# Front Speed in the Burgers Equation with a Random Flux

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**Abstract**

We study the large time asymptotic shock front speed in an inviscid Burgers equation with a spatially random flux function. This equation is a prototype for a class of scalar conservation laws with spatial random coefficents such as the well-known Buckley-Leverett equation for two-phase flows, and the contaminant transport equation in groundwater flows. The initial condition is a shock located at the origin (the indicator function of the negative real line). We first regularize the equation by a special random viscous term so that the resulting equation can be solved explicitly by a Cole-Hopf formula. Using the invariance principle of the underlying random processes, and the Laplace method, we prove that for large times the solutions behave like fronts moving at averaged constant speeds in the sense of distribution. However, the front locations are random, and we show explicitly the probability of observing the head or tail of the fronts. Finally, we pass to the inviscid limit, and establish the same results for the inviscid shock fronts.

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# 1 Introduction

We are interested in studying the initial value problem of the following Burgers equation with spatially random flux:

$$v_t + (\frac{1}{2}a(x,\omega)v^2)_x = 0, \quad x \in R^1, \tag{1.1}$$

with initial data $v(x,0) = I_{R_-}(x)$, the indicator function of the negative real line. Here $a(x,\omega)$ is a positive stationary random process. The complete list of assumptions $a$ satisfies will be stated as $A1$-$A5$ in the next section.

When $a$ is a constant, say one, it is well known that shock solutions are asymptotically stable and attract front-like initial data, see the classical work [11]. In applications to geoscience and other areas, however, shocks typically travel in spatially inhomogenous environment because of the natural formation of porous structures. Due to lack of experimental or field data, the spatially inhomogeneous environment is modeled as a random process. Conservation of mass then leads to a scalar conservation law with a random flux:

$$v_t + (f(v,x,\omega))_x = 0, \tag{1.2}$$

or its viscous analogue where the right hand side is instead a second-order elliptic term. Some of the equations of this form are: 1) the Buckley-Leverett equation for two-phase flows, see [10] and references therein; 2) the contaminant transport equation, see [25] and [4]; and 3) the Richards equation for infiltration problems, see [16] and [17], among others. The specific form of the nonlinear and random function $f$ depends on the problem at hand.

One of the fundamental issues discussed in these works is the shock dynamics in random media. This is usually a very difficult problem since it involves both nonlinearity and randomness. Equation (1.1) appears to be one of the few tractable cases where one can study rigorously the shock propagation in a random medium, here characterized by $a$. Burgers equation has been extensively studied recently in the literature as a model for turbulence and also for random shock asymptotics, see [1], [2], [20],[21], [22]; and [7], [23]. In [23], the present authors proved that under white noise initial perturbation, the viscous shock fronts move at the unperturbed speeds with their locations randomly distributed, and obey a central limit theorem in the large time limit. What happens to shocks in random media ? Do they propagate ? If so, at what speed ? What about their locations ? We will answer these questions in the context of equation (1.1). Our approach is to regularize (1.1) by adding the viscous term $\nu(\sqrt{a(x)}(\sqrt{a(x)}v)_x)_x$ to its right-hand side.

The resulting equation can be transformed into the standard viscous Burgers equation for another function, $u$, through a random change of spatial variables. The initial data for $u$ can be thought of as a random perturbation of the inviscid Burgers shock. The transformation of variables mentioned above is thus transferring the randomness from the coefficients of the equation to the initial data. The Cole-Hopf formula is then used to write down the solution in an explicit form. The formula is in the form of a ratio, involving five terms, which can be analyzed using probabilistic estimates, with the help of Laplace method, similarly to what was done in [23]. The new diffculty is that we now do not have the scaling property of the Brownian motion which in [23] was coming directly from the form of the initial perturbation (the white noise). Instead, we resort to an invariance principle in order to apply the Laplace method. The required invariance principle (i.e. a functional central limit theorem) holds for a class of $\varphi$-mixing processes $a(x,\omega)$; it is stated in detail in section 2. Finally, we pass to the $\nu \to 0$ limit of solutions to obtain results on the inviscid random shocks. It is known that such shocks are unique and satisfy the entropy conditions, [12]. We find that the inviscid shocks move at constant asymptotic speeds, and that the shock locations are random with their heads or tails seen with explicit probabilities as we probe the solution along the ray $x = ct + z\sqrt{t}$, where $c$ is the constant shock speed and $z$ is a real parameter.

The rest of the paper is organized as follows. In section 2, we state the main assumptions and the main theorem of the paper. Then we introduce the change of spatial variables and the Cole-Hopf representation of solutions. In section 3, we give the proof of the main theorem using invariance principle and Laplace method. Some of the technical results in the proof are relegated to section 4, the appendix. In particular, an invariance principle for hitting times, which may be of independent interest, is proven there (Theorem 4.1).

# 2 Main Theorem and the Cole-Hopf Solutions

The principal object studied in this paper is the inviscid Burgers equation with a random flux:

$$v_t + (\frac{1}{2}a(x,\omega)v^2)_x = 0. \tag{2.1}$$

We are interested in the long-time behavior of the solution to (2.1) with initial data of the front type:

$$v(x,0) = I_{R_-}(x). \tag{2.2}$$

Here $I_{R_-}$ denotes the indicator function of the negative real line and $a(x,\omega)$ is a stochastic process, satisfying the assumptions A1-A5 stated below. The assumptions A3 and A4 are stated in the way we use them and at this stage may appear somewhat technical. Below (see Remark 2.1) we describe a natural class of processes for which these assumptions are satisfied.

**A1.** Stationarity: the finite-dimensional distributions of the process $a(x,\omega)$ are invariant under translations of the variable $x$.

**A2**. Positivity: $a(x,\omega) > 0$ with probability one.

**A3**. Measurability and integrability of the inverse: paths of $a$ are measurable functions of $x$ and

$$E[\frac{1}{a(x)}] < \infty.$$

It follows that also

$$E[\frac{1}{\sqrt{a(x)}}] \stackrel{\text{def}}{=} \mu < \infty.$$

**A4**. Invariance principle: Let

$$\xi(x) = \int_0^x \frac{1}{\sqrt{a(y)}}\, dy.$$

Note that $\xi(x) < 0$ for $x < 0$. For each $x_0 > 0$, we have

$$(\frac{\xi(tx) - \mu tx}{\sigma\sqrt{t}})_{|x|\le x_0} \stackrel{\mathcal{D}}{\to} (W_x)_{|x|\le x_0}, \tag{2.3}$$

as $t \to \infty$, where $W = (W_x)_{x\in R}$ is the Wiener process and

$$\sigma^2 = 2\int_0^{+\infty} E[(\frac{1}{\sqrt{a(0)}} - \mu)(\frac{1}{\sqrt{a(x)}} - \mu)]\, dx < \infty.$$

$\mathcal{D}$ denotes converges of processes in law; see [3]. We stress that the finiteness of the last integral is part of the assumption. $\sigma^2$ is sometimes called the velocity autocorrelation function (of the process $\frac{1}{\sqrt{a}}$).

**A5**. Regularity: we assume that the paths of the process $a$ are Hölder continuous with some (positive) exponent. This will be used in the proof of the main theorem, to justify taking the zero viscosity limit. A well known probabilistic condition which implies Hölder continuity of sample paths is the Kolmogorov moment condition ([18], Theorem 25.2).

**Remark 2.1** *A large class of processes for which (2.3) holds is the class of stationary processes $a(x,\omega)$, satisfying the appropriate $\phi$-mixing condition. Here $\phi$ is a nonnegative* *function of a positive real variable, such that*

$$\lim_{t \to +\infty} \phi(t) = 0 \tag{2.4}$$

*and the $\phi$-mixing condition says that, for any $t > 0$ and for any $s$, whenever an event $E_1$ is in the $\sigma$-field generated by the random variables $a(x)$ with $-\infty \le x \le s$ and an event $E_2$ is in the $\sigma$-field generated by $a(x)$ with $s + t \le x \le +\infty$, we have*

$$|P[E_1 \cap E_2] - P[E_1]P[E_2]| \le \phi(t)P[E_1]. \tag{2.5}$$

*Roughly speaking, because of (2.4), (2.5) expresses a decay of correlations of the variables $a(x)$. More information on $\phi$-mixing processes can be found in [3], where it is, in particular, proven (pp. 178-179 of [3]) that the invariance principle (assumption A4) holds if*

$$\int_0^{+\infty} \sqrt{\phi(t)}\, dt < \infty. \tag{2.6}$$

It is well-known that the Burgers fronts are asymptotically stable for spatially decaying initial perturbation, [11]. The following main result of the paper shows that the front structure is also present in the random flux case. Throughout the paper the symbol $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution.

