0(x) = \chi_{B_\delta(0)}$. Then

$$u(ct,t) \le E\left[e^{f'(0)t} u_0(X^{ct}(t))\right] = e^{f'(0)t} P\left(X^{ct}(t) \in B_\delta(0)\right).$$

By the properties of I^+ we can choose $\epsilon > 0$ so that $cI^+(1/c) > f'(0) + \epsilon$ for all $c \in F$. Then the probability on the right can be estimated by Proposition 2.4 with $\kappa = 1$, so that

$$\lim_{t\to\infty} \frac{1}{t} \sup_{c\in F} \log u(ct,t) \le f'(0) - cI^+(\frac{1}{c}) < -\epsilon.$$

Since $\epsilon > 0$, the upper bound follows.

The lower bound can be proved as in [16], provided that we have the following bounds:

Lemma 4.1. *For any compact set $K \subset (c_+^*,\infty)$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{c\in K} u(ct,t) \ge -\max_{c\in K}(cI^+(\frac{1}{c}) - f'(0)).$$

For any compact set $K \subset (-\infty, c_-^*)$,

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{c\in K} u(ct,t) \ge -\max_{c\in K}\left(|c|I^-(\frac{1}{|c|}) - f'(0)\right).$$

The proof of this lemma follows from the arguments in [16], and the following important estimates. In the proof of Lemma 4.1 above and the lower bound in Theorem 1.1, these estimate plays the role of Corollary 1 in [16], which is applied at equations (61), (76), and (77) therein. The estimates are:

Lemma 4.2. *For any $v \in \mathbb{R}$ and $\eta > 0$,*

$$\lim_{t\to\infty} \sup_{|x|\le|v|t} P\left(\sup_{s\in[0,t]} |X^x(s) - x| \ge \eta t\right) = 0. \tag{4}$$

Also, for a given $M > 0$, there exists $\kappa_0 > 0$ sufficiently small so that

$$\limsup_{t\to\infty} \sup_{|x|\le vt} \frac{1}{t}\log P\left(\sup_{s\in[0,\kappa t]} |X^x(s) - x| \ge \eta t\right) \le -M \tag{5}$$

whenever $\kappa < \kappa_0$.

Proof of Lemma 4.2. From Theorem 2.3 and Remark 1, we know that

$$\limsup_{t\to\infty} \frac{1}{t}\log P\left(\frac{T^{vt}_{(v-\eta)t}}{t} < \kappa\right) \le -\eta I^+\left(\frac{\kappa}{\eta}\right)$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log P\left(\frac{T^{vt}_{(v+\eta)t}}{t} < \kappa\right) \le -\eta I^-\left(\frac{\kappa}{\eta}\right).$$

Because $a_0 = +\infty$ and $a_0^- = +\infty$ (by Proposition 2), the right hand sides of these equations are positive for all $\eta > 0$. This proves (4). Since $I^\pm(a) \to \infty$ as $a \to 0$, the right hand sides can be made arbitrarily large by taking κ sufficiently small. This implies that

$$\limsup_{t\to\infty} \sup_{|x|\le vt} \frac{1}{t}\log P\left(\sup_{s\in[0,\kappa t]} (x - X^x(s)) \ge \eta t\right) \le -M \tag{6}$$

and

$$\limsup_{t\to\infty} \sup_{|x|\le vt} \frac{1}{t}\log P\left(\sup_{s\in[0,\kappa t]} (X^x(s) - x) \ge \eta t\right) \le -M \tag{7}$$

with κ sufficiently small. This proves (5). $\square$

To complete the proof of Theorem 1.1, we now prove Proposition 2. We only prove that $a_0^+ = +\infty$, since the proof that $a_0^- = +\infty$ is similar. We divide the proof into two steps, stated in the following lemmas. Proposition 2 is an immediate consequence of Lemma 4.3, Lemma 4.4, and the equivalent statements for a_0^-. $\square$

Lemma 4.3. *Suppose that $E_Q[b(x,\hat{\omega})] = 0$. If there is a set of nonzero probability (with respect to Q) on which the limit*

$$\lim_{L\to\infty} \int_0^L e^{-\int_0^z b(s)\,ds}\,dz = +\infty \tag{8}$$

diverges, then the condition

$$a_0^+ = E_Q\left[E[T_0^1 \mathbb{I}_{T_0^1<\infty}]\right] = +\infty \tag{9}$$

holds.