**Theorem 2.1** *Let $2c = E[a^{-\frac{1}{2}}]^{-2}$ denote the square root-harmonic mean of the variables $a(x)$. Then as $t \to \infty$:*

1. $$v(\alpha t, t) \xrightarrow{\mathrm{d}} 0, \quad for\ \alpha > c; \tag{2.7}$$
2. $$\sqrt{a(\alpha t)}v(\alpha t, t) \xrightarrow{\mathrm{d}} \sqrt{2c}, \quad for\ \alpha < c; \tag{2.8}$$
3. $$\sqrt{a(ct + z\sqrt{t})}v(ct + z\sqrt{t}, t) \xrightarrow{\mathrm{d}} X, \tag{2.9}$$

*where $X$ is a random variable equal to $\sqrt{2c}$ with probability $\mathcal{N}(\frac{\mu^2}{\sigma}z)$ and equal to 0 with probability $1 - \mathcal{N}(\frac{\mu^2}{\sigma}z)$, where $\mathcal{N}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{s'^2}{2}}\, ds'$ is the error function.*

**Remark 2.2** *The first two parts of the theorem say, roughly speaking, that to leading order the shock speed in the presence of randomness equals c. We expect to prove that an analogous result holds (with some c) for a more general class of nonlinear conservation laws with noisy initial data (but no randomness in the coefficients of the equation) using scaling arguments and continuous dependence of solutions on the initial data. We propose to use this method to handle the viscous as well as the inviscid case. While more general, this method does not provide more detailed information about the front location contained in part 3 of the theorem (see also [23]).*

**Remark 2.3** *In the inviscid case, we also plan to generalize the strategy applied in this paper to other nonlinearities as follows. We represent the solution as a limit of approximate solutions, given by explicit expressions. The asymptotic behavior of these approximate solutions is then studied using bounds analogous to those developed here.*

**Remark 2.4** *For problems with more general (nonquadratic) nonlinearities, one can scale* $x$ *and* $t$ *by a small parameter* $\epsilon$ *and formulate a homogenization problem for the integrated conservation laws, namely the Hamilton-Jacobi equations. For the periodic case, such problems are well-studied ([13], [6]), and one can even formally derive part 1) and part 2) of Theorem 2.1 from the averaged Hamiltonian obtained in [13] and [6]. However, extending these results to the random setting is a challenging task, and it remains an interesting problem to study rigorously the connection between random homogenization of Hamilton-Jacobi equations and our results here.*

**Remark 2.5** *Burgers equation can be thought of as a hydrodynamic limit of the asymmetric simple exclusion process. In this context, we would like to mention that shock location has been studied for such a process, see [8] and references therein. These results are analogous to ours (see also [23]).*

In the proof of Theorem 2.1, we will make use of a regularized version of the equation (2.1):

$$v_t + (\frac{1}{2}a(x)v^2)_x = \nu(\sqrt{a(x)}(\sqrt{a(x)}v)_x)_x, \tag{2.10}$$

where $\nu > 0$. It is convenient to rewrite this equation in terms of the function

$$u = \sqrt{a(x)}v, \tag{2.11}$$

Equation for $u$ becomes:

$$\frac{u_t}{\sqrt{a(x)}} + (\frac{1}{2}u^2)_x = \nu(\sqrt{a(x)}u_x)_x, \tag{2.12}$$

To simplify the last equation, we change the space variable:

$$\xi = \int_0^x \frac{1}{\sqrt{a(x')}}\, dx', \tag{2.13}$$

Since this change of variables depends on the realization of the process $a$, we obtain this way a stochastic process $\xi(x,\omega)$, which has already been used to state assumption A4. Equation for $u$ in the variables $(\xi, t)$ becomes the standard viscous Burgers equation:

$$u_t + (\frac{1}{2}u^2)_\xi = \nu u_{\xi\xi}, \tag{2.14}$$

with the new initial condition:

$$u(\xi, 0) = \sqrt{a(x(\xi))} I_{R_-}(\xi). \tag{2.15}$$

It is known that the speed of an (unperturbed) shock of the Burgers equation is equal to its height divided by two. This offers an intuitive, if nonrigorous, explanation of the results of the theorem: the asymptotic speed of the front described arising from our random initial condition equals one half of its average height, calculated in the $\xi$ variable i.e.

$$\frac{1}{2} \lim_{L\to\infty} \frac{1}{L} \int_{-L}^{0} \sqrt{a(x(\xi))}\, d\xi,$$

which, after changing the variable of integration to $x$, gives

$$\frac{1}{2} \lim_{L\to\infty} \frac{-x(-L)}{L},$$

which is $\frac{1}{2}E[a^{-\frac{1}{2}}]^{-1}$. To get from this the front speed in the $x$ variable, we divide this value by $E[a^{-\frac{1}{2}}]$ in view of (2.13) and arrive at the speed $c$ in the main theorem. A similar, but more detailed argument, taking into account fluctuations of the total mass in a finite interval of the initial data, leads to a heuristic justification of the Gaussian statistics of the front location.

To prepare the proof of Theorem 2.1, we need to introduce the Cole-Hopf representation of the solution, rewrite it in a convenient form, and prove some auxiliary results about asymptotic behavior in distribution of its constituent terms.

The paths of the process $\xi(x,\omega)$ are (with probability one) continuous, strictly increasing functions of $x$. Therefore, they have continuous inverses, defining another process, $x(\xi,\omega)$. Our assumption A4 says that the process $\xi(x,\omega)$ satisfies an invariance principle. In appendix, we prove a theorem (Theorem 4.1), which will be used crucially in the proof of Theorem 2.1, and which says that the same is also true about the process $x(\xi)$. More precisely, Theorem 4.1 (which is stated using a different notation) implies that

$$\left(\frac{x(t\xi) - \frac{t\xi}{\mu}}{\mu^{-\frac{3}{2}}\sigma\sqrt{t}}\right)_{|\xi|\le\xi_0} \xrightarrow{\mathcal{D}} (W_\xi)_{|\xi|\le\xi_0}. \tag{2.16}$$

(see assumption A4 for the definition of $\sigma$). In the sequel, we will use the following notation for the process $x(\xi)$ with the mean subtracted:

$$\hat{x}(\xi) = x(\xi) - \frac{\xi}{\mu}. \tag{2.17}$$

The Cole-Hopf formula ([24]) for $u$ reads:

$$u(\xi,t) = \frac{\int_{-\infty}^{+\infty} \frac{\xi-\eta}{t} \exp[-\frac{G(\eta,\xi,t)}{2\nu}]\, d\eta}{\int_{-\infty}^{+\infty} \exp[-\frac{G(\eta,\xi,t)}{2\nu}]\, d\eta}, \tag{2.18}$$

where

$$\begin{aligned} G(\eta,\xi,t) &= \int_0^\eta u(\eta',0)\, d\eta' + \frac{(\xi-\eta)^2}{2t} \\ &= x(\eta) I_{R_-}(\eta) + \frac{(\xi-\eta)^2}{2t} \\ &= (\frac{\eta}{E[a^{-\frac{1}{2}}]} + \hat{x}(\eta)) I_{R_-}(\eta) + \frac{(\xi-\eta)^2}{2t}. \end{aligned} \tag{2.19}$$

(the second equality follows by changing the variable to $x(\eta')$, where $x$ denotes the inverse of $\xi$ and using the fact that the derivative of $\xi$ is $\frac{1}{\sqrt{a}}$; see (2.13). Let $u_l = \frac{1}{E[a^{-\frac{1}{2}}]}$. Even though, clearly, $u_l = \frac{1}{\mu} = \sqrt{2c}$, it is convenient in this context to use the suggestive notation $u_l$ (the "left state of $u$"; see (2.8)). The numerator of (2.18) is equal to:

$$\begin{aligned} &\int_{-\infty}^0 \frac{\xi-\eta}{t} \exp[\frac{-u_l\eta - \hat{x}(\eta) - (2t)^{-1}(\xi-\eta)^2}{2\nu}]\, d\eta \\ +\ &\int_0^\infty \frac{\xi-\eta}{t} \exp[-(2t)^{-1}\frac{(\xi-\eta)^2}{2\nu}]\, d\eta, \end{aligned} \tag{2.20}$$

which, with the substitution $y = \xi - \eta$, becomes

$$\begin{aligned} &\int_\xi^\infty \frac{y}{t} \exp[\frac{-(\xi-y)u_l - (2t)^{-1}y^2 - \hat{x}(\xi-y)}{2\nu}]\, dy + \int_{-\infty}^{\frac{\xi}{\sqrt{t}}} \eta e^{-\frac{\eta^2}{4\nu}}\, d\eta \\ =\ &\frac{1}{t}\int_\xi^\infty y \exp[\frac{-(\xi - \frac{u_l}{2}t)u_l - (2t)^{-1}(y-u_lt)^2 - \hat{x}(\xi-y)}{2\nu}]\, dy + \int_{-\infty}^{\frac{\xi}{\sqrt{t}}} \eta e^{-\frac{\eta^2}{4\nu}}\, d\eta. \end{aligned}$$