Lemma 4.4. *Suppose that $E_Q[b] = 0$ and that for some $\alpha \in \mathbb{R}$,*

$$\limsup_{z\to\infty} Q\left(\int_0^z b(s,\hat{\omega})\,ds \geq \alpha\right) < 1 \tag{10}$$

then condition (8) is satisfied.

Proof of Lemma 4.3. First, we claim that under condition (8), $P(T_0^1 < \infty) = 1$, almost surely with respect to Q. Thus, $E_Q[E[T_0^1 \mathbb{I}_{T_0^1<\infty}]] = E_Q[E[T_0^1]]$. To see this, consider the function $w(x,L) = P(T_0^x < T_L^x)$ for $x \in [0,L]$. This function solves the equation

$$\frac{1}{2}w_{xx} + b(x)w_x = 0$$

with boundary condition $w(0) = 1$ and $w(L) = 0$. Integrating the equation for w and using the boundary conditions, we find that

$$w(x,L) = 1 - \frac{1}{\int_0^L e^{-g(z)}\,dz}\int_0^x e^{-g(z)}\,dz,$$

where $g(z) = \int_0^z b(s)\,ds$. Therefore,

$$\lim_{L\to\infty} w(1,L) = 1$$

if and only if the integral $\int_0^L e^{-g(z)}\,dz$ diverges as $L \to \infty$, which is the condition in (8). Since the limit $\lim_{L\to\infty} w(1,L) = 1$ if and only if $P(T_0^1 < \infty) = 1$, this proves the claim.

Notice that for any $h > 0$,

$$\begin{aligned}\int_0^L e^{-\int_0^z b(s)\,ds}\,dz &= e^{-\int_0^h b(s)\,ds}\int_0^L e^{-\int_0^{z-h} b(s+h)\,ds}\,dz \\ &= e^{-\int_0^h b(s)\,ds}\int_0^{L-h} e^{-\int_0^z b(s,\tau_h\hat{\omega})\,ds}\,dz.\end{aligned} \tag{11}$$

This shows that the set where (8) holds is invariant under shifts $\hat{\omega} \to \tau_h\hat{\omega}$. So, if (8) holds on a set of nonzero measure, it must hold with probability one, by the ergodicity assumption on $b(x,\hat{\omega})$.

Now we show that $E_Q[E[T_0^1]]$ diverges. Let $v(x,L) = E[T_{[0,L]}^x]$, where $T_{[0,L]}^x$ is the first hitting time from $x \in [0,L]$ to either $x = 0$ or $x = L$. Then $v(x,L)$ solves

$$\frac{1}{2}v_{xx} + b(x)v_x = -1$$

with boundary conditions $v(0,L) = v(L,L) = 0$. This equation may be transformed to

$$\frac{1}{2}e^{-g(x)}\partial_x e^{g(x)}\partial_x v = -1$$

with $g(x) = \int_0^x b(s)ds$. The general solution is

$$v(x,L) = C_2 + C_1\int_0^x e^{-g(z)}\,dz + \int_0^x e^{-g(z)}\int_0^z -2e^{g(y)}\,dy\,dz.$$

Using the boundary condition we have

$$v(1,L) = \frac{2\int_0^L\int_0^z e^{g(y)-g(z)}\,dy\,dz}{\int_0^L e^{-g(z)}\,dz}\int_0^1 e^{-g(z)}\,dz - 2\int_0^1\int_0^z e^{g(y)-g(z)}\,dy\,dz.$$

As $L \to \infty$, consider the behavior of the term

$$C_1 = \frac{2 \int_0^L \int_0^z e^{g(y)-g(z)}\, dy\, dz}{\int_0^L e^{-g(z)}\, dz}.$$

The numerator is

$$\begin{aligned} N(L) &= 2\int_0^L e^{-g(z)} \int_0^z e^{g(y)}\, dy\, dz \\ &= 2\int_0^L e^{-2\int_0^z b(s)\,ds} \int_0^z e^{2\int_0^y b(s)\,ds}\, dy\, dz, \end{aligned}$$

and the denominator is

$$D(L) = \int_0^L e^{-g(z)}\, dz = \int_0^L e^{-2\int_0^z b(s)\,ds}\, dz.$$

All of the integrands are positive.