Changing variable to $x' = y - u_l t$, the numerator becomes

$$\begin{aligned} &\frac{1}{t}\int_{\xi-u_lt}^\infty (x' + u_lt) \exp[\frac{-(\xi - \frac{u_l}{2}t)u_l - (2t)^{-1}x'^2 - \hat{x}(\xi - x' - u_lt)}{2\nu}]\, dx' + \int_{-\infty}^{\frac{\xi}{\sqrt{t}}} \eta e^{-\frac{\eta^2}{4\nu}}\, d\eta \\ =\ &u_l \exp[-(\xi - \frac{u_l}{2}t)\frac{u_l}{2\nu}] \int_{\xi-u_lt}^\infty \exp[-(2t)^{-1}\frac{x'^2}{2\nu} - \frac{\hat{x}(\xi - x' - u_lt)}{2\nu}]\, dx' \\ +\ &t^{-1} \exp[-(\xi - \frac{u_l}{2}t)\frac{u_l}{2\nu}] \int_{\xi-u_lt}^\infty x' \exp[-(2t)^{-1}\frac{x'^2}{2\nu} - \frac{\hat{x}(\xi - x' - u_lt)}{2\nu}]\, dx' \\ +\ &\int_{-\infty}^{\frac{\xi}{\sqrt{t}}} \eta e^{-\frac{\eta^2}{4\nu}}\, d\eta. \end{aligned}$$

Finally, we introduce a new variable $\eta = \frac{x'}{\sqrt{t}}$ and rearrange the order of the terms to get

$$\int_{-\infty}^{\frac{\xi}{\sqrt{t}}} \eta e^{-\frac{\eta^2}{4\nu}}\, d\eta \;\; + \;\; \sqrt{t} u_l e^{-\frac{u_l}{2\nu}(\xi - \frac{u_l}{2} t)} \int_{\frac{\xi - u_l t}{\sqrt{t}}}^{\infty} e^{-\frac{\eta^2}{4\nu} - \frac{\hat{x}(\xi - \sqrt{t}\eta - u_l t)}{2\nu}}\, d\eta$$
$$+ \;\; e^{-\frac{u_l}{2\nu}(\xi - \frac{u_l}{2} t)} \int_{\frac{\xi - u_l t}{\sqrt{t}}}^{\infty} \eta e^{-\frac{\eta^2}{4\nu} - \frac{\hat{x}(\xi - \sqrt{t}\eta - u_l t)}{2\nu}}\, d\eta. \qquad (2.21)$$

The consecutive terms in the last expression will be denoted by $A_t, B_t$ and $C_t$ respectively. Proceeding in a similar way, we can write the denominator in the form $\frac{B_t}{u_l} + D_t$, where $B_t$ is as above and

$$D_t = \sqrt{t} \int_{-\infty}^{\frac{\xi}{\sqrt{t}}} e^{-\frac{\eta^2}{4\nu}}\, d\eta. \qquad (2.22)$$

# 3 Proof of Theorem 2.1

In the next two propositions we prove that a part of the expression for $u$ goes to zero. These propositions will be used in the proof of Theorem 2.1, where it will be important that the convergence takes place uniformly in $\nu$, in the appropriate sense defined below in (3.1). With this in mind, we adopt the following convention about constants: constants independent of $\nu$, but depending on the random parameter $\omega$ (i.e. on the realization of the random flux) will be denoted by $C(\omega)$, or simply by $C$. Constants independent of both $\nu$ and $\omega$ will be denoted by $c$. The actual value of $C$ or $c$ may vary from one line to another.

**Proposition 3.1** $\lim_{t\to\infty} \sup_\xi \frac{A_t}{\frac{B_t}{u_l} + D_t} \stackrel{\mathrm{d}}{=} 0$. *Moreover, convergence is uniform in* $\nu$ *in the sense that for every* $\epsilon > 0$*, as* $t \to \infty$*:*

$$P[\sup_{\nu \le \nu_0} |\frac{A_t}{\frac{B_t}{u_l} + D_t}| > \epsilon] \to 0, \qquad (3.1)$$

*for any* $\nu_0 > 0$. *(Note that convergence in distribution to* $0$ *is equivalent to convergence in probability to* $0$*).*

Proof: for positive $\xi$ we have

$$\frac{A_t}{D_t} \le c\sqrt{\nu} t^{-\frac{1}{2}}, \qquad (3.2)$$

with an absolute constant $c$. For negative $\xi$, we restrict the integration in the definition of $B_t$ to the interval $0 \le \eta \le 1$ and note that, since $\frac{\hat{x}(u)}{\sqrt{|u|}}$ converges in distribution to a normal random variable (by Theorem 4.1), with probability one there exists an ($\omega$-dependent) constant $C$ such that for all $u$

$$\hat{x}(u) \le C|u|^{\frac{2}{3}}. \tag{3.3}$$

Therefore the integral

$$\int_{\frac{\xi-u_l t}{\sqrt{t}}}^{\infty} e^{-\frac{\eta^2}{4\nu}-\frac{\hat{x}(\xi-\sqrt{t}\eta-u_l t)}{2\nu}}\, d\eta \tag{3.4}$$

can be bounded below by

$$e^{-\frac{1}{4\nu}}\int_0^1 \exp(-\frac{\hat{x}(\xi-\sqrt{t}\eta-u_l t)}{2\nu})\, d\eta \ge e^{-\frac{1}{4\nu}}e^{\frac{C}{\nu}|\xi-\sqrt{t}-u_l t|^{\frac{2}{3}}}, \tag{3.5}$$

for all $t$ and $\xi$. This implies that for almost all $\omega$ and $t \ge 1$ (uniformly in $\xi \le 0$), we have

$$B_t \ge \sqrt{t}u_l e^{-\frac{u_l}{2\nu}(\xi-\frac{u_l}{2}t)}e^{-\frac{1}{4\nu}}e^{-\frac{C}{\nu}(\xi-\sqrt{t}-u_l t)^{\frac{2}{3}}} \ge u_l e^{-\frac{u_l}{2\nu}(\xi-\frac{u_l}{2}t)-\frac{C}{\nu}|\xi-t-u_l t|^{\frac{2}{3}}}e^{-\frac{1}{4\nu}}.$$

The last expression clearly goes to $\infty$ uniformly in $\xi \le 0$ as $t \to +\infty$. Since $A_t$ is bounded from above by an absolute constant, it follows that

$$\sup_{\xi\le 0}\frac{A_t}{B_t} \to 0, \tag{3.6}$$

for almost all $\omega$. Combining (3.2) and (3.6) ends the proof.

**Proposition 3.2** $\lim_{t\to\infty}\sup_{\xi}\frac{C_t}{\frac{B_t}{u_l}+D_t} = 0$. *Moreover, the convergence is uniform in* $\nu \in (0,\nu_0)$ *in the sense described in the statement of Proposition 3.1.*

Proof:

$$\frac{C_t}{B_t} = \frac{1}{\sqrt{t}u_l}\frac{e^{\frac{-\hat{x}(\xi-u_l t)}{2\nu}}\int_{\frac{\xi-u_l t}{\sqrt{t}}}^{\infty}\eta e^{\frac{[\hat{x}(\xi-u_l t)-\hat{x}(\xi-u_l t-\sqrt{t}\eta)]}{2\nu}-\frac{\eta^2}{4\nu}}\, d\eta}{e^{\frac{-\hat{x}(\xi-u_l t)}{2\nu}}\int_{\frac{\xi-u_l t}{\sqrt{t}}}^{\infty} e^{\frac{[\hat{x}(\xi-u_l t)-\hat{x}(\xi-u_l t-\sqrt{t}\eta)]}{2\nu}-\frac{\eta^2}{4\nu}}\, d\eta.} \tag{3.7}$$

Changing the variable to $y = t^{-\frac{1}{6}}\eta$, we obtain

$$\frac{C_t}{B_t} = \frac{1}{\sqrt{t}u_l}\frac{t^{\frac{1}{3}}\int_{\frac{\xi-u_l t}{t^{\frac{2}{3}}}}^{\infty} y e^{\frac{t^{\frac{1}{3}}}{2\nu}[\frac{\hat{x}(\xi-u_l t)-\hat{x}(\xi-u_l t-t^{\frac{2}{3}}y)}{t^{\frac{1}{3}}}-\frac{y^2}{2}]}}{t^{\frac{1}{6}}\int_{\frac{\xi-u_l t}{t^{\frac{2}{3}}}}^{\infty} e^{\frac{t^{\frac{1}{3}}}{2\nu}[\frac{\hat{x}(\xi-u_l t)-\hat{x}(\xi-u_l t-t^{\frac{2}{3}}y)}{t^{\frac{1}{3}}}-\frac{y^2}{2}]}}. \tag{3.8}$$

We shall first consider the values of $\xi$ satisfying

$$\xi - \frac{2}{3}u_l t \le 0. \tag{3.9}$$

Stationarity of $a$ implies easily that

$$\frac{\hat{x}(\xi - u_l t) - \hat{x}(\xi - u_l t - t^{\frac{2}{3}} y)}{t^{\frac{1}{3}}} \stackrel{\mathcal{D}}{=} \frac{\hat{x}(t^{\frac{2}{3}} y)}{t^{\frac{1}{3}}}, \tag{3.10}$$

with equality in law of processes in the variable $y \in R$. Theorem 4.1 implies now that the processes

$$\frac{\hat{x}(\xi - u_l t) - \hat{x}(\xi - u_l t - t^{\frac{2}{3}} y)}{\sigma t^{\frac{1}{3}}}, \tag{3.11}$$

where $\sigma$ is defined in $A4$, converge in law to the Wiener process. It follows from the Skorohod representation theorem (Theorem 4.2) that there exists a probability space $(\Omega, \mathcal{F}, P)$ and processes $\tilde{\tau}^{(t)}, W$ on that space, such that for each $t$

$$\tilde{\tau}^{(t)} \stackrel{\mathcal{D}}{=} \frac{\hat{x}(\xi - u_l t) - \hat{x}(\xi - u_l t - t^{\frac{2}{3}} y)}{\sigma t^{\frac{1}{3}}}, \tag{3.12}$$

$W$ is a Wiener process and with $P$-probability one

$$\tilde{\tau}^{(t)}(y) \to W_y \tag{3.13}$$

uniformly in $y$ belonging to any compact interval. Using Lemma 4.1, we see that as long as $\frac{\xi - u_l t}{t^{\frac{2}{3}}} \to -\infty$, the ratio of the two integrals of (3.8) converges in distribution to $y_0$, where $y_0$ denotes the unique value of $y$ where the function $W_y - \frac{y^2}{2}$ attains its maximum. Existence and uniqueness of such a point were proven in [23], where they were used for asymptotic analysis of the expression

$$\frac{\int_{-\infty}^{+\infty} y e^{\frac{t^{\frac{1}{3}}}{2\nu}[W(y) - \frac{y^2}{2}]} \, dy}{\int_{-\infty}^{+\infty} e^{\frac{t^{\frac{1}{3}}}{2\nu}[W(y) - \frac{y^2}{2}]} \, dy}. \tag{3.14}$$

(3.8) implies now that, for almost all $\omega$, $\frac{C_t}{B_t}$ converges to zero uniformly in $\xi$ satisfying (3.9). To handle the values of $\xi$ for which

$$\xi - \frac{2}{3} u_l t > 0. \tag{3.15}$$

Note that $\xi - \frac{u_l}{2} t \to +\infty$, uniformly in $\xi$ satisfying (3.15). We will now use an argument similar to the one used in the proof of Proposition 3.1, to show that for almost all $\omega$ $\frac{C_t}{B_t}$ converges to zero uniformly in $\xi$ satisfying (3.15) as well. Namely, with probability one there exists an ($\omega$-dependent) constant $C$ such that (3.3) holds. This, together with subadditivity of the function $u \mapsto |u|^{\frac{2}{3}}$, implies that

$$\hat{x}(\xi - \sqrt{t}\eta - u_l t) \geq -C(|\xi|^{\frac{2}{3}} + t^{\frac{1}{3}} |\eta|^{\frac{2}{3}} + u_l^{\frac{2}{3}} t^{\frac{2}{3}}).$$

Therefore the $\xi$-dependent part of the integrand can be absorbed into the prefactor:

$$|C_t| \le e^{-\frac{u_l}{2\nu}(\frac{5}{6}\xi - \frac{u_l}{2}t)} \int_{\frac{\xi - u_l t}{\sqrt{t}}}^{\infty} \eta e^{-\frac{\eta^2}{4\nu} + \frac{C}{2\nu}(t^{\frac{1}{3}}|\eta|^{\frac{2}{3}} + u_l^{\frac{2}{3}} t^{\frac{2}{3}})}\, d\eta. \tag{3.16}$$

We now divide the integral in the last formula into two parts, corresponding to $|\eta| \le 1$ and $|\eta| \ge 1$. The first integral is clearly bounded above by

$$ce^{-\frac{u_l}{2\nu}(\frac{5}{6}\xi - \frac{u_l}{2}t)} e^{ct^{\frac{2}{3}}}$$

The last expression goes exponentially fast to zero, uniformly in $\xi$ satisfying (3.15), since for those $\xi$

$$\frac{5}{6}\xi - \frac{u_l}{2}t \ge \frac{1}{18}u_l t. \tag{3.17}$$

When $|\eta| \ge 1$, we have $|\eta|^{\frac{2}{3}} \le |\eta|$ and, consequently, the right-hand side of (3.16) is bounded above by

$$e^{-\frac{u_l}{2\nu}(\frac{5}{6}\xi - \frac{u_l}{2}t)} e^{\frac{C}{2\nu}u_l^{\frac{2}{3}} t^{\frac{2}{3}}} \int_{\frac{\xi - u_l t}{\sqrt{t}}}^{\infty} |\eta| e^{-\frac{\eta^2}{4\nu} + \frac{C}{2\nu} t^{\frac{1}{3}}|\eta|}\, d\eta.$$

The integral in the above formula can be estimated by first absorbing the factor $|\eta|$ into the exponential factor (by making $C$ bigger) and then using the identity

$$\int_R e^{-\frac{\eta^2}{4\nu} + \frac{C}{2\nu}s\eta}\, d\eta = \sqrt{4\pi\nu} e^{Cs}$$

with $s = t^{\frac{1}{3}}$. We obtain this way

$$|C_t| \le e^{-\frac{u_l}{2\nu}(\frac{5}{6}\xi - \frac{u_l}{2}t)} \sqrt{4\pi\nu} e^{\frac{C}{2\nu}u_l^{\frac{2}{3}} t^{\frac{2}{3}}} e^{ct^{\frac{1}{3}}}$$

the last expression clearly goes to zero uniformly in $\xi$ satisfying (3.15) (see (3.17)). Since for these $\xi$, $D_t$ can be uniformly bounded from below by $c\sqrt{t}$, where $c > 0$ is an absolute constant, the proof is finished.

**Proof of Theorem 2.1**: the strategy of the proof is to study the solution of the regularized equation (2.10) via its Cole-Hopf representation and then take the limit when $\nu \to 0$. It follows from the two preceding propositions, that we just need to study the limiting distribution of $\frac{B_t}{\frac{B_t}{u_l} + D_t}$.

Assume $\alpha > c$. In the $\xi$ coordinate, this means that in the representation (2.21) of $B_t$ the factor $e^{-\frac{u_l}{2\nu}(\xi - \frac{u_l}{2}t)}$ goes exponentially fast to zero, uniformly in $\nu$. We now use the bound (3.3) (true with probability one for some constant $C$) and proceeding exactly as in the proof of Proposition 3.2, we have, with probability one,

$$\int_R e^{-\frac{\eta^2}{4\nu}-\frac{\hat{x}(\xi-\sqrt{t}\eta-u_l t)}{2\nu}}\,d\eta \leq Ce^{Ct^{\frac{2}{3}}}. \tag{3.18}$$

This clearly implies that $B_t \to 0$ almost surely as $t \to \infty$. On the other hand, $D_t \to +\infty$ (at the order of $\sqrt{t}$), so

$$\frac{B_t}{\frac{B_t}{u_l}+D_t} \to 0 \tag{3.19}$$

almost surely and therefore the analog of part 1 of the theorem is proven for the solution $u_\nu$, $\nu > 0$, solving the regularized equation (2.10). Note that all the above convergence statements, including (3.19) hold uniformly in $\nu$. It follows that

$$\sup_{\nu\leq\nu_0} |u_\nu(\xi(\alpha t), t)| \xrightarrow{\text{d}} 0. \tag{3.20}$$

Thanks to assumption A5, we apply the results of [15] (Theorem 13 and Theorem 14) and [19] to conclude that for any given $t$, except for a set of $x$ consisting of countably many discontinuities of the first kind (shocks),

$$\lim_{\nu\to 0} u_\nu(x,t) = u_0(x,t). \tag{3.21}$$

Moreover, $u_0(x,t)$ is the unique weak solution satisfying entropy conditions. Notice that the continuous differentiability of $a(x)$ in $x$ in [15] is used in constructing characteristic curves. In our case, since we can make change of variables to get inviscid Burgers equation in the $\xi$ variables, continuity of $a(x)$ is enough. By [19], we also have unique entropy solutions.