We claim that the numerator must diverge as $L \to \infty$. To see this, note that

$$N(L) = 2\int_0^L \left(\frac{d}{dz} \log(h(z)) \right)^{-1} dz \tag{12}$$

where $h(z) = \int_0^z e^{g(y)}\, dy$. Using Jensen's inequality and the fact that the integrands are positive, we see that for $\epsilon \in (0, L)$

$$\begin{aligned} N(L) &\geq 2\int_\epsilon^L \left(\frac{d}{dz} \log(h(z)) \right)^{-1} dz \qquad \text{(integrands are positive)} \\ &\geq 2(L-\epsilon) \left(\frac{1}{(L-\epsilon)} \int_\epsilon^L \frac{d}{dz} \log(h(z))\, dz \right)^{-1} \qquad \text{(Jensen's inequality)} \\ &= \frac{2(L-\epsilon)^2}{\log(h(L)) - \log(h(\epsilon))} \end{aligned} \tag{13}$$

Since $E_Q[b] = 0$, it follows that $|g(z)|/z \to 0$ as $z \to \infty$. Therefore, $\log(h(L)) \leq C_2(1 + L)$, so that $N(L)$ must diverge to $+\infty$ as $L \to \infty$.

If $D(L)$ remains bounded as $L \to \infty$, then the quotient $N(L)/D(L)$ must diverge as $L \to \infty$, since we have shown that $N(L)$ diverges. On the other hand, suppose that $D(L) \to \infty$ as $L \to \infty$. Then L'Hôpital's rule implies that

$$\lim_{L\to\infty} \frac{N(L)}{D(L)} = \lim_{L\to\infty} \frac{N'(L)}{D'(L)} = 2 \lim_{L\to\infty} \int_0^L e^{g(y)}\, dy \tag{14}$$

The final limit exists, since the integral $\int_0^L e^{g(y)}\, dy$ is an increasing function of L.

The analysis above shows that

$$E[T_0^1] = \lim_{L\to\infty} E[T_{[0,L]}^1] \geq \lim_{L\to\infty} \int_0^L e^{g(y)}\, dy. \tag{15}$$

Since the integral on the right is an increasing function of L, Q-a.s., this implies that

$$\begin{aligned} E_Q[E[T_0^1]] &\geq \lim_{L\to\infty} \int_0^L E_Q[e^{g(y)}]\,dy \\ &\geq \lim_{L\to\infty} \int_0^L e^{E_Q[g(y)]}\,dy \qquad \text{(Jensen's inequality)} \\ &= \lim_{L\to\infty} \int_0^L e^0\,dy = +\infty \end{aligned} \tag{16}$$

Since $E_Q\left[E[T_0^1 \mathbb{I}_{T_0^1<\infty}]\right] = E_Q[E[T_0^1]]$ under condition (8), this proves the proposition. □

Proof of Lemma 4.4. Let $g(z) = \int_0^z b(s)\,ds$, and let γ denote the limit in (8):

$$\gamma = \lim_{L\to\infty} \int_0^L e^{-\int_0^z b(s)\,ds}\,dz = \lim_{L\to\infty} \int_0^L e^{-g(z)}\,dz.$$

For $\alpha \in \mathbb{R}$, define the random process $h_\alpha(s,\hat{\omega})$ as

$$h_\alpha(s,\hat{\omega}) = \begin{cases} 1 & \text{if } g(s,\hat{\omega}) \geq \alpha \\ 0 & \text{if } g(s,\hat{\omega}) < \alpha \end{cases},$$

and let $\{r_n(\hat{\omega})\}_{n=1}^\infty$ be the sequence of random variables

$$r_n(\hat{\omega}) = \int_n^{n+1} h_\alpha(s,\hat{\omega})\,ds.$$

The variable r_n is the proportion of time in the interval $[n, n+1]$ during which $g(s)$ is greater than or equal to α.

First, suppose that for some $\epsilon > 0$ and some set $A_\alpha \subset \hat{\mathcal{F}}$ of nonzero measure

$$\liminf_{n\to\infty} r_n(\hat{\omega}) < 1 - \epsilon \tag{17}$$

for all $\hat{\omega} \in A_\alpha$. Then for every $\hat{\omega} \in A_\alpha$, there is an increasing sequence of integers $\{n_k\}_{k=1}^\infty$ (depending on $\hat{\omega}$) such that $r_{n_k} < 1 - \epsilon/2$. Therefore,

$$|\{z \in [n_k, n_k+1] \mid \ g(z) < \alpha\}| > \epsilon/2,$$

and from this we conclude that

$$\begin{aligned} \int_0^L e^{-g(z)}\,dz &\geq \sum_{\substack{k>0 \\ n_k+1<L}} \int_{n_k}^{n_k+1} e^{-g(z)}\,dz \\ &\geq \sum_{\substack{k>0 \\ n_k+1<L}} \int_{n_k}^{n_k+1} e^{-g(z)} \chi_{\{g<\alpha\}}(z)\,dz \\ &\geq \sum_{\substack{k>0 \\ n_k+1<L}} \frac{\epsilon}{2} e^{-\alpha}. \end{aligned} \tag{18}$$

Since ϵ and α are constants, the sum diverges as $L \to \infty$. Hence $\gamma = +\infty$ for all $\hat{\omega} \in A_\alpha$, $Q(A_\alpha) > 0$.

If (17) does not hold, then we are in the situation where

$$\liminf_{n\to\infty} r_n = 1, \quad Q - a.s.$$

Since $r_n \in [0,1]$, Fatou's lemma implies that

$$\liminf_{n\to\infty} E_Q\left[r_n\right] = 1. \tag{19}$$

However, by definition of r_n and Fubini's Theorem,

$$\begin{aligned}\limsup_{n\to\infty} E_Q\left[r_n\right] &= \limsup_{n\to\infty} \int_n^{n+1} E_Q\left[h(s,\hat{\omega})\right]\, ds \\ &= \limsup_{n\to\infty} \int_n^{n+1} Q\left(g(s)\geq \alpha\right)\, ds < 1.\end{aligned}$$

The last inequality follows from our assumption (10). This contradicts (19). Therefore, $\int_0^L e^{-g(z)}\, dz$ must diverge as $L \to \infty$. □

5. **Estimating the KPP front speed.** In this section we prove Theorem 1.2 using the fact that c_+^* solves

$$c_+^* I^+(1/c_+^*) = \sup_{\lambda\leq 0}(\lambda - c_+^*\mu(\lambda)) = f'(0), \tag{20}$$

where $\mu(\lambda) = \mu^+(\lambda) = E_Q\left[\log E[e^{\lambda T_0^1}]\right] < 0$. From (20) it follows that

$$c_+^* = \inf_{\lambda<0} \frac{\lambda - f'(0)}{\mu(\lambda)} = \inf_{\lambda>0}\frac{\lambda + f'(0)}{|\mu(-\lambda)|}, \tag{21}$$

and from Lemma 2.1,

$$|\mu(-\lambda)| \leq C(|\lambda| + 1 + \delta E_Q\left[\sup_{x\in[-2,2]} |b(x,\hat{\omega})|\right]). \tag{22}$$

By combining (21) and (22), we see that

$$c_+^* \quad\geq\quad \frac{1}{C}\inf_{\lambda>0}\frac{f'(0)+\lambda}{\lambda + 1 + M\delta}.$$