It follows from (3.20) and (3.21) that

$$u_0(\xi(\alpha t), t) \xrightarrow{\text{d}} 0.$$

To prove the same thing for $v$, note that by (2.11),

$$v(\alpha t, t) = \frac{1}{\sqrt{a(\alpha t)}} u_0(\xi(\alpha t), t). \tag{3.22}$$

Since the random variables $\frac{1}{\sqrt{a(\alpha t)}}$ are tight (by stationarity of $a$), the product in (3.22) goes to zero in distribution. This ends the proof of part 1 of the theorem.

Similarly, if $\alpha < c$, $B_t$ grows exponentially fast with probability one (since the exponential prefactor in (2.21) does), while $D_t$ grows at most like $\sqrt{t}$ (if at all). Hence in this case

$$\frac{B_t}{\frac{B_t}{u_l} + D_t} \to u_l \tag{3.23}$$

and part 2 is proven for a positive $\nu$. Just as in the proof of part 1, it suffices to note now that the convergence is uniform in $\nu$ and part 2 of the theorem follows. Note that, unlike in part 1, $u_\nu$ does not converge to zero and therefore, we do not obtain convergence of $v(\alpha t, t)$. In fact, (2.11) shows that $v(\alpha t, t)$ fluctuates as $t \to \infty$. Let now

$$x = ct + z\sqrt{t}. \tag{3.24}$$

We want to find the distribution of

$$\frac{B_t}{\frac{B_t}{u_l} + D_t}$$

in the limit when $t \to \infty$. We know that $D_t$ behaves as $\sqrt{t}$ times a constant, proportional to $\sqrt{\nu}$. Roughly speaking, $B_t$ is either exponentially large or exponentially small and depending on which one of these two things happens, the above ratio is close to $u_l$ or to 0. This will be seen from the calculation below. Let $y \in (0, u_l)$ (note that clearly $0 < \frac{B_t}{u_l^{-1}B_t + D_t} < u_l$). We have:

$$\begin{aligned} P[\frac{B_t}{u_l^{-1}B_t + D_t} \leq y] = P[\frac{B_t}{D_t} \ &\leq \ \frac{u_l y}{u_l - y}] = P[\frac{\log B_t}{\sqrt{t}} - \frac{\log D_t}{\sqrt{t}} \leq \frac{\log \frac{u_l y}{u_l - y}}{\sqrt{t}}] \\ &= \ P[\nu\frac{\log B_t}{\sqrt{t}} - \nu\frac{\log D_t}{\sqrt{t}} \leq \nu\frac{\log \frac{u_l y}{u_l - y}}{\sqrt{t}}]. \end{aligned} \tag{3.25}$$

Now, $\nu\frac{\log \frac{u_l y}{u_l - y}}{\sqrt{t}} \to 0$ and, since $D_t$ is of the order $\sqrt{\nu t}$, we also have $\nu\frac{\log D_t}{\sqrt{t}} \xrightarrow{\mathrm{d}} 0$ as well. Both convergence statements hold uniformly in $\nu$ in the sense explained in (3.1). The limit of the probability in (3.25) is therefore equal to

$$\lim_{t\to\infty} P[\frac{\nu \log B_t}{\sqrt{t}} \leq 0]$$

Similarly to [23] we now write

$$B_t = p(t)\tilde{B}_t, \tag{3.26}$$

where $p(t) = e^{-\frac{u_l}{2\nu}(\xi - \frac{u_l}{2}t) - \frac{\hat{x}(\xi - u_l t)}{2\nu}}$ and

$$\tilde{B}_t = u_l\sqrt{t}\int_{\frac{\xi - u_l t}{\sqrt{t}}}^{\infty} e^{-\frac{\eta^2}{4\nu} + \frac{1}{2\nu}[\hat{x}(\xi - u_l t) - \hat{x}(\xi - u_l t - \sqrt{t}\eta)]}\, d\eta. \tag{3.27}$$

Changing the variable of integration, as in the proof of Proposition 3.2, to $y = t^{-\frac{1}{6}}\eta$, we obtain

$$\tilde{B}_t = t^{\frac{1}{6}} \int_{\frac{\xi - u_l t}{t^{\frac{2}{3}}}}^{\infty} e^{\frac{t^{\frac{1}{3}}}{2\nu}[\frac{\hat{x}(\xi-u_l t)-\hat{x}(\xi-u_l t-t^{\frac{2}{3}}y)}{t^{\frac{1}{3}}}-\frac{y^2}{2}]}\, dy. \tag{3.28}$$

Now,

$$\frac{\hat{x}(\xi - u_l t) - \hat{x}(\xi - u_l t - t^{\frac{2}{3}} y)}{\mu^{-\frac{3}{2}}\sigma t^{\frac{1}{3}}}$$

converges in distribution to the Wiener process in the variable $y$, on any finite interval of $y$. Using the Skorohod representation theorem (Theorem 4.2), we can find a probability space $(\Omega, \mathcal{F}, P)$ and processes $\tilde{\tau}^{(t)}(y,\omega), W(y,\omega)$ such that $W$ is a Wiener process,

$$\tilde{\tau}^{(t)}(y,\omega) \stackrel{\mathcal{D}}{=} \frac{\hat{x}(\xi - u_l t) - \hat{x}(\xi - u_l t - t^{\frac{2}{3}} y)}{\sigma t^{\frac{1}{3}}}$$

and for almost every $\omega$

$$\tilde{\tau}^{(t)}(y,\omega) \to W(y,\omega)$$

uniformly in $y$ belonging to any compact interval. Lemma 4.1 now implies that as $t \to \infty$, the distribution of

$$t^{-\frac{1}{3}} \log \tilde{B}_t$$

converges to that of a constant times

$$\sup_y\left(\frac{y^2}{2} - W_y\right).$$

(That the assumptions of the lemma are satisfied, follows easily from the assumption A4.) It follows that

$$\nu t^{-\frac{1}{2}} \log \tilde{B}_t \xrightarrow{\mathrm{d}} 0$$

uniformly in $\nu$, and we just need to study the behavior of $\nu t^{-\frac{1}{2}} \log p(t)$. Now,

$$\nu t^{-\frac{1}{2}} \log p(t) = -\frac{1}{2}\left[\frac{u_l(\xi - \frac{u_l}{2}t)}{\sqrt{t}} + \frac{\hat{x}(\xi - u_l t)}{\sqrt{t}}\right], \tag{3.29}$$

where $\xi = \xi(ct + z\sqrt{t})$. Since $c = \frac{1}{2\mu^2}$ and $u_l = \frac{1}{\mu}$, substituting (3.24), we get from the central limit theorem for $\xi$ in assumption $A4$ that:

$$\frac{u_l(\xi(ct + z\sqrt{t}) - \frac{u_l}{2}t)}{\sqrt{t}} \xrightarrow{\mathrm{d}} z + \frac{\sigma}{\mu^2\sqrt{2}} W_1,$$

where here and in the sequel, $W_1$ is used to denote a Gaussian random variable with mean zero and unit variance. Also, using the central limit theorem for $\hat{x}$ ((2.16) with $b = \frac{1}{2\mu}$), we obtain:

$$\frac{\hat{x}(\xi - u_l t)}{\sqrt{t}} \xrightarrow{\text{d}} \frac{\sigma}{\mu^2\sqrt{2}} W_1.$$

In order to study the limiting distribution of the sum in (3.29), we need to know the joint distribution of the variables $\frac{u_l(\xi(ct+z\sqrt{t})-\frac{u_l}{2}t)}{\sqrt{t}}$ and $\frac{\hat{x}(\xi-u_l t)}{\sqrt{t}}$ in the limit when $t \to \infty$. We claim that the two-dimensional random variables

$$(\frac{u_l(\xi(ct + z\sqrt{t}) - \frac{u_l}{2}t)}{\sqrt{t}}, \frac{\hat{x}(\xi - u_l t)}{\sqrt{t}})$$

converge in distribution to a two-dimensional Gaussian with independent coordinates of mean $z$ and 0 respectively. To prove this claim, we express the joint distribution function of the coordinates through a finite-dimensional distribution of the process $\xi = \xi(ct + z\sqrt{t})$:

$$P[(\frac{u_l(\xi(ct + z\sqrt{t}) - \frac{u_l}{2}t)}{\sqrt{t}} \le y_1\,; \frac{\hat{x}(\xi - u_l t)}{\sqrt{t}} \le y_2] = P[\frac{\xi(ct + z\sqrt{t}) - \mu ct - \mu z\sqrt{t}}{\sigma\sqrt{t}}$$

$$\le \frac{y_1}{u_l} - \mu z + (\frac{u_l}{2} - \mu c)\sqrt{t}; x(\xi(ct + z\sqrt{t}) - u_l t) \le \frac{\xi - u_l t}{\mu} + y_2\sqrt{t}] \tag{3.30}$$

Using the fact that $\mu c = \frac{u_l}{2}$ and rewriting the event

$$\{x(\xi(ct + z\sqrt{t}) - u_l t) \le \frac{\xi(ct + z\sqrt{t}) - u_l t}{\mu} + y_2\sqrt{t}\} \tag{3.31}$$

as

$$\{\xi(\frac{\xi(ct + z\sqrt{t}) - u_l t}{\mu} + y_2\sqrt{t}) \ge \xi(ct + z\sqrt{t}) - u_l t\} = \tag{3.32}$$

$$\{\frac{\xi(\frac{\xi(ct+z\sqrt{t})-u_l t}{\mu} + y\sqrt{t}) - \xi + u_l t - \mu y_2\sqrt{t}}{\sigma\sqrt{t}} \ge \frac{-\mu y_2}{\sigma}\}, \tag{3.33}$$

we obtain, using the invariance principle of Assumption A4, the following expression for the joint distribution function (3.30) in the limit $t \to \infty$:

$$P[W_{\mu c} \le \frac{y_1}{u_l} - \mu z\,; W_{-\mu c} \ge \frac{-\mu y_2}{\sigma}]. \tag{3.34}$$

The last expression clearly factors, since the processes $(W_x)_{x\ge 0}$ and $(W_x)_{x\le 0}$ are independent, and this proves the claimed independence. Note that we also recovered the formula for the variance of each coordinate. In fact, the calculation above can be generalized to provide an alternative proof that the finite-dimensional distributions of the rescaled process $\hat{x}$ converge to those of the Wiener process (this fact is a part of Theorem 4.1). Going back to the main line of the proof and adding the variances of the two limiting normal distributions, we see that the sum

$$\frac{u_l(\xi(ct+z\sqrt{t})-\frac{u_l}{2}t)}{\sqrt{t}}+\frac{\hat{x}(\xi-u_lt)}{\sqrt{t}}$$

converges in distribution to a Gaussian random variable with mean $z$ and variance $\frac{\sigma^2}{\mu^4}$. Hence,

$$P[\nu t^{-\frac{1}{2}}\log p(t)\leq 0]\to P[-\frac{1}{2}(z+\frac{\sigma}{\mu^2}W_1)\leq 0]=P[W_1\geq -\frac{\mu^2}{\sigma}z]=\mathcal{N}(\frac{\mu^2}{\sigma}z),$$

where $\mathcal{N}(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s}e^{-\frac{s'^2}{2}}\,ds'$ is the error function. This ends the proof for $\nu>0$. Because in this case $u(ct+z\sqrt{t},t)$ does not converge to a constant, a more careful argument is necessary to carry out the $\nu\to 0$ limit. Since the above estimates were uniform in $\nu$, we have actually shown that for every $y\in(0,u_l)$,

$$P[\inf_{\nu\leq\nu_0}u_\nu(ct+z\sqrt{t},t)\leq y]\to\mathcal{N}(\frac{\mu^2}{\sigma}z).$$

In exactly the same way, we can show that

$$P[\inf_{\nu\leq\nu_0}u_\nu(ct+z\sqrt{t},t)>y]\to 1-\mathcal{N}(\frac{\mu^2}{\sigma}z).$$

Taking the limit $\nu\to 0$ we obtain from these two relations

$$\liminf_{t\to\infty}P[u_0(ct+z\sqrt{t},t)\leq y]\geq\mathcal{N}(\frac{\mu^2}{\sigma}z),\quad a.e.\ in\ z. \tag{3.35}$$

and

$$\liminf_{t\to\infty}P[u_0(ct+z\sqrt{t},t)>y]\geq 1-\mathcal{N}(\frac{\mu^2}{\sigma}z),\quad a.e.\ in\ z. \tag{3.36}$$

Hence

$$1\leq\liminf_{t\to\infty}P[u_0(ct+z\sqrt{t},t)\leq y]+\liminf_{t\to\infty}P[u_0(ct+z\sqrt{t},t)>y]\leq$$
$$\liminf_{t\to\infty}(P[u_0(ct+z\sqrt{t},t)\leq y]+P[u_0(ct+z\sqrt{t},t)>y])=1,\quad a.e.\ in\ z,$$

so that (3.35) and (3.36) are satisfied as equalities. If there exists a sequence $t_n\to\infty$ such that

$$\lim_{n\to\infty}P[u_0(ct_n+z\sqrt{t_n},t_n)\leq y]\geq\mathcal{N}(\frac{\mu^2}{\sigma}z)+\epsilon,\quad a.e.\ in\ z,$$

then, since also

$$\liminf_{n\to\infty} P[u_0(ct_n + z\sqrt{t_n}, t_n) > y] \geq 1 - \mathcal{N}(\frac{\mu^2}{\sigma} z), \quad a.e.\ in\ z$$

adding the last two equations, we get a contradiction. Therefore

$$\lim_{t\to\infty} P[u_0(ct + z\sqrt{t}, t) \leq y] = \mathcal{N}(\frac{\mu^2}{\sigma} z), \quad a.e.\ in\ z,$$

and

$$\lim_{t\to\infty} P[u_0(ct + z\sqrt{t}, t) > y] = 1 - \mathcal{N}(\frac{\mu^2}{\sigma} z), \quad a.e.\ in\ z.$$

Since the right hand side is continuous in $z$, these two equalities are valid for all $z$, and so together with (2.11) prove the last part of the theorem.

# 4 Appendix

We prove the results invoked in the proof of the Theorem 2.1 here.

**Lemma 4.1** *Let $\varphi_\lambda(u) \in C(R^1)$, $\varphi_\lambda(u) \to \varphi(u)$, uniformly on compact sets of $u$ as $\lambda \to \infty$; and $C_1 u^2 \leq |\varphi_\lambda(u)| \leq C_2 u^2$ for some positive constants $C_i$, $i = 1, 2$, uniformly in $\lambda \to \infty$. The limiting function $\varphi(u) \in C(R^1)$, $\varphi(u) < \varphi(u_0)$, $\forall u \neq u_0$. Here $c_0$ and $C$ are positive constants. Then for the probability measures $\mu_\lambda$ with densities $\frac{\exp\{\lambda\varphi_\lambda(u)\}du}{\int_{R^1}\exp\{\lambda\varphi_\lambda(u)\}du}$, we have as $\lambda \to +\infty$:*

$$1. \qquad \mu_\lambda \xrightarrow{d} \delta(u_0), \quad \text{the unit mass at } u_0; \tag{4.1}$$

$$2. \qquad \text{the expected value } E_{\mu_\lambda}(u) \to u_0; \tag{4.2}$$

$$3. \qquad \lambda^{-1} ln \int_{R^1} \exp\{\lambda\varphi_\lambda(u)\}du \to \varphi(u_0). \tag{4.3}$$

**Proof:** Let $\psi(u) \in C^\infty(R^1)$, $|\psi(u)| \leq C(1 + u^2)^m$, for some $m > 0$. By the assumption on $\varphi_\lambda(u)$, $\forall\ \delta > 0$, $\exists\ \Lambda_1 = \Lambda_1(\delta)$ such that if $\lambda \geq \Lambda_1$, any maximal point $u_\lambda$ of $\varphi_\lambda(u)$ lies in $[u_0 - \delta, u_0 + \delta]$, and so:

$$\begin{aligned} & \int \psi(u)d\mu_\lambda - \psi(u_0) = \int (\psi(u) - \psi(u_0))d\mu_\lambda \\ = & \int_{|u-u_0|\leq 4\delta} (\psi(u) - \psi(u_0))d\mu_\lambda + \int_{|u-u_0|\geq 4\delta} (\psi(u) - \psi(u_0))d\mu_\lambda \\ = & \ I + II, \end{aligned} \tag{4.4}$$

The first term is bounded as:

$$|I| \leq \sup_{|u-u_0|\leq 4\delta} |\psi(u) - \psi(u_0)| \equiv \omega(\delta, u_0). \tag{4.5}$$

Let us denote $\omega_0 = \omega_0(\delta, u_0) = \limsup_{\lambda\to\infty} \sup_{|u-u_0|\leq 4\delta} |\varphi_\lambda(u) - \varphi_\lambda(u_0)|$. Now the second term can be written as:

$$\begin{aligned} II &= \int_{|u-u_0|\leq 4\delta} (\psi(u) - \psi(u_\lambda))d\mu_\lambda + \int_{|u-u_0|\geq 4\delta} (\psi(u_\lambda) - \psi(u_0))d\mu_\lambda \\ &\leq 2\omega(\delta; u_0) + \int_{|u-u_0|\geq 4\delta} (\psi(u_\lambda) - \psi(u_0)) \frac{\exp\{\lambda(\varphi_\lambda(u) - \varphi_\lambda(u_\lambda))\}du}{\int_{R^1} \exp\{\lambda(\varphi_\lambda(u) - \varphi_\lambda(u_\lambda))\}du}. \end{aligned} \tag{4.6}$$