If $1+\delta M > f'(0)$, then

$$\frac{d}{d\lambda}\left[\frac{f'(0)+\lambda}{(\lambda+1+M\delta)}\right] > 0 \tag{23}$$

for all $\lambda > 0$, so the infimum of the quotient is attained at $\lambda = 0$. Otherwise, the infimum is attained in the limit $\lambda \to \infty$. This implies that c_+^* is bounded below by

$$c_+^* \geq \frac{1}{C}\min\left(1, \frac{f'(0)}{1+\delta M}\right).$$

Now we bound c_+^* from above by bounding $\mu(\lambda)$ from above. Since $\lambda < 0$,

$$\begin{aligned}E_P\left[e^{\lambda T_0^1}\right] &= \int_0^1 P(T_0^1 < -\frac{1}{|\lambda|}\log r)\, dr \\ &= 1 - \int_0^1 P(T_0^1 > -\frac{1}{|\lambda|}\log r)\, dr\end{aligned} \tag{24}$$

and

$$P(T_0^1 > r) = P(\inf_{s\in[0,r]} X^1(s) \geq 0).$$

Therefore, to bound $\mu(\lambda)$ from above, we must bound $P(T_0^1 > r)$ from below, which means that the stopping times T_0^1 can be large with high probability. We expect

T_0^1 to be large when the wind is blowing to the right in the interval $[0,1]$. This suggests that for $h>0$ and $\epsilon>0$, we define the set $A_h^\epsilon \subset \hat{\Omega}$ by

$$A_h^\epsilon = \{\hat{\omega} \mid \ b(x,\hat{\omega}) \geq h \ \forall x \in [0,\epsilon]\}\,.$$

Since $E[b] = 0$ and $b(x,\hat{\omega})$ is almost surely continuous, the set A_h^ϵ has nonzero measure if ϵ and h are sufficiently small. Since $\lambda < 0$ and $T^1 \geq T_0^\epsilon$ when $\epsilon \leq 1$, we can bound $\mu(\lambda)$ by

$$\mu(\lambda) \leq E_Q\left[\chi_{A_h^\epsilon} \log E[e^{\lambda T_0^\epsilon}]\right] \leq 0.$$

by choosing ϵ and h sufficiently small. We will assume without loss of generality that $Q(A_h^1) > 0$ and $\epsilon = 1$.

For any $s_1 \geq 0$, the process $X^1(t)$ satisfies

$$X^1(t) = X^1(s_1) + \int_{s_1}^t b(X^1(s))\,ds + W(t) - W(s_1)$$

for all $t > s_1$. If $\hat{\omega} \in A_h^1$ and $X^1(s) \in [0,1]$ for $s \in [s_1,s_2]$, then

$$X^1(t) \geq X^1(s_1) + (t-s_1)\delta h + W(t) - W(s_1) \tag{25}$$

for all $t \in [s_1,s_2]$. So for $s_2 = T_0^1$, we have $X^1(s_2) = 0$ and (25) implies that

$$W(s_2) - W(s_1) \leq -X^1(s_1) - (s_2-s_1)\delta h. \tag{26}$$

From (26), it follows that for $\hat{\omega} \in A_h^1$, $X^1(t)$ will stay to the right of the point $z=0$ for all $t < r$ unless $W(s)$ makes a relatively large leap (to the left) during some interval in $[0,r]$. If $W(s)$ does not make a relatively large jump over some interval $s \in [0,r]$, then $X^1(s)$ cannot overcome the opposing drift while $X(s) \in [0,1]$. More precisely,

$$\begin{aligned} P(T_0^1 < r) &= P(\inf_{s\in[0,r]} X^1(s) \leq 0) \qquad (27)\\ &\leq P\left(W(s_2) - W(s_1) \leq -1 - \delta h(s_2-s_1), \ \text{ for some } \ [s_1,s_2] \subset [0,r]\right)\\ &= P\left(W(s_2) - W(s_1) \geq 1 + \delta h(s_2-s_1), \ \text{ for some } \ [s_1,s_2] \subset [0,r]\right) \end{aligned}$$

To bound this probability, we need an estimate on the modulus of continuity of $W(s)$ in over the interval $[0,r]$. We will let $J_{r,\delta,h}$ denote the set we want to bound:

$$J_{r,\delta,h} = \{\omega \mid W(s_2) - W(s_1) \geq 1 + \delta h(s_2-s_1), \ \text{ for some } \ [s_1,s_2] \subset [0,r]\}\,. \tag{28}$$

This is the set where $W(s)$ makes relatively large jumps over some interval.