By our assumption on $\varphi_\lambda$, we see that for any given $\delta > 0$, there exist constants $K_i = K_i(\delta) > 0$, $i = 1, 2$, $\Lambda_1 = \Lambda_1(\delta)$, such that if $\lambda \geq \Lambda_1$ and $u \notin [u_0 - 3\delta, u_0 + 3\delta]$,

$$-K_2(\delta)|u - u_\lambda|^2 \leq \varphi_\lambda(u) - \varphi_\lambda(u_\lambda) \leq -K_1(\delta)|u - u_\lambda|^2. \tag{4.7}$$

On the other hand, for any $\delta_1 > 0$, there is $\Lambda_2(\delta)$ such that if $\lambda \geq \Lambda_2$, we have:

$$\begin{aligned} & \int \exp\{\lambda(\varphi_\lambda(u) - \varphi_\lambda(u_\lambda))\}du \\ \geq & \int_{u\in[u_0-\delta_1, u_0+\delta_1]} \exp\{-4\lambda\omega_0(\delta_1, u_0)\}du \\ = & 2\delta_1 \exp\{-4\lambda\omega_0(\delta_1, u_0)\}. \end{aligned} \tag{4.8}$$

Hence:

$$\begin{aligned} & |II| \leq 2\omega(\delta, u_0) \\ + & 2^{-1}\delta_1^{-1} \exp\{4\lambda\omega_0(\delta_1, u_0)\} \int_{|u-u_0|\geq 4\delta} (1 + |u - u_0|^2)^m \exp\{-\lambda K(\delta)|u - u_\lambda|^2\}du \\ \leq & 2\omega(\delta; u_0) + \delta_1^{-1} \exp\{4\lambda\omega_0(\delta_1, u_0)\}C(\delta, m) \int_{|u-u_\lambda|\geq 3\delta} \exp\{-\frac{\lambda}{2}K(\delta)|u - u_\lambda|^2\}du \\ \leq & 2\omega(\delta, u_0) + \delta_1^{-1} \exp\{4\lambda\omega_0(\delta_1, u_0)\}C(\delta, m) \int_{|u-u_\lambda|\geq 3\delta} \exp\{-\frac{3\lambda}{2}K(\delta)\delta|u - u_\lambda|\}du \\ \leq & 2\omega(\delta, u_0) + \delta_1^{-1} \exp\{4\lambda w_0(\delta_1, u_0)\}C(\delta, m)\frac{4}{3\delta K(\delta)\lambda} \exp\{-\frac{9\lambda}{2}K(\delta)\delta^2\}. \end{aligned} \tag{4.9}$$

Choosing $\delta_1$ small enough so that $8\omega_0(\delta_1, u_0) < 9\delta^2 K(\delta)$, and letting $\lambda \to \infty$, we have:

$$\limsup_{\lambda\to\infty} |II| \leq 2w(\delta, u_0),$$

while

$$\limsup_{\lambda\to\infty} |I| \le \omega(\delta; u_0).$$

Finally, sending $\delta \to 0$, we conclude that

$$\int \psi(u)d\mu_\lambda \to \psi(u_0).$$

This is, in particular, true for all bounded $\psi$, which implies part 1 of the lemma, and also for $\psi(u) = u$:

$$\int u d\mu_\lambda \to u_0. \tag{4.10}$$

which proves part 2. For part 3, we only need to show that:

$$\lambda^{-1} ln \int \exp\{\lambda(\varphi_\lambda(u) - \varphi(u_0))\}du \to 0. \tag{4.11}$$

The above integral can be decomposed into the sum of two integrals over $|u-u_0| \le 4\delta$ and its complement, which we denote by $I_{\lambda,1}$ and $I_{\lambda,2}$ respectively. Notice that for $|u-u_0| \le 4\delta$, there exists a positive constant $K'(\delta)$, $K'(\delta) \to 0$ as $\delta \to 0$, such that:

$$|\varphi_\lambda(u) - \varphi(u_0)| = |\varphi_\lambda(u) - \varphi_\lambda(u_0) + \varphi_\lambda(u_0) - \varphi(u_0)| \le K'(\delta).$$

We now have the upper bound for $I_{\lambda,1}$:

$$I_{\lambda,1} = \int_{|u-u_0|\le 4\delta} e^{\lambda(\varphi_\lambda(u)-\varphi(u_0))} \le 4\delta e^{\lambda K'(\delta)}. \tag{4.12}$$

Similarly, we have the lower bound $4\delta e^{-\lambda K'(\delta)}$. Now let us bound $I_{\lambda,2}$ from above using (4.7) as:

$$\begin{aligned} I_{\lambda,2} &= \int_{|u-u_0|>4\delta} \exp\{\lambda(\varphi_\lambda(u) - \varphi(u_0))\}du \\ &= \int_{|u-u_0|>4\delta} \exp\{\lambda(\varphi_\lambda(u) - \varphi_\lambda(u_\lambda))\} \exp\{\lambda(\varphi_\lambda(u_\lambda) - \varphi(u_0))\}\, du \\ &\le \exp\{\lambda(\varphi_\lambda(u_\lambda) - \varphi(u_0))\} \int_{|u-u_0|>4\delta} \exp\{-\lambda K_1(\delta)|u-u_\lambda|^2\}du \\ &\le \exp\{\lambda(\varphi_\lambda(u_\lambda) - \varphi(u_0))\} \int_{R^1} \exp\{-\lambda K_1(\delta)u^2\}du \\ &\le \exp\{\lambda(\varphi_\lambda(u_\lambda) - \varphi(u_0))\} \frac{C}{\sqrt{\lambda K_1(\delta)}}. \end{aligned} \tag{4.13}$$

Similarly, we have the lower bound for $I_{\lambda,2}$ with $K_2$ replacing $K_1$. Combining (4.12), (4.13), and the analogous lower bounds, and using the arbitrary smallness of $\delta$, we arrive at (4.11). The proof is complete.

**Theorem 4.1** *Let*

$$T(b) = \inf\{x \geq 0 \,:\, \xi(x) = b\}$$

*Under above assumptions we have for each* $b_0 > 0$

$$(\frac{T(tb) - \frac{tb}{\mu}}{\mu^{-\frac{3}{2}}\sigma\sqrt{t}})_{0\leq b\leq b_0} \xrightarrow{\mathcal{D}} (W_b)_{0\leq b\leq b_0}$$

**Remark 4.1** *The theorem is thus roughly saying that the invariance principle (2.3) for the rescalings of the process* $\xi$ *implies an invariance principle for the rescalings of the hitting times process. Central limit theorems for hitting times are known in similar contexts (see e.g. [5], p.116). While the present theorem may also exist in literature, a precise reference is not known to the authors.*

Proof:
The first observation is that the hitting times $T(tb)$ satisfy a law of large numbers:

$$\frac{T(tb)}{t} \xrightarrow{\text{P}} \frac{b}{\mu}.$$

We will show it in a stronger form: for any $b_0$ and for any $\eta > 0$

$$P[\exists b \,:\, 0 \leq b \leq b_0; \frac{1}{t}|T(tb) - \frac{tb}{\mu}| > \eta] \to 0, \tag{4.14}$$

as $t \to \infty$. We have, by positivity of $a$:

$$P[\exists b \,:\, 0 \leq b \leq b_0; \frac{T(tb) - \frac{tb}{\mu}}{t} > \eta] = P[\exists b \,:\, 0 \leq b \leq b_0 : \xi(t(\frac{b}{\mu} + \eta)) \leq tb]$$

$$= P[\inf_{0\leq b\leq b_0} \frac{\xi(t(\frac{b}{\mu} + \eta)) - \mu t(\frac{b}{\mu} + \eta)}{\sqrt{t}} \leq -\eta\mu\sqrt{t}]$$

The last quantity clearly goes to zero when $t$ goes to infinity, since by (2.3) the processes $\frac{\xi(t(\frac{b}{\mu}+\eta))-\mu t(\frac{b}{\mu}+\eta)}{\sqrt{t}}$ converge in law to the process $(\sigma W_{\frac{b}{\mu}+\eta})_{|b|\leq b_0}$, while the numbers $-\eta\mu\sqrt{t}$ go to $-\infty$. A similar argument shows that

$$P[\exists b \,:\, 0 \leq b \leq b_0; \frac{1}{t}T(tb) - \frac{tb}{\mu} < -\eta] \to 0,$$

and (4.14) is proven. Next, we show that for a fixed $b$

$$\frac{T(tb) - \frac{tb}{\mu}}{\mu^{-\frac{3}{2}}\sigma\sqrt{t}} \xrightarrow{\text{d}} W_b. \tag{4.15}$$