Now we bound the size of the set $J_{r,\delta,h}$ through an estimate on the Hölder continuity of the sample paths of W. Fix $\gamma > 0$, $\delta > 0$, $\alpha \in (0,1/2)$. It is known that the sample paths of the Wiener process are almost surely α-Hölder continuous for any $\alpha \in (0,1/2)$. For a given δ, h, α, and γ, let $\beta = \beta(\delta,h,\alpha,\gamma) > 0$ be defined by

$$\beta = \inf_{s\in[0,\gamma]} \frac{1+\delta h s}{s^\alpha}. \tag{29}$$

Therefore, $\beta s^\alpha \leq 1 + \delta h s$ for all $s \in [0,\gamma]$. Also, if

$$|W|_{\alpha,[0,\gamma]} \triangleq \sup_{s_1,s_2\in[0,\gamma]} \frac{|W(s_2)-W(s_1)|}{|s_2-s_1|^\alpha} < \beta \tag{30}$$

then

$$W(s_2) - W(s_1) \leq 1 + \delta h(s_2-s_1) \tag{31}$$

for any interval $[s_1, s_2] \subset [0, \gamma]$. For an integer $n > 0$, define the set

$$G_{\beta,n,\gamma} = \left\{ \omega | \ \ |W|_{\alpha,[j\gamma,(j+1)\gamma]} \leq \beta \ \ \forall j = 0, 1, \ldots, n-1 \right\}. \tag{32}$$

This is the set of paths that have α-Hölder seminorm bounded by β on each of the n blocks $[j\gamma, (j+1)\gamma]$, $j = 0, 1, \ldots, n-1$. Because the increments of the Wiener process are independent, the random variables $|W|_{\alpha,[j\gamma,(j+1)\gamma]}$, $j = 0, 1, \ldots, n$, are independent. Therefore, the measure of the set $G_{\beta,n,\gamma}$ is given by

$$\begin{aligned} P\left(G_{\beta,n,\gamma}\right) &= \prod_{j=0}^{n-1} P\left(|W|_{\alpha,[j\gamma,(j+1)\gamma]} \leq \beta\right) \\ &= P\left(G_{\beta,1,\gamma}\right)^n = (K_{\beta,\gamma})^n. \end{aligned} \tag{33}$$

where $K_{\beta,\gamma} = P\left(G_{\beta,1,\gamma}\right)$ is a positive constant that depends only on β, γ, and α. Note that for α and γ fixed, $K_{\beta,\gamma} \to 1$ as $\beta \to +\infty$.

For γ fixed, we can choose β so that (29) is satisfied. Then, by combining (27), (28), (31) and (32), we see that

$$P(T_0^1 < r) \leq P\left(J_{r,\delta,h}\right) \leq 1 - P(G_{\beta,n,\gamma}) = 1 - (K_{\beta,\gamma})^n \tag{34}$$

whenever n is chosen to the be smallest integer greater than r/γ. Therefore,

$$P(T_0^1 > r) \geq (K_{\beta,\gamma})^n. \tag{35}$$

Next, we use this information to bound the integral in (24). If $\gamma = \frac{1}{|\lambda|}$ and $r = \frac{|\log(r')|}{|\lambda|}$,

$$P(T_0^1 > -\frac{1}{|\lambda|}\log r') = P(T_0^1 > r) \geq (K_{\beta,\gamma})^n \tag{36}$$

where n is the smallest integer greater than $r/\gamma = |\log(r')|$. Therefore,

$$(K_{\beta,\gamma})^n \geq K_{\beta,\gamma}(K_{\beta,\gamma})^{-\log(r')} = K_{\beta,\gamma}(r')^{-\log(K_{\beta,\gamma})}. \tag{37}$$

Since $K < 1$, the constant $p = -\log(K_{\beta,\gamma})$ is positive. Therefore, plugging this information into (24), we see that for $\hat{\omega} \in A_h^1$,

$$\begin{aligned} E_P\left[e^{\lambda T_0^1}\right] &= 1 - \int_0^1 P(T_0^1 > -\frac{1}{|\lambda|}\log r')\,dr' \\ &\leq 1 - K_{\beta,\gamma}\int_0^1 (r')^p dr' \\ &= 1 - \frac{K_{\beta,\gamma}}{p+1} = 1 - \frac{K_{\beta,\gamma}}{1 - \log(K_{\beta,\gamma})} \end{aligned} \tag{38}$$