Let

$$\zeta(x) = \xi(x) - \mu x,$$

so that, in particular, $E[\zeta(x)] = 0$. We have:

$$\frac{T(tb) - \frac{tb}{\mu}}{\sqrt{t}} = -\frac{1}{\mu}\frac{tb - \mu T(tb)}{\sqrt{t}} = -\frac{1}{\mu}\frac{\zeta(T(tb))}{\sqrt{t}}, \tag{4.16}$$

since, by definition, $\xi(T(tb)) = tb$. The argument of $\zeta$ in the expression on the right-hand side of (4.16) is a random time $T(tb)$, which is not far from $\frac{tb}{\mu}$. It is therefore natural to expect that $\zeta(T(tb))$ does not differ much from $\zeta(\frac{tb}{\mu})$. In fact, as we will now show,

$$\frac{\zeta(T(tb)) - \zeta(\frac{tb}{\mu})}{\sqrt{t}} \xrightarrow{\mathrm{d}} 0. \tag{4.17}$$

Indeed, for any $\epsilon, \eta > 0$, we have

$$\begin{aligned} P[|\zeta(T(tb)) - \zeta(\frac{tb}{\mu})| > \epsilon\sqrt{t}] \quad &\leq \quad P[\sup_{0\leq b\leq b_0} |T(tb) - \frac{tb}{\mu}| \geq \eta t] \\ &+ \quad P[\sup_{|c-\frac{tb}{\mu}|\leq \eta t} |\zeta(c) - \zeta(\frac{tb}{\mu})| > \epsilon\sqrt{t}]. \end{aligned} \tag{4.18}$$

The first term goes to zero when $t \to \infty$, in view of (4.14). Rewriting the second term as

$$P[\sup_{|c-\frac{tb}{\mu}|\leq \eta t} \frac{|\zeta(c) - \zeta(\frac{tb}{\mu})|}{\sigma\sqrt{t}} > \frac{\epsilon}{\sigma}]$$

and using the invariance principle for $\xi$ as in (2.3), we see that the second term in (4.18) converges to

$$P[\sup_{|y-\frac{b}{\mu}|\leq \eta} |W(y) - W(\frac{b}{\mu})| > \frac{\epsilon}{\sigma}].$$

Taking now $\eta$ to zero, we obtain (4.15). Since, by (2.3) again,

$$\frac{\zeta(\frac{tb}{\mu})}{\sigma\sqrt{t}} \xrightarrow{\mathrm{d}} W_{\frac{b}{\mu}},$$

(4.17) follows. This proves that one-dimensional distributions converge to those of the Wiener process. A similar calculation shows convergence of arbitrary finite-dimensional distributions to those of the Wiener process. We only sketch the argument, since, apart from the notation, it does not contain any new elements. Given $b_1, \ldots, b_n \in [0, b_0]$, we have by (4.17):

$$\frac{\zeta(T(tb_i)) - \zeta(\frac{tb_i}{\mu})}{\sqrt{t}} \xrightarrow{\mathrm{d}} 0$$

for $i = 1, \ldots, n$. This implies that the limit of the distribution of the random vector $\frac{1}{\mu^{-\frac{3}{2}}\sigma\sqrt{t}}(T(tb_i) - \frac{tb_i}{\mu})$ is the same as that of $\frac{1}{\mu^{-\frac{3}{2}}\sigma\sqrt{t}}\zeta(\frac{tb_i}{\mu})$, provided that the latter exists. It follows from the invariance principle for the process $\xi$ that the limit in fact exists and is the finite-dimensional distribution of the Wiener process, as claimed.

To complete the proof it remains to show that the family of processes $(\tau^{(t)}(b))_{0\le b\le b_0} = (\frac{1}{\sqrt{t}}(T(tb) - \frac{tb}{\mu}))_{0\le b\le b_0}$ is relatively compact in the topology of convergence in law. We are using here Theorem 8.1 of [3], which is used in the sequel as our principal reference for questions related to convergence in law. According to a standard criterion of tightness (Theorem 8.2 of [3]), it is enough to show that the following two conditions hold:

1. For each positive $\eta$ there exists an $a$ such that

$$P[|\tau^{(t)}(0)| > a] \le \eta.$$

2. For each positive $\epsilon$ and $\eta$, there exists a $\delta_0$ with $\delta \in (0, \delta_0)$ and a $t_0$ such that $|\tau^{(t)}(b) - \tau^{(t)}(c)| \ge \epsilon$

$$P[\sup_{b\le c\le b+\delta} |\tau^{(t)}(b) - \tau^{(t)}(c)| \ge \epsilon] \le \eta$$

for all $t \ge t_0$.

Condition 1 is obviously satisfied, since $\tau^{(t)}(0) = 0$ for each $t$. To prove that condition 2 also holds, let us fix a $\theta > 0$. If there exist $b, c \in [0, \delta_0]$ such that $|\tau^{(t)}(b) - \tau^{(t)}(c)| \ge \epsilon$, then with $b' = \frac{T(tb)}{t}$ and $c' = \frac{T(tc)}{t}$, we have

$$|\zeta(tb') - \zeta(tc')| = \mu\sqrt{t}|\tau^{(t)}(b) - \tau^{(t)}(c)| \ge \epsilon\mu\sqrt{t}.$$

Also:

$$|b'-c'| = \frac{1}{t}|T(tb)-T(tc)| \le \frac{1}{t}(|T(tb)-\frac{tb}{\mu}|+|\frac{tb}{\mu}-\frac{tb}{\mu}|+|\frac{tc}{\mu}-T(tc)|) \le \frac{1}{t}(\theta t+\frac{t\delta}{\mu}+\theta t) = 2\theta+\frac{\delta}{\mu}.$$

Therefore:

$$P[\sup_{b\le c\le b+\delta} |\tau^{(t)}(b) - \tau^{(t)}(c)| \ge \epsilon] \le$$

$$P[\sup_{0\le b\le b_0} |T(tb) - \frac{tb}{\mu}| \le \theta t; \sup_{|b'-c'|\le 2\theta+\frac{\delta}{\mu}} |\zeta(tb') - \zeta(tc')| \ge \epsilon\mu\sqrt{t}] + P[\sup_{0\le b\le b_0} |T(tb) - \frac{tb}{\mu}| \ge \theta t].$$

The first term on the right-hand side is obviously bounded by

$$P[\sup_{|b'-c'|\leq 2\theta+\frac{\delta}{\mu}} |\zeta(tb')-\zeta(tc')| \geq \epsilon\mu\sqrt{t}]. \tag{4.19}$$

and this probability converges to

$$P[\sup_{|b'-c'|\leq 2\theta+\frac{\delta}{\mu}} |W_b - W_c| \geq \mu\epsilon],$$

by virtue of the invariance principle for $\xi$ in (2.3). Choosing $\delta$ small enough and taking $\theta = \delta$ (say), we can make this limit smaller than $\frac{\eta}{4}$, since almost all paths of the Wiener process, are uniformly continuous on the compact interval $[0, b_0]$. For $t$ large enough, (4.19) is thus bounded by $\frac{\eta}{2}$. As we have seen in (4.15), for any fixed $\theta$ the second term goes to zero, so choosing $t_0$ large enough we can make the second term in (4.19) smaller than $\frac{\eta}{2}$ as well. Theorem 8.2 of [3] together with the convergence of the finite-dimensional distributions, proven above, implies the desired statement of the theorem. The proof is complete.

In the proof of Theorem 4.1, we use a theorem by Skorohod, which we state here in a special case, suitable for our application. A general version can be found, together with a proof in [18] (Theorem 86.1).

**Theorem 4.2** *Suppose that* $(\tau^{(t)}(b))_{0\leq b\leq b_0}$ *is a family of stochastic processes, converging in law to a process* $\tau$ *as* $t \to \infty$*:*

$$\tau^{(t)} \overset{\mathcal{D}}{\to} \tau.$$

*Then there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and processes* $\tilde{\tau}^{(t)}, \tilde{\tau}$ *such that:*

$$\tilde{\tau}^{(t)} \overset{\mathcal{D}}{=} \tau^{(t)};$$

$$\tilde{\tau} \overset{\mathcal{D}}{=} \tau$$

*and with* $P$*-probability one*

$$\tilde{\tau}^{(t)}(b) \to \tilde{\tau}(b)$$

*uniformly in* $0 \leq b \leq b_0$*.*

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