Finally, with $\gamma = |\lambda|^{-1}$,

$$\begin{aligned} \mu(\lambda) &\leq E\left[\chi_{A_h^1}\log E_P\left[e^{\lambda T_0^1}\right]\right] \\ &\leq Q\left(A_h^1\right)\log\left(1 - \frac{K_{\beta,\gamma}}{1 - \log(K_{\beta,\gamma})}\right). \end{aligned} \tag{39}$$

The term $Q\left(A_h^1\right)$ is a constant that depends only on h, and the properties of the stochastic field $b(x, \hat{\omega})$. For h fixed, $\alpha \in (0, 1/2)$ fixed, and $\gamma = |\lambda|^{-1}$ fixed, β defined by (29) depends only on δ. As a function of δ, $\beta(\delta) \to \infty$ as $\delta \to \infty$. Therefore, as a function of δ,

$$\lim_{\delta\to\infty} K_{\beta(\delta),\gamma} = 1 \tag{40}$$

Using this limit in (39), we see that for each $\lambda < 0$ fixed, μ diverges to $-\infty$ as $\delta \to \infty$:

$$\lim_{\delta \to +\infty} \mu(\lambda, \delta) = -\infty. \tag{41}$$

In fact, we can estimate the rate (with respect to δ) at which μ diverges. This will give us the bound (7). For δ sufficiently large, the infimum in the definition of β (29) is attained at the point

$$s_0 = \frac{\alpha}{1 + \delta h(1-\alpha)} = O(1/\delta) \tag{42}$$

Therefore, as a function of δ, $\beta(\delta) = O(\delta^{\alpha})$. Using the theorem of Garsia, Rodemich, and Rumsey [11] one can show that

$$P(|W|_{\alpha,[0\gamma]} \le \beta) = 1 - O(e^{-C\beta^2}), \tag{43}$$

for some $C > 0$. Hence $K_{\beta,\gamma} = 1 - O(e^{-C\beta^2}) = 1 - O(e^{-C\delta^{2\alpha}})$. By Taylor expansion, it now follows that

$$\log\left(1 - \frac{K_{\beta,\gamma}}{1 - \log(K_{\beta,\gamma})}\right) \le -C\delta^{2\alpha}. \tag{44}$$

Plugging this bound and (39) into the variational formula (21), we now see that for any $\alpha \in (0, 1/2)$ there is a constant $C(\alpha)$ such that

$$c_+^*(\delta) \le \frac{C}{\delta^{2\alpha}}. \tag{45}$$

This completes the proof of Theorem 1.2. □

6. **Conclusions.** By hitting time and large deviation analysis of the associated stochastic flows, we have shown that KPP front speeds are almost surely deterministic and finite in a mean zero stationary ergodic drift satisfying certain mixing and extremal properties, true in particular for locally Lipschitz continuous Gaussian processes with enough decay of correlations. The front speeds obey variational principles and power law of decay in the limit of large root mean square amplitude of the drift process. In contrast, diffusion (homogenization) approximation based on second order moments of the stochastic flow trajectories may give a rather different decay law. The existence of finite front speeds in random drifts is more robust than that of finite effective diffusion in random environments.

Acknowledgments. J.N. is grateful for support through an NSF postdoctoral fellowship at Stanford University. J.X. is partially supported by NSF grant DMS-0549215. We would like to thank M. Cranston and D. Stroock for helpful discussions.

REFERENCES

[1] R. Adler, "An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes," Institute of Math Stat, Lecture Notes-Monograph Series, **12**, 1990.

[2] M. A. Akcoglu and U. Krengel, *Ergodic theorems for superadditive processes*, J. Reine Angew. Math., **323** (1981), 53–67.

[3] P. Billingsley, "Convergence of Probability Measures," Wiley, 1968.

[4] T. Brox, *A one-dimensional diffusion process in a Wiener medium*, Ann. Prob., **14** (1986) , 1206–1218.

[5] P. Clavin and F. Williams, *Theory of premixed-flame propagation in large-scale turbulence*, J. Fluid Mech., **90** (1979), 598–604.

[6] F. Comets, N. Gantert and O. Zeitouni, *Quenched, annealed and functional large deviations for one-dimensional random walk in random environment*, Probab. Theory Relat. Fields, **118** (2000), 65–114.

[7] J. Conlon and C. Doering, *On traveling waves for the Stochastic FKKP equation*, Jour. Stat Physics, **120** (2005), 421–477.

[8] A. Dembo and O. Zeitouni, "Large Deviations Techniques and Applications," 2nd ed., Springer: New York, 1998.

[9] M. Freidlin, "Functional Integration and Partial Differential Equations," Ann. Math. Stud. **109**, Princeton University Press, Princeton, NJ, 1985.

[10] M. Freidlin, *Propagation of a concentration wave in the presence of random motion associated with the growth of a substance*, Soviet Math Dokl. (English Translation), **20** (1979), 503–507.

[11] A.M. Garsia, E. Rodemich and H. Rumsey Jr., *A real variable lemma and the continuity of paths of some Gaussian processes*, Indiana Univ. Math. Journal, **20** (1970), 565–578.

[12] J. Gärtner and M. Freidlin, *The propagation of concentration waves in periodic and random media*, Soviet Math Dokl. (English Translation), **20** (1979), 1282–1286.

[13] P.-L. Lions and P. E. Souganidis, *Homogenization of viscous Hamilton-Jacobi equations in stationary ergodic media*, Comm. Partial Diff. Eqn., **30** (2005), 335–375.

[14] A. Majda and P. E. Souganidis, *Flame fronts in a turbulent combustion model with fractal velocity fields*, Comm Pure Appl Math, **LI** (1998), 1337–1348.

[15] C. Mueller and R. Sowers, *Random traveling waves for the KPP equation with noise*, J. Functional Analysis, **128** (1995), 439–498.

[16] J. Nolen and J. Xin, *A variational principle for KPP front speeds in temporally random shear flows*, Comm. Math. Phys., **269** (2007), 493–532.

[17] J. Nolen and J. Xin, *A variational principle based study of KPP minimal front speeds in random shears*, Nonlinearity, **18** (2005), 1655–1675.

[18] G. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Colloquia Mathematica Societatis János Bolyai, **27** (1979), 835–873.

[19] N. Peters, "Turbulent Combustion," Cambridge University Press, 2000.

[20] P. Ronney, *Some open issues in premixed turbulent combustion*, in: "Modeling in Combustion Science" (J. D. Buckmaster and T. Takeno, Eds.), Lecture Notes In Physics, **449**, Springer-Verlag, Berlin, (1995), 3–22.

[21] L. Ryzhik and A. Zlatoš, *KPP pulsating front speed-up by flows*, Comm. Math. Sci., **5** (2007), 575–593.

[22] S. Schumacher, *Diffusions with random coefficients*, in: "Particle Systems, Random Media and Large Deviations" (R. Durrett Ed.), American Mathematical Society, Providence, 1985.

[23] W. Shen, *Traveling waves in diffusive random media*, J. Dynamics Diff. Eqs., **16** (2004), 1011–1060.

[24] Ya. Sinai, *The limit behavior of a one-dimensional random walk in a random medium*, Theo. Probl. and Appls., **27** (1982), 256–268.

[25] M. Taleb, *Large deviations of Brownian motion in a drifted Brownian potential*, Ann. Prob., **29** (2001), 1173–1204.

[26] J. Xin, *KPP front speeds in random shears and the parabolic Anderson problem*, Methods and Applications of Analysis, **10** (2003), 191–198.

[27] V. Yakhot, *Propagation velocity of premixed turbulent flames*, Comb. Sci. Tech, **60** (1988), 191.

[28] O. Zeitouni, "Random Walks in Random Environment," Lecture Notes in Math., **1837**, Springer, Berlin, 2004.

Received November 2007; revised April 2008.

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